DEMONSTRATIO MATHEMATICA
Vol. XXV No 1-2 1992

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SPECIAL PROBLEMS OF SURFACE THEORY IN THE EUCLIDEAN 3-DIMENSIONAL SPACE

1. Introduction

In the proof of the theorem of Hilbert which asserts that there does not exist an isometric immersion of the Lobachevski plane L^2 in the 3-dimensional Euclidean space E^3, as explained in [1] is used the property of asymptotic lines on a hypothetic, complete (with respect to a distance function defined by a Riemannian metric of constant Gauss curvature $K=-1$) surface in E^3, that every two asymptotic lines which belong to different families have a point in common. This property, established by the argument that if there exists a diffeomorphism f of the Euclidean plane E^2 referred to coordinates (u_1,u_2) on open set $U\subset E^2$ referred to coordinates (v_1,v_2), then there exists a diffeomorphism F of E^2 on U, which transforms coordinate straightlines u_2=const, u_1=const on intersections of coordinate straightlines v_2=const, v_1=const with U, together with the fact that asymptotic lines on the hypothetical surface define a Chebyshev net enable us to construct an increasing sequence of Chebyshev rectangles which exhaust the surface. The existence of the diffeomorphism f implies that the Jacobi determinant of the transformation $v_1=f_1(u_1,u_2)$, $v_2=f_2(u_1,u_2)$, $f=(f_1,f_2)$, is different from zero for every $(u_1,u_2)\in E^2$. The set $U\subset E^2$ provided with a complete metric is the underlying manifold of the hypothetical surface referred to the asymptotic lines. The argument that asymptotic lines can be prolonged in both directions to curves of infinite length is not sufficient to prove the existence of the diffeomorphism F which preserves coordinate straightlines or in other words to establish the behavior of the asymptotic lines like that of straightlines of E^2 parallel to the coordinate axes Ou_1, Ou_2. If we take e.g. the part of every such straightline (u_2=const, u_1=const) contained in the unit disk $D^2 \subset E^2$ with center at the origin and provide D^2 with a complete metric, then we get in D^2 two families of curves (u_2=const, u_1=const) which have not the property that every two curves which belong to different families have a point in common, although every curve of every of these two families has infinite length in both directions with respect to the complete metric. Hence a diffeomorphism of E^2 on D^2 cannot preserve coordinate straightlines.

In this paper we do not use geometric properties of asymptotic lines and we omit any constructions by means of these curves on surfaces of negative Gauss curvature at every point of a considered surface.

We prove that among surfaces of Gauss curvature $K \le -c^2$, $c \ne 0$, these of constant Gauss curvature are distinguished by the following property: the surface area of a connected surface with constant Gauss curvature $K=-c^2$, $c \ne 0$, is less than or equal $\frac{2\pi}{c^2}$ (Theorem 1). This theorem is proved by the assumption that the considered surface is referred to the lines of curvature. In Theorem 2 as an addition to Theorem 1 we prove that this restriction can be removed. As a consequence of Theorems 1 and 2 we get the theorem of Hilbert (Theorem 3). The theorem of Efimov [2] (Theorem 4) which asserts that there does not exist a complete surface in E^3 with Gauss curvature bounded from above by a negative constant: $K \le -c^2$, $c \ne 0$, cannot be proved by a similar estimation of the surface area. We prove the theorem of Efimov by the contrary argument and the underlying idea of the proof can be stated as follows. On a complete, connected surface in E^3 with Gauss curvature $K<0$ there exists an open, connected subset with infinite surface area which by means of the Gauss map is diffeomorphic with a subset of the unit sphere $S^2 \subset E^3$ of surface area less than or equal 2π. This implies that on this open subset the Gauss curvature cannot be bounded from above by a negative constant. This proof is possible, since we get a direct relation between the first and third quadratic forms of the surface, and this enable us to define the Gauss map as a mapping from the underlying manifold $M^2 \subset E^2$ of the surface to the sphere S^2 without any reference to the surface itself.

To get this results we refer the system of equations of surface theory to the lines of curvature and describe a solution of the theorem egregium, the Gauss equation and the Codazzi-Mainardi equations in these coordinates (Propositions 1, 2 and 3).

The derivatives with respect to variables u and v are marked by numbers 1 and 2 after the coma. All functions are supposed to be differentiable, this means of class C^∞.

2. Preliminaries

Let

(2.1) $$x:M^2 \longrightarrow E^3,\ M^2 \subset E^2,$$

denote an isometric immersion of a Riemannian, connected manifold (M^2,g), where g denotes the Riemannian metric of M^2, into the Euclidean 3-dimensional space E^3. We suppose that the Gauss curvature of the metric g is different from zero:

(2.2) $$K(u,v) \neq 0 \quad \text{for every} \quad (u,v) \in M^2$$

and that $x(u,v)$, v=const, and $x(u,v)$, u=const, are lines of curvature of the surface $x(M^2) \subset E^3$. The 2-dimensional plane E^2 is referred to Cartesian coordinates (u,v) with the origin 0. We denote by

(2.3) $$ds^2 = g_{11}du^2 + g_{22}dv^2$$

the Riemannian metric of M^2, and by

(2.4) $$d\sigma^2 = G_1du^2 + G_2dv^2$$

the spherical metric of the third quadratic form. The manifold M^2 provided with the metric (2.4) is denoted by (M^2,G). By

(2.5) $$n.d^2x = L_{11}du^2 + L_{22}dv^2$$

we denote the second quadratic form of the surface $x(M^2)$. The theorema egregium in arbitrary orthogonal coordinates has the form

$$(2.6)\qquad -\left[\frac{(\sqrt{g_{11}})_{,2}}{(\sqrt{g_{22}})}\right]_{,2} - \left[\frac{(\sqrt{g_{22}})_{,1}}{(\sqrt{g_{11}})}\right]_{,1} = K\sqrt{g_{11}g_{22}} .$$

The Gauss equation and the Codazzi-Mainardi equations have the form

$$(2.7)\qquad L_{11}L_{22} = Kg_{11}g_{22} ,$$

$$(2.8)\qquad L_{11,2} = \frac{1}{2}\,\frac{g_{11,2}}{g_{11}}\,L_{11} + \frac{1}{2}\,\frac{g_{11,2}}{g_{22}}\,L_{22} ,$$

$$(2.9)\qquad L_{22,1} = \frac{1}{2}\,\frac{g_{22,1}}{g_{11}}\,L_{11} + \frac{1}{2}\,\frac{g_{22,1}}{g_{22}}\,L_{22} .$$

From (2.7), (2.8) and (2.9) we get

$$(2.10)\qquad L_{11,1} = \left(\frac{K_{,1}}{K} + \frac{g_{11,1}}{g_{11}} + \frac{1}{2}\,\frac{g_{22,1}}{g_{22}}\right)L_{11} - \frac{1}{2}\,\frac{g_{22,1}}{Kg_{11}^2g_{22}}\,L_{11}^3 ,$$

$$(2.11)\qquad L_{11,2} = \frac{1}{2}\,\frac{g_{11,2}}{g_{11}}\,L_{11} + \frac{1}{2}\,Kg_{11,2}g_{11}\,\frac{1}{L_{11}} .$$

