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SUBDIRECTLY IRREDUCIBLE LEFT NORMAL BANDOIDS, I

Introduction

This paper is a continuation of the authors work [1] and [2] devoted to investigation of algebras called left normal bandoids. In [1] the lattice of varieties of such algebras was described and in [2] some construction methods were discussed. In this paper we begin our study of subdirectly irreducible left normal bandoids. We use the notation and terminology of [2]. Our numbering begins with Section 4. References in Sections 1 through 3 are to the relevant parts of [2].

In Section 4 a congruence relation of the bandoid constructed in Section 3 is discribed that plays an important role in the sequel. In Section 5 some sufficient conditions are given for this bandoid to be subdirectly irreducible. Finally, in Section 6 it is shown that certain subalgebras of subdirectly irreducible bandoids considered in Section 5 are subdirectly irreducible as well.

In subsequent papers it will be shown that the subdirectly irreducible left normal bandoids constructed in this paper are in fact all finite subdirectly irreducible left normal bandoids.

For more information concerning left normal bandoids and general algebraic concepts we refer the reader to [1], [2], and references there.

4. An important congruence relation

Let $\underline{L}$ be a complete lattice satisfying the distributive law (3.1) and let R be a subset of $\leq_L$ such that $(1, 1) \in R$. We introduce a congruence relation on $\underline{B}(L, R)$ that will play an important role in the next section. Let us define the relation ρ on $\underline{B}(L, R)$ as follows:

(4.1) $x \rho y$ iff $1xy_1 = 1y_1$ and $1yx_1 = 1x_1$

for every $x_1, y_1 \in B(L,R)$ with $x_1 \leq x$, $y_1 \leq y$.

To prove that ρ is a congruence relation on $\underline{B}(L, R)$ we need some lemmas.

Lemma 4.2. For every x in B(L, R) the left multiplication $L(1) : (xT(B(L, R)), \cdot) \longrightarrow (1T(B(L, R)), \cdot)$; $z \mapsto 1 \cdot z$ is a monomorphism.

Proof. By Proposition 1.3 it is enough to show that the mapping L(1) is one to one. Let $z, t \in xT(B(L, R))$, $z = y_{ab}$, $t = s_{cd}$ and $1z = 1t$. By Remark 1.1 $zt = tz$. Since $zt \in S_{ab}$ and $tz \in S_{cd}$, it follows that $(a,b) = (c,d)$. So $t = s_{ab}$. By definitions (2.2) and (3.3),

$$1z = 1 \cdot \phi_{ab11} z = \phi_{ab11} z = \sum (v \in (b] \mid va = y)$$

and

$$1t = 1 \cdot \phi_{ab11} t = \phi_{ab11} t = \sum (v \in (b] \mid va = s).$$

Since $1z = 1t$ we have

$$\sum (v \in (b] \mid va = y) = \sum (v \in (b] \mid va = s).$$

Consequently

$$a \cdot \sum (v \in (b] \mid va = y) = a \cdot \sum (v \in (b] \mid va = s).$$

Thus by distributivity (3.1) of $\underline{L}$ we conclude that $y = s$, and in consequence $z = t$. ∎

Lemma 4.3. For every $x_1, x_2, \ldots, x_n \in B(L, R)$ and $2 \leq i \leq n$ the following identity holds:

$$x_1 x_2 \ldots x_n = x_1 x_2 \ldots x_i 1 x_{i+1} \ldots x_n.$$

Proof. By Propositon 1.9,

$$1x_1 x_2 \ldots x_n = 1x_1 x_2 \ldots x_i 1 x_{i+1} \ldots x_n.$$

Hence by Lemma 4.2,

$$x_1x_2\ldots x_n = x_1x_2\ldots x_i1x_{i+1}\ldots x_n. \qquad \blacksquare$$

Lemma 4.4. For every x,y in $B(L, R)$, if $x\rho y$ then

(i) $(\{x,y\},\cdot)$ is a left zero semigroup

and

(ii) $1x = 1y$.