Let $\{P_i\}_{i\in N}$, where N denotes the positive integers, denotes an at most countable covering of M^2 by open rectangles $P_i\subset M^2$ with sides parallel to the coordinate axes 0u and 0v of E^2. As a consequence of the equations (2.10) and (2.11) we get

$$(2.12)\qquad L_{11}^2 = \frac{K^2g_{11}^2g_{22}}{\int_{u_i}^{u} Kg_{22,1}d\xi + \psi_i(v)} ,$$

$$L_{11}^2 = g_{11}\left[\int_{v_i}^{v} Kg_{11,2}d_\eta + \varphi_i(u)\right],$$

where $(u_i,v_i)\in P_i$ denotes a fixed point, $(u,v)\in P_i$ and $\varphi_i(u)$, $\psi_i(v)$ are positive functions. For every $(u,v)\in P_i\cap P_k$ we have

$$(2.13)\qquad \int_{u_i}^{u} K(\xi,v)g_{22,1}(\xi,v)d\xi =$$

$$= \int_{u_i}^{u_k} K(\xi,v)g_{22,1}(\xi,v)d\xi + \int_{u_k}^{u} K(\xi,v)g_{22,1}(\xi,v)d\xi.$$

Setting

$$(2.14)\qquad \psi_{ik}(v) = \int_{u_i}^{u_k} K(\xi,v)g_{22,1}(\xi,v)d\xi$$

we get for every $(u,v)\in P_i\cap P_k$

$$(2.15)\qquad \int_{u_i}^{u} K(\xi,v)g_{22,1}(\xi,v)d\xi + \psi_i(v) = \int_{u_k}^{u} K(\xi,v)g_{22,1}(\xi,v)d\xi + \psi_k(v),$$

where

$$(2.16)\qquad \psi_k(v) = \psi_i(v) + \psi_{ik}(v).$$

Similarly, for every $(u,v)\in P_i\cap P_k$ we have

$$(2.17)\qquad \int_{v_i}^{v} K(u,\eta)g_{11,2}(u,\eta)d\eta+\varphi_i(u) = \int_{v_k}^{v} K(u,\eta)g_{11,2}(u,\eta)d\eta + \varphi_k(u),$$

where

$$(2.18)\qquad \varphi_k(u) = \varphi_i(u) + \varphi_{ik}(u)$$

and

$$(2.19)\qquad \varphi_{ik}(u) = \int_{v_i}^{v_k} K(u,\eta)g_{11,2}(u,\eta)d\eta.$$

From (2.15) and (2.17) it follows that by means of the transition functions (2.14) and (2.19) the definition of L^2_{11} by (2.12) is valid on the whole of M^2. From (2.12) we get

$$(2.20)\qquad |K|\sqrt{g_{11}g_{22}} = \left[\int_{u_i}^{u} Kg_{22,1}d\xi+\psi_i(v)\right]^{\frac{1}{2}}\left[\int_{v_i}^{v} Kg_{11,2}d\eta+\varphi_i(u)\right]^{\frac{1}{2}}$$

for every $(u,v)\in P_i$, $i\in N$.

3. The spherical metric

We set

$$(3.1)\qquad G_{i1}= \int_{v_i}^{v} Kg_{11,2}d\eta+\varphi_i(u), \qquad G_{i2}= \int_{u_i}^{u} Kg_{22,1}d\xi+\psi_i(v),$$

where $(u,v)\in P_i$, $i\in N$. The formula (2.6) can be written in the form

$$(3.2)\qquad -\left(\frac{Kg_{11,2}}{|K|\sqrt{g_{11}g_{22}}}\right)_{,2} - \left(\frac{Kg_{22,1}}{|K|\sqrt{g_{11}g_{22}}}\right)_{,1} = 2|K|\sqrt{g_{11}g_{22}}\ .$$

From (2.20), (3.1) and (3.2) we get

$$(3.3)\qquad -\left(\frac{(\sqrt{G_{i1}})_{,2}}{(\sqrt{G_{i2}})}\right)_{,2} - \left(\frac{(\sqrt{G_{i2}})_{,1}}{(\sqrt{G_{i1}})}\right)_{,1} = \sqrt{G_{i1}G_{i2}}$$

for every $(u,v)\in P_i$, $i\in N$. From (2.6) and (3.3) it follows that (G_{i1},G_{i2}) is a Riemannian metric of Gauss curvature 1, written in orthogonal coordinates $(u,v)\in P_i\subset M^2$. From (2.15), (2.17) and (3.1) it follows that

$$(3.4)\qquad G_{i1}=G_{k1},\ G_{i2}=G_{k2}\quad \text{for}\quad (u,v)\in P_i\cap P_k.$$

Hence

$$(3.5)\qquad G=(G_1,G_2),\ \text{where}\ G_1(u,v)=G_{i1}(u,v),\ G_2(u,v)=G_{i2}(u,v)$$

for every $(u,v)\in P_i$, $i\in N$, is by means of the identifications (3.4) a Riemannian metric of Gauss curvature 1 on M^2 which is called the spherical metric of $g=(g_{11},g_{22})$. The formulas (3.1) are equivalent to

$$(3.6)\quad g_{11} = \varphi_{i1}(u) + \int_{v_i}^{v} \frac{G_{i1,2}}{K}\,d\eta,\quad g_{22} = \psi_{i1}(v) + \int_{u_i}^{u} \frac{G_{i2,1}}{K}\,d\xi,$$

where $(u,v)\in P_i\subset M^2$ and φ_{i1}, ψ_{i1} are positive functions for every $i\in N$. From (3.6) it follows that every Riemannian metric (g_{11},g_{22}) of a surface $x(M^2)\subset E^3$ referred to the lines of curvature is determined by its spherical metric (G_1,G_2). However, not every solution of (3.3) determines a Riemannian metric (g_{11},g_{22}), but only such solutions that (2.20) is satisfied. The theorema egregium (2.6) not only in the coordinates (u,v) determined by lines of curvature but also in any other orthogonal coordinates. Thus, (2.20) can be viewed as a condition imposed on the orthogonal coordinates (u,v) such that they are coordinates defined by lines of curvature of a surface $x(M^2)\subset E^3$. More precisely this can be formulated as follows. From (3.1) and (3.6) it follows that (2.20) is equivalent with

$$(3.7)\qquad K^2\left[\varphi_{i1}(u) + \int_{v_i}^{v} \frac{G_{i1,2}}{K}\, d\eta\right]\left[\psi_{i1}(v) + \int_{u_i}^{u} \frac{G_{i2,1}}{K}\, d\xi\right]=G_{i1}G_{i2},$$

where $(u,v)\in P_i$, $i\in N$. Let

$$(3.8)\qquad \bar{G}=(\bar{G}_1,\bar{G}_2)$$

denote the Riemannian metric of the unit sphere $S^2\subset E^3$ in any orthogonal coordinates $(\bar{u},\bar{v})$.

We have the following

Proposition 1. Let (M^2,G) denote the Riemannian manifold with the spherical metric G defined by (3.5). There exists an isometric immersion (Gauss map)

$$(3.9)\qquad F:M^2\longrightarrow S^2,\ F=(F_1,F_2),$$

of (M^2,G) into $(S^2,\bar{G})$ such that

$$(3.10)\qquad G_1=\bar{G}_1F^2_{1,1} + \bar{G}_2F^2_{2,1},\quad G_2=\bar{G}_1F^2_{1,2}+\bar{G}_2F^2_{2,2}\ .$$

The mapping (3.9) is up to an orthogonal transformation of E^3, which transforms S^2 onto itself, uniquely defined.

Proof. By $\{D_i\}_{i\in N}$, we denote a covering of M^2 by disks D_i with radii r_i, $0<r_i\le\frac{\pi}{2}$, where r_i is defined by means of the distance function of the Riemannian metric G. We suppose that every disk D_i is provided with a polar geodesic coordinate system with the pole at the center of D_i. There exists such a numeration of the disks D_i, $i\in N$, that for every $k\in N$, $k>1$, there exists such an $i\in N$, $1\le i\le k-1$, that $D_k\cap D_i\ne\emptyset$. Let $\bar{D}_1\subset S^2$ denotes a disk of radius r_1 with the center at $(0,0,1)\in S^2\subset E^3$ and provided with a polar, geodesic coordinate system. We define $F:D_1\longrightarrow\bar{D}_1$ setting that corresponding points of D_1 and $\bar{D}_1$ have the same polar geodesic coordinates. The disk $\bar{D}_2\subset S^2$ with radius r_2 we choose such that the transformation of polar geodesic coordinates in $\bar{D}_1\cap\bar{D}_2\ne\emptyset$ coincide with that in $D_1\cap D_2$. We define $F:D_1\cup D_2\longrightarrow\bar{D}_1\cup\bar{D}_2$ setting that corresponding points in D_2 and $\bar{D}_2$ have the same polar geodesic coordinates. Step by step we define (3.9) in this way.

Proposition 2. Let $\bar{K}(\bar{u},\bar{v})$ denote a function different from zero on an open subset of S^2 and $(\bar{u},\bar{v})$ denote orthogonal coordinates on this subset. It there exists an immersion (3.9) such that for every $i \in N$ the functions (3.10) restricted to $P_i \subset M^2$ and

$$K(u,v) = \bar{K}(F_1(u,v), F_2(u,v)), \quad (u,v) \in P_i, \tag{3.11}$$

satisfy (3.7), then (G_1, G_2) defined by (3.10) is a spherical metric, i.e., there exists a Riemannian metric (g_{11}, g_{22}) on M^2 such that (2.6) with Gauss curvature (3.11) is satisfied.

Proof. We define (g_{11}, g_{22}) by (3.6). From (3.8) and (3.10) it follows that (G_1, G_2) restricted to $P_i \subset M^2$ satisfies (3.3). Hence, from (3.1) equivalent to (3.6) it follows that (3.2) equivalent to (2.6) is satisfied.

The immersion F from Proposition 2 is called a solution of (3.7), where (G_1, G_2) is defined by (3.10) and $K(u,v)$ by (3.11). Such a solution has the properties explained in Proposition 1. A solution F of (3.7) defines a surface $x(P_i) \subset E^3$ for every $i \in N$.

Namely, we have

Proposition 3. Let F denotes a solution of (3.7). For every $i \in N$ there exists a surface $x(P_i) \subset E^3$ such that $x(u,v)$, v=const, $x(u,v)$, u=const, are lines of curvature on $x(P_i)$.

Proof. From Proposition 2 we get that there exists a metric (g_{11}, g_{22}) in orthogonal coordinates $(u,v) \in P_i$ with Gauss curvature (3.11). From (2.12) we get L_{11}, where (g_{11}, g_{22}) is defined by (3.6), and from (2.7) we get L_{22} on $P_i \subset M^2$. From the fundamental theorem of surface theory it follows that there exists a surface $x(P_i) \subset E^3$ referred to lines of curvature such that (g_{11}, g_{22}), (L_{11}, L_{22}) are respectively the first and the second quadratic forms of this surface.

4. Surfaces of constant Gauss curvature

a) $K=-c^2$, $c \neq 0$. In this case from (2.20) we get up to a scale transformation of the form

$$\bar{u} = \int_{u_i}^{u} \sqrt{\varphi_i(\xi)}\, d\xi, \quad \bar{v} = \int_{v_i}^{v} \sqrt{\psi_i(\eta)}\, d\eta, \quad (u,v) \in P_i, \ i \in N, \tag{4.1}$$

the formula

(4.2) $g_{11} + g_{22} = \frac{1}{c^2}$.

We substitute (4.2) into (2.6) setting

(4.3) $g_{11} = \frac{1}{c^2} \sin^2 \frac{k}{2}$, $g_{22} = \frac{1}{c^2} \cos^2 \frac{k}{2}$, $0<k<\pi$.

We get

(4.4) $k_{,22} - k_{,11} = \sin k$.

After the change of coordinates

(4.5) $\bar{u} - \bar{v} = u$, $\bar{u} + \bar{v} = v$

(4.4) takes the form

(4.6) $\bar{k}_{,12} = \sin\bar{k}$, $\bar{k}(\bar{u},\bar{v}) = k(\bar{u}-\bar{v},\bar{u}+\bar{v})$, $0<\bar{k}<\pi$.

Setting $K=-c^2$, $-c^2g_{11}(u,v_i)+\varphi_i(u)=1$ into (2.12) we get

(4.7) $L_{11} = -L_{22} = \frac{1}{2c} \sin k$.

From (3.1) we get that the spherical metric of (4.3) is

(4.8) $G_1 = 1 - c^2g_{11}$, $G_2 = 1 - c^2g_{22}$.

b) $K=c^2$, $c\neq 0$. From (2.20) we get

(4.9) $c^4g_{11}g_{22} = \left(c^2g_{11}+\bar{\varphi}_i(u)\right)\left(c^2g_{22}-\bar{\psi}_i(v)\right)$, $(u,v)\in P_i$, $i\in N$,

where $\bar{\varphi}_i(u)$, $\bar{\psi}_i(v)$ are positive functions. After the scale transformation (4.1) we get from (4.9)

(4.10) $g_{22} - g_{11} = \frac{1}{c^2}$.