Proof. Let $x\rho y$. By (4.1) it follows that

$$1xy = 1y \tag{4.4.1}$$

and consequently $y1xy = y1y$. Hence by Lemma 4.3 we get $yxy = yy$ and by Proposition 1.9, $yx = y$. Analogously we show that

$$xy = x. \tag{4.4.2}$$

Therefore $(\{x, y\}, \cdot)$ is a left zero semigroup.

The equality (ii) is an immediate consequence of (4.4.1) and (4.4.2). ∎

Proposition 4.5. The relation ρ is a congruence relation on $\underline{B}(L, R)$.

Proof. It is evident that the relation ρ is reflexive and symmetric. To prove the transitivity let us assume that $x \rho y$, $y \rho z$, $x_1 \le x$ and $z_1 \le z$. Then

$$\begin{aligned}
1xz_1 &= 1x1z_1 && \text{by Proposition 1.9}\\
&= 1x1yz_1 && \text{by (4.1), since } y \rho z \text{ and } z_1 \le z\\
&= 1xyz_1 && \text{by Proposition 1.9}\\
&= 1yz_1 && \text{by (4.1), since } x \rho y \text{ and } yz_1 \le y\\
&= 1z_1 && \text{since } y \rho z \text{ and } z_1 \le z.
\end{aligned}$$

Analogously we show that $1zx_1 = 1x_1$. Therefore $x \rho z$, and consequently ρ is transitive.

Now it remains to show that $x \rho y$ implies $zx \rho zy$ and $xz \rho yz$, for every $x, y, z \in B(L, R)$. Let $x, y, z \in B(L, R)$ and $x \rho y$. To show that $zx \rho zy$, let us assume that $y_1 \le zy$ and $x_1 \le zx$. Then

$$\begin{aligned}
zx &= z1x && \text{by Lemma 4.3}\\
&= z1y && \text{since } x \rho y \text{ and by Lemma 4.4(ii)}\\
&= zy && \text{by Lemma 4.3.}
\end{aligned}$$

Consequently $1(zx)y_1 = 1(zy)y_1$. But since $y_1 \le zy$, $(zy)y_1 = y_1$.

Therefore

$$(4.5.1) \qquad 1(zx)y_1 = 1y_1.$$

Analogously we prove that

$$(4.5.2) \qquad 1(zy)x_1 = 1x_1.$$

By (4.5.1), (4.5.2) it follows that

$$(4.5.3) \qquad zx\rho zy.$$

It remains to show that $xz\ \rho\ yz$. Let $y_2 \le yz$ and $x_2 \le xz$. Then

$$\begin{aligned}
1(xz)y_2 &= 1(xz)(xy_2) && \text{by Proposition 1.8}\\
&= (1xz)(1xy_2) && \text{by Proposition 1.3}\\
&= (1xz)(1y_2) && \text{since } x\ \rho\ y \text{ and } y_2 \le yz \le y\\
&= (1yxz)(1y_2) && \text{since } x\ \rho\ y \text{ and } xz \le x\\
&= (1y1xyz)(1y_2) && \text{by Proposition 1.9}\\
&= (1y1yz)(1y_2) && \text{since } x\ \rho\ y \text{ and } yz \le y\\
&= (1yz)(1y_2) && \text{by Proposition 1.9}\\
&= 1(yz)y_2 && \text{by Proposition 1.3}\\
&= 1y_2 && \text{since } y_2 \le yz\ .
\end{aligned}$$

Similarly we prove that $1(yz)x_2 = 1x_2$. Thus $xz\ \rho\ yz$, which finishes the proof. ■

Remark 4.6. Let $x, y \in B(L, R)$. Then $x\ \rho\ y$ if and only if

(i) $1x = 1y$

and

(ii) the left multiplications :

$$L(x)\colon (yT(B(L,R)),\cdot) \longrightarrow (xT(B(L,R)),\cdot) \quad \text{and}$$
$$L(y)\colon (xT(B(L,R)),\cdot) \longrightarrow (yT(B(L,R)),\cdot)$$

are semilattice isomorphisms.