Setting in (4.10)

(4.11) $g_{11} = \frac{1}{c^2} \mathrm{sh}^2 \frac{k}{2}$, $g_{22} = \frac{1}{c^2} \mathrm{ch}^2 \frac{k}{2}$

we get from (2.6)

(4.12) $k_{,22} - k_{,11} = \mathrm{sh}k$, $k>0$ (or $k<0$).

After the change of coordinates (4.5) we get from (4.12)

(4.13) $\bar{k}_{,12} = \mathrm{sh}\bar{k}$, $\bar{k}(\bar{u},\bar{v}) = k(\bar{u}-\bar{v},\bar{u}+\bar{v})$, $\bar{k}>0$ (or $\bar{k}<0$).

Similarly as in (4.7) the second quadratic form is defined by

(4.14) $L_{11} = L_{22} = \frac{1}{2c} \mathrm{sh}k$.

The spherical metric of (4.11) is

(4.15) $G_1 = c^2g_{11}+1$, $G_2 = c^2g_{22}-1$.

The spherical metric (4.8) has the following property, which is not satisfied by the metric (4.15). There does not exist a surface $x(M^2)\subset E^3$ of Gauss curvature 1 such that $x(u,v)$, v=const, and $x(u,v)$, u=const, are uniquely defined lines of curvature and

(4.16) $ds^2 = G_1du^2 + G_2dv^2$,

where (G_1,G_2) in (4.16) is the metric (4.8). Indeed, from (4.3) and (4.8) we get

(4.17) $G_1 + G_1 = 1$,

while every metric with the above mentioned properties satisfies (4.10).

Complex curves $\hat{u}$, $\hat{v}$ defined by

(4.18) $(\hat{u}+\hat{v})i = u,\ \hat{u}-\hat{v}=v,\ i=\sqrt{-1}$,

are called complex asymptotic lines of a surface $x(M^2)\subset E^3$ of Gauss curvature $K=c^2$, referred to the lines of curvature. In the coordinates $(\hat{u},\hat{v})$ the first and second quadratic forms are defined by

(4.19) $ds^2 = \frac{1}{c^2}(d\hat{u}^2 - 2\mathrm{ch}\hat{k}d\hat{u}d\hat{v} + d\hat{v}^2),\ n.d^2x = -\frac{1}{c^2}\mathrm{sh}\hat{k}d\hat{u}d\hat{v}$,

where $\hat{k}(\hat{u},\hat{v}) = k((\hat{u}+\hat{v})i,\hat{u}-\hat{v})$. From (4.19) it follows that the complex asymptotic lines form on a surface of Gauss curvature $K=c^2$, referred to the lines of curvature a Chebyshev net.

5. Surfaces of negative Gauss curvature in E^3

Before considering the general case we prove the following

Theorem 1. The surface area of every surface $x(M^2)\subset E^3$, where x denotes an immersion (2.1), of a connected manifold $M^2\subset E^2$, of Gauss curvature $K=-c^2$, $c\neq 0$, is less than or equal $\frac{2\pi}{c^2}$.

Proof. The surface area $A_g(M^2)$ of $x(M^2)$ is

(5.1) $A_g(M^2) = \int_{M^2} \sqrt{g_{11}g_{22}}\, dudv$.

From (4.3) and (5.1) we get

(5.2) $A_g(M^2) = \frac{1}{2c^2}\int_{M^2} \sin k dudv,\ 0<k<\pi$.

Since $M^2\subset E^2$ is open, it is a measurable set. Hence, there exists an increasing sequence $\{M^2_n\}_{n\in N}$ of manifolds $M^2_n\subset M^2$ with boundary ∂M^2_n such that

$$(5.3) \qquad \sup_{n\geq 1} A_g(M_n^2) = A_g(M^2), \quad \bigcup_{n=1}^{\infty} M_n^2 = M^2,$$

and ∂M_n^2 is a polygon for every n=1,2,.... From (5.2) we get

$$(5.4) \qquad A_g(M_n^2) = \frac{1}{2c^2} \int_{M_n^2} \sin k_n du dv, \quad 0<k_n<\pi,$$

where $k_n(u,v)$ is the function $k(u,v)$ restricted to M_n^2. For $(u,v)\in E^2 \setminus M_n^2$ we set

$$(5.5) \qquad k_n(u,v) = \begin{cases} \pi, & \text{if } v^2-u^2\geq 0, \\ 0, & \text{if } u^2-v^2>0. \end{cases}$$

By means of the extension (5.5) the formula (5.4) can be written in the form

$$(5.6) \qquad A_g(M_n^2) = \frac{1}{2c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin k_n du dv, \quad 0<k_n<\pi.$$

After the change of coordinates (4.5) in (5.6) we get

$$(5.7) \qquad A_g(M_n^2) = \frac{1}{c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin \bar{k}_n d\bar{u} d\bar{v}, \quad 0<\bar{k}_n<\pi.$$

Since 0 and π are solutions of (4.6), we get from (4.6), (5.5) and (5.7)

$$(5.8) \qquad A_g(M_n^2) = \frac{1}{c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{k}_{n,12} d\bar{u} d\bar{v} .$$

From (5.5) and (5.8) it follows

$$(5.9) \qquad A_g(M_n^2) = \frac{1}{c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{k}_{n,12} d\bar{u} d\bar{v} =$$

$$= \lim_{a\to\infty} \frac{1}{c^2} \left[\bar{k}_n(a,a) - \bar{k}_n(a,-a) - \bar{k}_n(-a,a) - \bar{k}_n(-a,-a)\right] \leq \frac{2\pi}{c^2} .$$

From (5.3) and (5.9) it follows

$$(5.10) \qquad A_g(M^2) = \sup_{n\geq 1} A_g(M_n^2) \leq \frac{2\pi}{c^2} .$$

A similar estimation to that of (5.10) for surfaces of positive, constant Gauss curvature $K=c^2$, $c\neq 0$, is not valid.

Indeed, from (4.11) and (5.1) we get

$$A_g(M^2) = \frac{1}{2c^2}\int_{M^2} \text{shk}\,du\,dv,\ k>0. \tag{5.11}$$

Since there exist unbounded solutions of (4.12), it follows that (5.11) can be unbounded.

The Theorem 1 was proved by the assumption that the surface $x(M^2)\subset E^3$ is referred to the lines of curvature. We prove that this assumption can be neglected. We have

Theorem 2. If

$$y:N^2\longrightarrow E^3,\ N^2\subset E^2, \tag{5.12}$$

is an isometric immersion of a connected, Riemannian manifold (N^2,g), where g is a Riemannian metric of negative Gauss curvature: $K(s,t)<0$ for every $(s,t)\in N^2$, then there exists a manifold $M^2\subset E^2$ and a diffeomorphism

$$s=\alpha(u,v),\ t=\beta(u,v),\ (s,t)\in N^2,\ (u,v)\in M^2, \tag{5.13}$$

of M^2 on N^2 such that

$$x(u,v)=y(\alpha(u,v),\beta(u,v)) \tag{5.14}$$

has the properties of the immersion (2.1).

Proof. The system of partial differential equations of the lines of curvature of the surface $y(N^2)$ has the form

$$\frac{\partial s}{\partial u} = f_1B_{11},\ \frac{\partial t}{\partial u} = f_1B_{12},\ \frac{\partial s}{\partial v} = f_2B_{21},\ \frac{\partial t}{\partial v} = f_2B_{22}. \tag{5.15}$$

Since the Gauss curvature is negative, it follows that the orthogonal vectors (B_{11},B_{12}), (B_{21},B_{22}) are different from the zero vector at every point of N^2. The functions f_i, $i=1,2$, defined on N^2 are integrating factors. The proof that integrating factors $f_i\neq 0$ exist (on the whole of N^2) belongs to the theory of partial differential equations and we omit it here. The system (5.15) is completely integrable. Hence, by the theorem of Frobenius, for every $(s_0,t_0)\in N^2$, $(u_0,v_0)\in E^2$ there exists a rectangle $P_0\subset E^2$ with center (u_0,v_0) and uniquely defined functions $\alpha_0(u,v)$, $\beta_0(u,v)$, $(u,v)\in P_0$, such that $\alpha_0(u_0,v_0)=s_0$, $\beta_0(u_0,v_0)=t_0$, which satisfy the system (5.13). We denote

(5.16) $\quad Q_0 = \{(s,t) \mid s=\alpha_0(u,v),\ t=\alpha_0(u,v),\ (u,v)\in P_0\}.$

The curves $x_0(u,v)$, v=const, $x_0(u,v)$, u=const, where $F_0(u,v)=y(\alpha_0(u,v),\beta_0(u,v))$, are lines of curvature of the surface $y(Q_0)\subset y(N^2)$. There exists an at most countable covering $\{Q_i\}_{i\in N}$ of N^2 such that

(5.17) $\quad Q_i = \{(s,t) \mid s=\alpha_i(u,v),\ t=\beta_i(u,v),\ (u,v)\in P_i\},$

where $\alpha_i(u,v)$, $\beta_i(u,v)$, $(u,v)\in P_i$, are uniquely defined solutions of the system (5.15) in the rectangle P_i with center (u_i,v_i). The curves $x_i(u,v)$, v=const, $x_i(u,v)$, u=const, where $x_i(u,v)=y(\alpha_i(u,v),\beta_i(u,v))$, are lines of curvature of $y(Q_i)$. If $Q_i\cap Q_k\neq\emptyset$, then we can choose the center $(u_k,v_k)\in E^2$ such that $\alpha_i(u,v)=\alpha_k(u,v)$, $\beta_i(u,v)=\beta_k(u,v)$, if and only if $(u,v)\in$ $\in P_i\cap P_k$. Step by step we can improve the definition of P_k, $k\in N$, choosing if necessary a new center (u_k,v_k), such that the intersections of the Q_i, $i\in N$, coincide with that of the P_i, $i\in N$. After that we define

(5.18) $$(\alpha(u,v),\beta(u,v)) = (\alpha_i(u,v),\alpha_i(u,v)),\ (u,v)\in P_i,$$
$$M^2=\bigcup_{i\in N} P_i.$$

Obviously $M^2\subset E^2$ is diffeomorphic with N^2 and $x(M^2)$, where x is defined by (5.14), is the surface $y(N^2)$ referred to the lines of curvature and of Gauss curvature $K(\alpha(u,v),\beta(u,v))$.

As a consequence of Theorems 1 and 2 we get the theorem of Hilbert.

Theorem 3. There does not exist an isometric immersion of a complete, connected, Riemannian manifold (L^2,g) with Gauss curvature $K=-c^2$, $c\neq 0$, into E^3.

Proof. There exists an atlas on L^2 with a single chart (E^2,h), $h:L^2\longrightarrow E^2$. Let us suppose contrary to the assertion of Theorem 3 that there exists an isometric immersion

(5.19) $\quad y:E^2\longrightarrow E^3$

of (L^2,g) written in the chart (E^2,h) into E^3. From Theorems 1 and 2 it follows that the surface area of the surface $y(E^2)\subset E^3$

is less than or equal $\frac{2\pi}{c^2}$. This is possible only if L^2 is a covering space of $y(E^2)$, but the only surfaces with this property are closed, orientable surfaces of genus ≥ 2. Such a surface provided with a Riemannian metric of constant Gauss curvature $K=-c^2$, $c\neq 0$, cannot be imbedded isometrically in E^3.

Example 1. The surface of revolution in E^3 of the conic type of Gauss curvature $K=-c^2$, $c\neq 0$, has the equation

$$y(u,v)=b\,\mathrm{sh}\,cu(e_1\cos cv+e_2\sin cv) + e_3\int_0^u\sqrt{1-b^2c^2\mathrm{ch}^2 c\eta}\,d\eta, \tag{5.20}$$

where $oe_1e_2e_3$ is an orthonormal frame in E^3, b and c are positive numbers such that $0<bc<1$ and M^2 is defined by

$$1<\mathrm{ch}\,cu<\frac{1}{bc}\ ,\quad 0<v<\frac{2\pi}{c}\ . \tag{5.21}$$

The Riemannian metric of (5.20) is

$$ds^2 = du^2 + b^2c^2\mathrm{sh}^2 cu\,dv^2. \tag{5.22}$$

We define

$$x(u,v) = y(u,v) + bcve_4, \tag{5.23}$$

where u satisfies the first inequalities of (5.21) and $-\infty<v<\infty$. By $oe_1e_2e_3e_4$ we denote an orthonormal frame in E^4. The Riemannian metric of (5.23) is

$$ds^2 = du^2 + b^2c^2\mathrm{ch}^2 cu\,dv^2. \tag{5.24}$$

Hence, (5.23) defines in E^4 a surface of Gauss curvature $K=-c^2$, $c\neq 0$, and unbounded surface area. We have

$$\int_{u_0}^{u_1}\int_{-\infty}^{\infty}\sqrt{g_{11}g_{22}}\,du\,dv = bc\int_{u_0}^{u_1}\int_{-\infty}^{\infty}\mathrm{ch}\,cu\,du\,dv = \infty, \tag{5.25}$$

where $0<bc<1$ and the numbers u_0, u_1, $0<u_0<u_1$, satisfy the first inequalities of (5.21).