Proof. ($\Rightarrow$) By Proposition 1.3 the mappings $L(x)$ and $L(y)$ are semilattice homomorphisms. So it suffices to show that they are one to one and onto.

Let $x_1, x_2 \in xT(B(L,R))$ and $yx_1 = yx_2$. Then $1yx_1 = 1yx_2$. Since $x\ \rho\ y$, $x_1 \le x$ and $x_2 \le x$, we have that $1x_1 = 1x_2$. Hence, by Lemma 4.2, $x_1 = x_2$. Therefore $L(y)$ is one-to-one .

Let $y_1 \in yT(B)$. Note that since $x\ \rho\ y$ and $y_1 \le y$, $1xy_1 =$

$1y_1$. Hence $y1xy_1 = y1y_1$. Using Lemma 4.3 we get $yxy_1 = yy_1 = y_1$. Therefore $y_1 = L(y)(xy_1)$ and consequently $L(y)$ is onto. Analogously we prove that $L(x)$ is one to one and onto. So the condition (ii) is satisfied. The condition (i) follows immediately from Lemma 4.4(ii) and the assumption that $x \rho y$.

($\Leftarrow$) Let $x_1 \le x$ and $y_1 \le y$. Note that

$$
\begin{aligned}
1y_1 = 1yy_1 &= (1y)(yy_1) && \text{by (B3)}\\
&= (1x)(yy_1) && \text{by (i)}\\
&= (1x)y_1 && \\
&= (1y_1)(xy_1) && \text{by (B5)}\\
&= (1y_1)(1xy_1) && \text{by Proposition 1.8}\\
&= (1xy_1)(1y_1) && \text{by Remark 1.1}\\
&= (1xy_1)y_1 && \text{by Proposition 1.8}\\
&= (1xy_1)(yxy_1) && \text{since by Proposition 1.9, } xyxy_1=xy_1\\
& && \text{and } L(x) \text{ is one to one}\\
&= (1y)(xy_1) && \text{by (B5)}\\
&= (1x)(xy_1) && \text{by (i)}\\
&= 1xy_1 && \text{by (B3).}
\end{aligned}
$$

Analogously we show that $1yx_1 = 1x_1$. By (4.1), $x \rho y$, which completes the proof. ∎

For an algebra $\underline{A}$ and a congruence relation ρ on $\underline{A}$ the symbol $\underline{A}^{\rho}$ denotes the quotient algebra, and a^{ρ} the congruence class of a.

An immediate consequence of Proposition 4.5 is the following

Corollary 4.7. $\underline{B}(L, R)^{\rho}$ is a left normal bandoid. ∎

Remark 4.8. For each z^{ρ}_{xy} in $B(L, R)^{\rho}$, the following holds:

$$\left(z^{\rho}_{xy}T(B(L, R)^{\rho}), \cdot\right) \cong \left(z_{xy}T(B(L, R)^{\rho}), \cdot\right) \cong ((z], \cdot).$$

The mappings $\varphi : ((z], \cdot) \longrightarrow \left(z_{xy}T(B(L, R)), \cdot\right)\ ;\ u \longmapsto u_{xy}$

and $\psi : \left(z_{xy}T(B(L, R)\ ), \cdot\right) \longrightarrow \left(z^{\rho}_{xy}T(B(L, R)^{\rho}), \cdot\right);$

$$u_{xy} \mapsto u^{\rho}_{xy}$$

are semilattice isomorphisms.

Proof. Note that $z_{xy}T(B(L, R)) = \{u_{xy}: u \in (z]\}$. Therefore, as an easy consequence of (3.2), we get that φ is an isomorphism. We will show that ψ is an isomorphism as well. Obviously ψ is a homomorphism. It remains to check that ψ is one to one and onto.

Let $s_{xy}, t_{xy} \in z_{xy}T(B(L, R))$ and $s^{\rho}_{xy} = t^{\rho}_{xy}$. Then by Lemma 4.4$(\{s_{xy}, t_{xy}\},\cdot)$ is a left zero semigroup, i.e. $s_{xy} = s_{xy}t_{xy}$ and $t_{xy}=t_{xy}s_{xy}$. But s_{xy},t_{xy} lie in the same orbit $z_{xy}T(B(L,R))$ and hence, by Remark 1.1, $s_{xy}\cdot t_{xy} = t_{xy}\cdot s_{xy}$. Therefore $s_{xy} = t_{xy}$ and ψ is one to one.

It remains to prove that φ maps onto the set $z^{\rho}_{xy}T(B(L,R)^{\rho})$.

Let $a^{\rho}_{bc} \in z^{\rho}_{xy}T(B(L, R)^{\rho})$. Then by Remark 1.6, $a^{\rho}_{bc}= z^{\rho}_{xy}\cdot a^{\rho}_{bc}$ and consequently $a^{\rho}_{bc}=(z_{xy}\cdot a_{bc})^{\rho}$, i.e. $a^{\rho}_{bc} = \psi(z_{xy}\cdot a_{bc})$. By Remark 1.6 $z_{xy}\cdot a_{bc}\in z_{xy}T(B(L,R)$, which completes the proof. ∎

Let us note that, in a finite left normal bandoid, each orbit with the bandoid partial order defined in Proposition 1.5 forms a lattice. As an immediate consequence of Remark 4.8 we obtain:

Corollary 4.9. If L is finite then the lattice $(L,\leq_L)$ is isomorphic to the lattice $\left(1^{\rho}T(B(L,R)^{\rho}),\leq\right)$.

Example 4.10. Let $\underline{L}$ and R be as in Example 3.14. The bandoid $\underline{B}(L,R)^{\rho}$ is presented in the picture below.

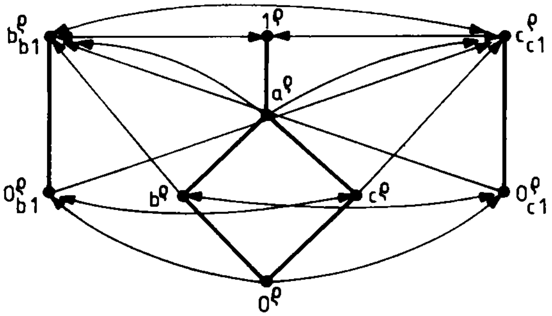

Figure 4.11. ∎

Example 4.12. Let $\underline{L}$ and R be as in Example 3.17 . The bandoid $\underline{B}(L, R)^{\rho}$ is pictured below.

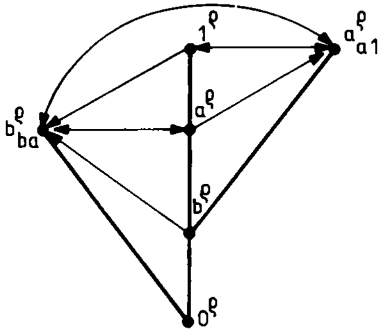

Figure 4.13. ∎

5. Some subdirectly irreducible left normal bandoids

The aim of this section is to prove Theorem 5.6. This Theorem shows that if $\underline{L}$ and R satisfy certain conditions, then the bandoid $\underline{B}(L,R)^{\rho}$ constructed in the previous section is subdirectly irreducible.

In the whole section let $\underline{B} = (B,\cdot)$ be a finite left normal bandoid.

Lemma 5.1. For all distinct elements x, y in B the principal congruence $\Theta(x, y)$ on $\underline{B}$ contains a principal congruence $\Theta(u, w)$ such that

(i) $u < v$

or

(ii) $(\{u, v\},\cdot)$ is a left zero semigroup, $uT(B)\setminus vT(B)=\{u\}$ and $vT(B)\setminus uT(B) = \{v\}$.

Proof. Let x, y be distinct elements of B.

If $x \neq xy$ then, by (B2), $xy<x$ and for $u=xy$, $v=x$ we have that $u<v$ and $\Theta(u,v) = \Theta(xy,x) = \Theta(xy,xx) = \Theta(xx,xy) \leq \Theta(x,y)$.