Moreover, let

$$f_3 = e_3\cos\gamma + e_4\sin\gamma,\quad 0\leq\gamma<\frac{\pi}{2}\ ,$$
$$f_4 = -e_3\sin\gamma + e_4\cos\gamma.$$

We take a projection of the surface define by (5.23) in E^3 spanned by the orthonormal frame $oe_1e_2f_3$. We get

(5.26) $$x_\gamma(u,v) = b\,\mathrm{sh}cu(e_1\cos cv + e_2\sin cv) + f_3\left(\int_0^u\sqrt{1-b^2c^2\mathrm{ch}^2c\eta}\,d\eta\cos\gamma + bcv\sin\gamma\right),$$

where u satisfies the first inequalities of (5.21) and $-\infty<v<\infty$. The Riemannian metric of (5.26) has the form

(5.27) $$ds^2 = g_{\gamma 11}(u)du^2 + 2g_{\gamma 12}(u)dudv + g_{\gamma 22}(u)dv^2,$$

where

(5.28) $$g_{\gamma 11} = \cos^2\gamma + b^2c^2\mathrm{ch}^2cu\sin^2\gamma,$$

(5.29) $$g_{\gamma 12} = bc\sqrt{1-b^2c^2\mathrm{ch}^2cu}\,\sin\gamma\cos\gamma,$$

(5.30) $$g_{\gamma 22} = b^2c^2(\mathrm{sh}^2cu + \sin^2\gamma).$$

The Gauss curvature of (5.27) has the form

(5.31) $$K_\gamma(u,v) = -c^2f_\gamma(u),$$

where $f_\gamma(u)$ is an analytic function of u and γ and $f_0(u)=1$. Hence, for every $\varepsilon>0$ there exists such a γ, $0<\gamma<\frac{\pi}{2}$, that

(5.32) $$|f_\gamma(u)-1|<\varepsilon$$

for every u which satisfies the first inequalities of (5.21). From (5.26) it follows easily that there exist such numbers $u_{\gamma 0}$, $u_{\gamma 1}$, $0<u_{\gamma 0}<u_{\gamma 1}$, which satisfy the first inequalities of (5.21) and (5.26) is an imbedding of the set

(5.33) $$M^2 = \{\ (u,v)\in E^2\ |\ u_{\gamma 0}<u<u_{\gamma 1},\ -\infty<v<\infty\ \}$$

in E^3. From (5.28), (5.29) and (5.30) we get

(5.34) $$\int_{u_{\gamma 0}}^{u_{\gamma 1}}\int_{-\infty}^{\infty}\sqrt{g_{\gamma 11}g_{\gamma 22}-g_{\gamma 12}^2}\,dudv = \infty$$

and as follows from (5.31) and (5.32) the Gauss curvature $K_\gamma(u,v)$ of (5.27) can be chosen arbitrarily close to $-c^2$.

Now we proceed to the theorem of Efimov. As we see from Example 1 the argument of Theorem 1 cannot be applied to the proof of this theorem. Let

(5.35) $$\bar{z}:N^2\longrightarrow E^3$$

denote an immersion of a 2-dimensional, connected manifold N^2

in E^3 with the following properties: a) the Gauss curvature of the Riemannian metric $\bar{g}$ induced from E^3 on $\bar{z}(N^2)$ is negative at every point of N^2, b) the metric space $(N^2, dist_{\bar{g}})$, where $dist_{\bar{g}}$ denotes the distance function defined by $\bar{g}$, is complete. From Theorem 8.1 in [6] it follows that for every $p \in N^2$

$$(5.36) \qquad exp_p : T_p(N^2) \longrightarrow N^2$$

is a covering mapping of the tangent space $T_p(N^2)$ to N^2 at p on N^2. By means of (5.36) we define in $T_p(N^2)$ the Riemannian metric $(exp_p)^* \bar{g} = g$. Then $(T_p(N^2), g)$ is a complete, Riemannian manifold with negative Gauss curvature at every point and (5.36) is an isometric immersion of $(T_p(N^2), g)$ on $(N^2, \bar{g})$. The immersion

$$(5.37) \qquad z = \bar{z} \circ exp_p$$

of $(T_p(N^2), g)$ in E^3 is called a covering immersion of (5.35). Thus, for every immersion (5.35) which satisfies the properties a) and b) there exists a covering immersion (5.37) with the same properties and the surfaces $\bar{z}(N^2)$ and $z(T_p(N^2))$ are identical. Hence, it suffices to consider covering immersions. Let $h: T_p(N^2) \longrightarrow E^2$ denote an isomorphism. The immersion (5.37) written in the chart (E^2, h) we denote by

$$(5.38) \qquad y: E^2 \longrightarrow E^3,$$

where E^2 in (5.38) is referred to the coordinates (s,t). From Theorem 2 it follows that there exists a manifold $M^2 \subset E^2$ and a diffeomorphism (5.13) of M^2 on E^2 such that

$$(5.39) \qquad x(u,v) = y(\alpha(u,v), \beta(u,v)), \ (u,v) \in M^2,$$

is the immersion (5.38) referred to the lines of curvature. The Riemannian metric of M^2 induced by (5.13) we denote again by g.

The metric space $(M^2, dist_G)$, where G denote the spherical metric of g defined by (3.1) and (3.5), cannot be complete. Indeed, the Gauss curvature of G is 1 and 2-dimensional, complete manifold with Gauss curvature 1 is compact (see § 7.3 in [3]), while M^2 is an open set. Now, as an immediate consequence of the theorem of Hopf and Rinow [4] we get the following

Lemma 1. There exists a unit vector $e\in E^2$ and a point $(u_0,v_0)\in M^2$, where M^2 is the manifold in (5.39), such that the geodesic ray of (M^2,G) with origin (u_0,v_0) and tangent to e at (u_0,v_0) has finite length.

Example 2. The universal covering space CS^2 of $S^2 \setminus \{p,q\}$, where p and q are antipodal points of the unit sphere S^2, is different from $S^2 \setminus \{p,q\}$. We have diameter $(CS^2)=\infty$, while diameter$(S^2)=\pi$ and for every $t\in CS^2$ we have $\text{dist}(t,p)<\pi$, $\text{dist}(t,q)<\pi$, where p,q, are points "at infinity" of CS^2.

If (M^2,dist_g) is a complete, connected, metric space, then (M^2,g) is called a complete, connected, Riemannian manifold and $x(M^2)\subset E^3$ a complete, connected surface.

Now, the theorem of Efimov follows from

Theorem 4. If $x(M^2)\subset E^3$, where x is an immersion (2.1), is a complete, connected surface with negative Gauss curvature K, then

(5.40) $\sup K(u,v) = 0$ for $(u,v)\in M^2$.