There is a similar situation in the case $y \neq yx$.

Now let us assume that $x=xy$ and $y=yx$. Note that the set $xT(B)\setminus yT(B)$ is nonempty since $x \in xT(B)\setminus yT(B)$. Indeed, if $x \in yT(B)$ then by Remark 1.1 $xy=yx$ and hence $x=y$, a contradiction.

Let $\mathrm{Min}(xT(B)\setminus yT(B))$ be the set of all minimal elements in $(xT(B)\setminus yT(B),\leq)$, and let $u \in \mathrm{Min}(xT(B)\setminus yT(B))$.

Note that $(yu)T(B)\backslash xT(B)$ is nonempty since $yu\in(yu)T(B)$ and $yu\notin xT(B)$. Indeed, by Remark 1.1 $yu\in xT(B)$ implies that $(yu)u=u(yu)$. Consequently, by Corollary 1.11, $yu=uy$. But by Proposition 1.8 $uy=uxy$, and since $xy=x$ and $u\in xT(B)$, $uy=ux=xu=u$. So we have $yu=u$. By Remark 1.6 it follows that $u\in yT(B)$, contradicting the fact that $u\in Min(xT(B)\backslash yT(B))$.

Let $v \in Min((yu)T(B)\backslash xT(B))$. Then we have

$$\begin{aligned} vu &= v((yu)u) && \text{by Proposition 1.8, since } v\in(yu)T(B) \\ &= v(yu) && \text{by Corollary 1.11} \\ &= v && \text{since } v\in(yu)T(B)\text{, by Remarks 1.1, 1.6.} \end{aligned}$$

Therefore

$$(5.1.1) \qquad vu = v.$$

We will show that $uv=u$ as well. By (B2), $uv\leq u$. Suppose that $uv\neq u$. Since u is minimal in $(xT(B)\backslash yT(B),\leq)$, we have $uv\leq y$. Thus v and uv lie in the same orbit $yT(B)$. By Remark 1.1 it follows that $v(uv)=(uv)v$. Consequently, by Corollary 1.11, $vu=uv$ and by (5.1.1) $v=uv$. So we have $v=uv\leq u\leq x$. Hence $v\leq x$, i.e. $v\in xT(B)$, which gives a contradiction. Therefore

$$(5.1.2) \qquad (\{u,\ v\},\cdot) \text{ is a left zero semigroup.}$$

Now we will show that $uT(B)\backslash vT(B)=\{u\}$. Suppose that $u_1\in uT(B)\backslash vT(B)$ and $u_1\neq u$. Then $u_1<u$. Since u is minimal in $(xT(B)\backslash yT(B),\leq)$, we have $u_1\in yT(B)$. Also $v\in yT(B)$, whence by Remark 1.1, $vu_1=u_1v$. Thus by Proposition 1.8 and (5.1.2), $vu_1=u_1v=u_1(uv)=u_1u=u_1$. Hence $u_1\leq v$, i.e. $u_1\in vT(B)$, a contradiction.

Analogously we show that $vT(B)\backslash uT(B)=\{v\}$. It remains to prove that $\Theta(u,v) \subseteq \Theta(x,y)$. Indeed

$$\Theta(x,\ y) \supseteq \Theta(xu,\ yu) = \Theta(u,\ yu) \supseteq \Theta(uv,\ (yu)v) = \Theta(u,\ v),$$

which completes the proof. ∎

Recall that in a finite left normal bandoid, each orbit with the bandoid partial order defined in Proposition 1.5 is a lattice. In what follows the symbol $JI(L)$ denotes the set of all join irreducible elements of a lattice $\underline{L}$.

Lemma 5.2. If there are a, b in B such that

(i) $b < a$

and for all x, y in B

(ii) $x<y$ implies $ax\neq ay$,

(iii) if x,y satisfy the condition 5.1(ii), then $ax\neq ay$,

(iv) if $x\in JI(aT(B))\setminus\{0\}$ and $y<x$, then there exist a mapping $\alpha \in L(B)$ such that $\alpha y \leq b$ and $\alpha x = a$,

then $\underline{B}$ is subdirectly irreducible and $\Theta(a, b)$ is the monolith of $\underline{B}$.