Proof. Let $\overline{E^2}$ denote the plane E^2 completed by points "at infinity" which correspond to unit vectors on E^2. Hence, $\overline{E^2}$ is diffeomorphic with the closed, unit disk $\overline{D^2}\subset E^2$. By $\overline{M^2}$ we denote the closure of M^2 in $\overline{E^2}$. Let $p(s)\in M^2$, $0<s\le s_0$, denote the geodesic ray of finite length from Lemma 1 parameterized by the arc length parameter s such that $p(s_0)=(u_0,v_0)$ and

(5,41) $$\lim_{s\to 0} \text{dist}_G(p(s),\overline{M^2} \setminus M^2) = 0.$$

We have

(5.42) $$\lim_{s\to 0} F(p(s)) = q\in S^2,$$

where F denote the mapping (3.9). Since F is an isometric immersion of (M^2,G) in S^2, the curve $F(p(s))$, $0<s\le s_0$, describes an arc of a great circle $S^1\subset S^2$, and therefore the limit (5.42) exists. We prove that also

(5.43) $$\lim_{s\to 0} p(s) = p_0 = F^{-1}(q)\in\overline{M^2} \setminus M^2$$

exists. Let us suppose contrary that $p(s)$ has two distinct limit points $p_0, p_1 \in \overline{M^2} \setminus M^2$. We prove that in this case the geodesic ray $p(s)$, $0<s\le s_0$, has infinite length. There exists a decreasing sequence $(s_k)_{1\le k<\infty}$ such that $\lim s_k=0$ if $k\to\infty$ and

(5.44) $\lim_{k\to\infty} p(s_{2k}) = p_0, \quad \lim_{k\to\infty} p(s_{2k+1}) = p_1.$

Since $p_0 \neq p_1$ it follows from (5.44) that there exists a number κ such that

(5.45) $\liminf_{k\to\infty} \mathrm{dist}_G(p(s_{2k}),p(s_{2k+1}))>\kappa>0.$

From (5.45) it follows that there exists such a subsequence $(k_n)_{1\le n<\infty}$ that

(5.46) $\mathrm{dist}_G(p(s_{2k_n}),p(s_{2k_n+1}))>\kappa$ for $n=1,1,\ldots$

From (5.46) we get

(5.47) $\sum_{n=1}^{\infty} \mathrm{dist}_G(p(s_{2k_n}),p(s_{2k_n+1})) = \infty$

contrary to the assertion of Lemma 1. This proves (5.43).

We choose s_0 such that $0<s_0\le\frac{\pi}{2}$. Let $\overline{D}(q,s_0)\subset S^2$ denote a closed disk with radius s_0 and center q. By $D(p_0,s_0)\subset \subset F^{-1}(\overline{D}(q,s_0))\subset M^2$ we denote this component of $F^{-1}(\overline{D}(q,s_0))$ which contains the geodesic ray $p(s)$, $0<s\le s_0$. On $D(p_0,s_0)$ the mapping F is bijective and therefore a diffeomorphism. Hence, the boundary $\partial D(p_0,s_0)$ of $D(p_0,s_0)$ in M^2 is a differentiable curve without selfintersections. Now we investigate $D(p_0,s_0)\subset \subset M^2$ with respect to the Riemannian metric $g=(g_{11},g_{22})$. Every two distance functions dist_{g_1}, dist_{g_2} defined by positive definite Riemannian metrics g_1, g_2 on M^2 are topologically equivalent (see Proposition 3.5 in [5]). In particular dist_G and dist_g are topologically equivalent. Hence, the mapping F defined by (3.9) and restricted to $D(p_0,s_0)$, being a homeomorphism of the metric space $(D(p_0,s_0),\mathrm{dist}_G)$ is also a homeomorphism of the metric space $(D(p_0,s_0),\mathrm{dist}_g)$. Therefore, from $\mathrm{dist}(p_0,\partial D(p_0,s_0))=s_0$ it follows that there exists a number λ, $0<\lambda\le\infty$, such that

(5.48) $\liminf_{s\to 0} \mathrm{dist}_g(p(s),\partial D(p_0,s_0)) = \lambda$

since otherwise $q=F(p_0)$ would be a cluster point of $F(\partial D(p_0,s_0))$.

We have

Lemma 2. By the assumptions of Theorem 4 there exists a geodesic ray $r_a(t)\in D(p_0,s_0)$, $0\le t<\infty$, of (M^2,g) such that $a\in E^2$ denotes the unit vector tangent to $r_a(t)$ at $r_a(0)\in\partial D(p_0,s_0)$ and

(5.49) $\lim_{t\to\infty} r_a(t) = p_0$,

where t, $0\le t<\infty$, denote the arc length of $r_a(t)$.

Proof of Lemma 2. By $c(s_0,s)$ we denote the shortest geodesic segment which joins $p(s_0)$ and $p(s)$ in (M^2,g). By the theorem of Hopf and Rinow $c(s_0,s)$ exists. Let $(s_k)_{1\le k<\infty}$ denote a zero sequence. From (5.43) it follows

(5.50) $\lim_{k\to\infty} p(s_k) = p_0$.

By a_k we denote the unit tangent vector to $c(s_0,s_k)$ at $p(s_0)$. There exists a convergent subsequence (a_{k_i}) of (a_k) such that

(5.51) $\lim_{i\to\infty} a_{k_i} = a$

is a unit vector. By $r_a(t)$ we denote the deodesic ray of (M^2,g), tangent to a at $p(s_0)$ and parameterized by the arc length t. By the theorem of Hopf and Rinow we have $0\le t<\infty$. Since a geodesic ray $r_a(t)$ defines exactly one point "at infinity" which corresponds to a, we have

(5.52) $\lim_{t\to\infty} r_a(t) = p_1\in\overline{M^2}\setminus M^2$.

Let us suppose contrary to (5.49) that $p_1\ne p_0$. There exist neighborhoods U_0 of p_0 and U_1 of p_1 in $\overline{E^2}$ such that $U_0\cap U_1=\emptyset$. There exists a neighborhood V_1 of $r_a(t)$, $0\le t<\infty$, such that $U_1\subset V_1$ and $V_1\cap U_0=\emptyset$. Hence, there exists such an integer i_0 that for $i>i_0$ the geodesic ray $r_{a_{k_i}}(t)$, $0\le t<\infty$, tangent to a_{k_i} at $p(s_0)$, which contains the geodesic segment $c(s_0,s_{k_i})$, is contained in V_1. Therefore $p(s_{k_i})\in V_1$ for $i>i_0$. On the other hand from (5.50) it follows that there exists an integer j_0

such that for $i>j_0$ we have $p(s_{k_i})\in U_0$. For $i>\max(i_0,j_0)$ we get $p(s_{k_i})\in U_0\cap V_1$ contrary to the fact that these sets are disjoint. This contradiction proves (5.49).

From (5.48) and (5.49) it follows that there exists such a number $t_0\geq 0$ that $r_a(t)$ belongs to the interior of $D(p_0,s_0)$ for every $t>t_0$ and $r_a(t_0)\in\partial D(p_0,s_0)$. We set $t_0=0$. This ends the proof of Lemma 2.

We have

(5.53) $$\int_{D(p_0,s_0)} \sqrt{g_{11}g_{22}}\, dudv = \infty.$$

Proof of (5.53). Let $t_1>0$. There exists such a number $\mu>0$ that

(5.54) $$\operatorname{dist}_g(r_a(t),\partial D(p_0,s_0))\geq\mu \quad \text{for every} \quad t\geq t_1.$$

Indeed, otherwise we get

(5.55) $$\liminf_{t\to\infty} \operatorname{dist}_g(r_a(t),\partial D(p_0,s_0)) = 0.$$

From (5.48) and (5.49) it follows that (5.55) is different from zero. This contradiction proves (5.54). Since (M^2,g) is a complete manifold with negative Gauss curvature at every point, it follows that on M^2 there exists a polar, geodesic coordinate system $(\bar{u},\bar{v})$, $0\leq\bar{u}<\infty$, $0\leq\bar{v}<2\pi$, with the only singularity at the pole $\bar{u}=0$, such that

(5.56) $$ds^2 = g_{11}du^2 + g_{22}dv^2 = d\bar{u}^2 + B(\bar{u},\bar{v})d\bar{v}^2,$$

where $\bar{v}=0$ is the equation of the geodesic ray $r_a(t)$ such that $r_a(t_1)=(0,0)$. From (5.54) it follows that for every $\bar{u}$, $0\leq\bar{u}<\infty$, there exist two numbers $\bar{v}_1(\bar{u})$, $\bar{v}_2(\bar{u})$ such that

(5.57) $$\int_{\bar{v}_1(\bar{u})}^{\bar{v}_2(\bar{u})} B(\bar{u},\bar{v})d\bar{v} \geq 2\mu.$$

From (5.57) we get

$$(5.58)\qquad \int_{D(p_0,s_0)} \sqrt{g_{11}g_{22}}\,dudv \geq \int_0^{\infty} d\bar{u} \int_{\bar{v}_1(\bar{u})}^{\bar{v}_2(\bar{u})} B(\bar{u},\bar{v})\,d\bar{v} = \infty.$$

From (5.58) follows (5.53).

On the other hand from the definition of $D(p_0,s_0)$ it follows

$$(5.59)\qquad \int_{D(p_0,s_0)} |K|\sqrt{g_{11}g_{22}}\,dudv = \int_{D(p_0,s_0)} \sqrt{G_1G_2}\,dudv \leq 2\pi.$$

From (5.53) and (5.59) follows (5.40).

6. Special surfaces

Let (2.1) denote an immersion such that

$$(6.1)\qquad \frac{L_{11}}{g_{11}} = \alpha\,\frac{L_{22}}{g_{22}}\,,\quad \alpha\neq 0,1,\ \alpha=\text{const}.$$

From (2.2) and (2.7) it follows that L_{11}, L_{22} in (6.1) are different from zero.

We have the following

Theorem 5. Let us suppose that (g_{11},g_{22}), (L_{11},L_{22}) besides (2.7), (2.8) and (2.9) satisfy (6.1) for every $(u,v)\in M^2$. Then the Gauss curvature satisfies the following condition: c) for every $(u_0,v_0)\in M^2$ and every neighborhood $U\subset M^2$ of (u_0,v_0) there exists such a point $(u_1,v_1)\in U$ that

$$(6.2)\qquad K(u_0,v_0)\neq K(u_1,v_1).$$

If the Gauss curvature satisfies c), then K satisfies the equation

$$(6.3)\qquad \alpha\left(\frac{\frac{\kappa_{,2}}{2}}{\kappa^{\alpha+1}}\right)_{,2} + \left(\frac{\frac{\kappa_{,1}}{2\alpha}}{\kappa^{\alpha+1}}\right)_{,1} = -(\alpha+1)\kappa^{\frac{\alpha-1}{\alpha+1}}$$

where $\kappa^{\frac{\alpha-1}{\alpha+1}} = \sqrt{K}$, $\alpha>0$, $\alpha\neq 1$, if $K>0$ and $\kappa^{\frac{\alpha-1}{\alpha+1}} = \sqrt{-K}$, $\alpha<0$, $\alpha\neq -1$, if $K<0$. For $\alpha=-1$ we get for all minimal surfaces

$$(6.4)\qquad \Delta\kappa=2e^{-\kappa},\quad \kappa = \log\frac{1}{\sqrt{-K}}\,,$$

where Δ denote the Laplace operator.

Proof. Let us suppose contrary to (6.2) that there exists an open set $U \subset M^2$ such that

(6.5) $K(u,v) = K_0 \neq 0$ for every $(u,v) \in U$, K_0=const.

From (2.7) and (6.1) we get

(6.6) $$\frac{L_{11}^2}{g_{11}^2} = \alpha K.$$

From (2.12), (3.1) and (6.6) we get

(6.7) $$Kg_{22} = \alpha G_2 \ , \ Kg_{11} = \frac{1}{\alpha} G_1 \ ,$$

where $\alpha>0$ if $K>0$ and $\alpha<0$ if $K<0$. From (3.1), (6.5) and (6.7) we get

(6.8) $$g_{11} = \varphi(u), \ g_{22} = \psi(v).$$

After a scale transformation of the form (4.1) we get $g_{11}=g_{22}=1$ and therefore $K_0=0$ contrary to (2.2). This proves c).

The set

(6.9) $$A = \{(u,v) \in M^2 \mid K_{,1}^2 + K_{,2}^2 = 0\}$$

is closed and non dense in M^2, i.e. every point of A is a limit point of the open set $B=M^2 \setminus A$, since, otherwise there would exist an open set $U \subset M^2$ such that $K_{,1}(u,v)=K_{,2}(u,v)=0$ for every $(u,v) \in U$ and therefore $K(u,v)$=const for $(u,v) \in U$ contrary to (6.2). Differentiating (6.7) we get

(6.10) $$K_{,1} g_{22} = (\alpha-1) K g_{22,1} \ , \ K_{,2} g_{11} = \left(\frac{1}{\alpha} - 1\right) K g_{11,2} \ .$$

The solution of (6.10) is up to a scale transformation of the form (4.1)

(6.11) $$g_{11} = |K|^{\frac{\alpha}{1-\alpha}} \ , \ g_{22} = |K|^{\frac{1}{\alpha-1}} \ .$$

For $(u_0,v_0) \in A$ we define g_{11}, g_{22} as limit values of (6.11) if $(u,v) \in B$ tends to (u_0,v_0). From (2.6) and (6.11) we get (6.3) and (6.4).

Conversely, we have

Theorem 6. If a function $K(u,v)$ satisfies in M^2 the equation (6.3) for $\alpha \neq 0,1,-1$ or (6.4) for $\alpha=-1$, then $K(u,v)$ satisfies the condition c) and there exists a unique solution of (2.7), (2.8), (2.9) and (6.1) in M^2 such that $K(u,v)$ is the Gauss curvature of (2.1).

Proof. From (6.3) and (6.4) it follows $K(u,v)\neq 0$ for every $(u,v)\in M^2$. We define (g_{11},g_{22}) by (6.11). From (2.7) and (2.11) we get (L_{11},L_{22}) such that (6.1) is satisfied. Now, as in the proof of Theorem 5 it follows that K satisfies c).

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Received September 9, 1988.