Proof. Let a, b be elements of B satisfying the conditions (i)\(iv). We will show that $\Theta(a, b)$ is the least nontrivial congruence on $\underline{B}$. Let $x,y\in B$ and let $x<y$ or x,y satisfy the condition 5.1(ii). By Lemma 5.1 it suffices to show that $\Theta(a, b) \subseteq \Theta(x, y)$.

Note that by (ii) and (iii), $ax\neq ay$. There exists $t\in JI(aT(B))$ such that $t\leq ay$ and $t\leq ax$ or such that $t\leq ax$ and $t\leq ay$. By symmetry we may assume that $t\in JI(aT(B))$, $t\leq ay$ and $t\leq ax$. Then $t=t(ay)$ and $t(ax)<t$. By (iv) there exists $\alpha\in L(B)$ such that $\alpha(t(ax))\leq b$ and $\alpha t=a$. So we have $\alpha(t(ax))\leq b<a=\alpha t$. Therefore

$$\Theta(a, b) \subseteq \Theta(\alpha(t(ax)), \alpha t) \subseteq \Theta(t(ax), t) \subseteq \Theta(ax, ay) \subseteq \Theta(x, y). \qquad \blacksquare$$

Let $\underline{L}$ be a finite distributive lattice and R be a subset of $\leq_L$ containing the element (1,1). Let us consider the bandoid $\underline{B}(L,R)^{\rho}$ defined in the previous section. The following lemmas hold.

Lemma 5.3. Let $x^{\rho},y^{\rho}\in B(L,R)^{\rho}$ and $x^{\rho}< y^{\rho}$. Then $1^{\rho}x^{\rho}\neq 1^{\rho}y^{\rho}$.

Proof. Suppose on the contrary that $1^{\rho}x^{\rho}= 1^{\rho}y^{\rho}$. To arrive at a contradiction, we will prove that $x^{\rho}= y^{\rho}$.

First note that $x^{\rho}<y^{\rho}$ implies

(5.3.1) $$(yx)^{\rho}= x^{\rho}.$$

Hence by Lemma 4.4(i)

(5.3.2) $$1yx = 1x.$$

Since $1^{\rho}x^{\rho}= 1^{\rho}y^{\rho}$, i.e. $(1x)^{\rho}= (1y)^{\rho}$, by Lemma 4.4(ii) we have that $11x = 11y$. Consequently, by (B2),

(5.3.3) $$1x = 1y.$$

From (5.3.2) and (5.3.3) it follows that $1y = 1yx$. Hence, by

Lemma 4.2 we obtain

(5.3.4) $$y = yx .$$

To prove $x^\rho = y^\rho$ let us assume that $x_1 \le x$ and $y_1 \le y$. Then we have

$$\begin{aligned} 1xy_1 &= 1xy_1x && \text{by Proposition 1.9} \\ &= 1xy_1yx && \text{by Proposition 1.8} \\ &= 1y_1yx && \text{since } y_1yx \le yx \text{ and by (5.3.1) } yx \ \rho \ x \\ &= 1y_1y && \text{by (5.3.4)} \\ &= 1y_1 && \text{since } y_1 \le y. \end{aligned}$$

It remains to show that $1yx_1 = 1x_1$.

Now

$$\begin{aligned} 1yx_1 &= 1yxx_1 && \text{since } x_1 \le x \\ &= 1(yx)(xx_1) && \text{by (B3)} \\ &= 1(yx)x_1 && \text{since } x_1 \le x \\ &= 1x_1 && \text{since } x_1 \le x \text{ and } yx \ \rho \ x. \end{aligned}$$

This completes the proof that $x^\rho = y^\rho$ and gives a contradiction to $x^\rho < y^\rho$. Therefore $1^\rho x^\rho \ne 1^\rho y^\rho$. ∎

Lemma 5.4. Let $x^\rho, y^\rho \in B(L, R)^\rho$ and x^ρ, y^ρ satisfy 5.1(ii). Then

$$1^\rho x^\rho \ne 1^\rho y^\rho .$$

Proof. Suppose on the contrary that $1^\rho x^\rho = 1^\rho y^\rho$, i.e. $1x \ \rho \ 1y$. Then by Lemma 4.4(ii), $11x = 11y$. Hence, by (B2), we obtain

(5.4.1) $$1x = 1y.$$

Since $(\{x^\rho, y^\rho\}, \cdot)$ is a left zero semigroup, we have $x \ \rho \ xy$ and $y \ \rho \ yx$. Thus, by Lemma 4.4(ii) it follows that

(5.4.2) $$1x = 1xy \quad \text{and} \quad 1y = 1yx.$$

We will prove that $x^\rho = y^\rho$, which will give a contradiction to the fact that x^ρ, y^ρ satisfy 5.1(ii). Let $x_1 \le x$ and $y_1 \le y$. If $x_1 = x$, then

$$\begin{aligned} 1yx_1 = 1yx &= 1y && \text{by (5.4.2)} \\ &= 1x && \text{by (5.4.1)} \\ &= 1x_1 . \end{aligned}$$

Analogously, if $y_1 = y$, then $1xy_1 = 1y_1$. It remains to show that for $x_1 < x$ and $y_1 < y$, $1yx_1 = 1x_1$ and $1xy_1 = 1y_1$.

Let $x_1 < x$. Then also $x_1^\rho < x^\rho$. Indeed, if $x_1^\rho = x^\rho$, then by Lemma 4.4(i) $xx_1 = x$, whence $x_1 = x$, contradicting $x_1 < x$.

Since x^ρ is minimal in $(x^\rho T(B(L,R)^\rho)\setminus y^\rho T(B(L, R)^\rho),\le)$ we have that $x_1^\rho \le y^\rho$, i.e. $yx_1 \rho\, x_1$. Hence, using Lemma 4.4, we get $1yx_1 = 1x_1$. Analogously, if $y_1 < y$, then $1xy_1 = 1y_1$. Consequently $x\ \rho\ y$, i.e. $x^\rho = y^\rho$, contrary to the assumption that x^ρ, y^ρ satisfy 5.1(ii). ∎

Recall that by Corollary 4.9, $(1^\rho T(B(L, R)^\rho),\le)$ is a lattice isomorphic to $\underline{L}$. Also by Remark 4.8, 1^ρ and 0^ρ are the greatest and the least elements respectively in $(1^\rho T(B(L,R)^\rho),\le)$.

Lemma 5.5. Let $\underline{L}$ be a finite distributive lattice with exactly one coatom c, and let R be a subset of $\le_L$ such that
*) for every $t \in JI(L)\setminus\{0\}$ there exist elements $t=x_1,x_2,\ldots,x_n$, $z_1,z_2,\ldots,z_n=1$ satisfying the following for all $i=1,2,\ldots,n$ and $j=1,2,\ldots,n-1$:

(i) $(x_i,z_i) \in R$

and (ii) $x_1 \le_L x_{j+1} \le_L z_j$.

Then for all $y \in JI(1^\rho T(B(L,R)^\rho))\setminus\{0^\rho,1^\rho\}$ and $z<y$, there exist $\alpha \in L(B(L,R)^\rho)$ such that $\alpha y=1^\rho$ and $\alpha z<c^\rho$.

Proof. Let $y \in JI(1^\rho T(B(L, R)^\rho))\setminus\{1^\rho,0^\rho\}$ and let $z<y$. By Remark 4.8, $y=s^\rho$ for some $s \in JI(L)\setminus\{1,0\}$ and $z=u^\rho$ for some $u \in L$ with $u<_L s$. Let $s = x_1,x_2,\ldots,x_n,z_1,z_2,\ldots,z_n = 1$ be elements of L such that $(x_i,z_i) \in R$ and $x_1 \le_L x_{j+1} \le_L z_j$ for every $i=1,2,\ldots,n$ and $j=1,2,\ldots,n-1$. Write

$$\bar{x}_i := (x_i)_{x_i,z_i} \qquad \text{for } i=1,2,\ldots,n.$$

Let $\alpha := L(1^\rho)\circ L(\bar{x}_n^\rho)\circ L(\bar{x}_{n-1}^\rho)\circ\ldots\circ L(\bar{x}_1^\rho)$. We want to show that $\alpha y = 1^\rho$ and $\alpha z<c^\rho$. To prove this, note first that the following holds for $i=1,2,\ldots,n$:

(5.5.1) $\quad 1\bar{x}_i = \phi_{x_i z_i 11}\bar{x}_i$ by Definition 2.2

$= \sum(v \in (z_i] \mid v\cdot x_i = x_i)$ by (3.3)

$= z_i$ since $x_i \le_L z_i$ and thus $z_i x_i = x_i$.

Furthemore, for $i=1,2,\dots,n-1$ we have

(5.5.2) $\quad \bar{x}_{i+1}\bar{x}_i = \bar{x}_{i+1}1\bar{x}_i$ by Lemma 4.3

$= \bar{x}_{i+1}z_i$ by (5.5.1)

$= \bar{x}_{i+1}\phi_{11x_{i+1}z_{i+1}}z_i$ by Definition 2.2

$= \bar{x}_{i+1}\bar{x}_{i+1}$ by (3.3), since $x_{i+1}\cdot z_i = x_{i+1}$

$= \bar{x}_{i+1}$ by (B1).

By Definition 2.2, (3.3) and (B1) it follows that

(5.5.3) $\quad \bar{x}_1 s = \bar{x}_1 x_1 = \bar{x}_1\phi_{11x_1z_1}x_1 = \bar{x}_1\bar{x}_1 = \bar{x}_1$.

Moreover

(5.5.4) $\quad 1\bar{x}_n = 1\phi_{x_n z_n 11}x_n = \sum(v\in(z_n] \mid v\cdot x_n = x_n)$

$= z_n$ since, $x_n \le_L z_n$ and hence $z_n\cdot x_n = x_n$.

Putting together (5.5.1), (5.5.2), (5.5.3) and (5.5.4) we obtain

$$1\bar{x}_n\bar{x}_{n-1}\dots\bar{x}_1 s = 1.$$

Thus

$$1^\rho\ \bar{x}_n^\rho\ \bar{x}_{n-1}^\rho\dots\bar{x}_1^\rho\ \ s^\rho = 1^\rho$$

and since $s^\rho = y$ we get

$$1^\rho\ \bar{x}_n{}^\rho\ \bar{x}_{n-1}^{\ \rho}\dots\bar{x}_1{}^\rho\ y = 1^\rho.$$

By definition of left multiplication it follows that

$$(L(1^\rho)\circ L(\bar{x}_n{}^\rho)\circ L(\bar{x}_{n-1}^{\ \rho})\circ\dots\circ L(\bar{x}_1{}^\rho))y = 1^\rho.$$

Hence, by definition of α

(5.5.6) $\quad \alpha y = 1^\rho.$

It remains to prove that $\alpha z < c^\rho$. First we show that the following condition (5.5.7) is satisfied for each $k=1,2,\dots,n-1$:

(5.5.7) $$1\bar{x}_k\bar{x}_{k-1}\ldots\bar{x}_1 u \geq s.$$

The proof is by induction on k. Let k = 1. Then

$$\begin{aligned} s1\bar{x}_1u &= s\bar{x}_1u && \text{by Lemma 4.3} \\ &= s\cdot\phi_{x_1z_111}(\bar{x}_1\cdot\phi_{11x_1z_1}u) && \text{by Definition 2.2} \\ &= s\cdot\textstyle\sum(v \in (z_1] \mid vx_1 = x_1u) && \text{by (3.3)} \\ &= su && \text{since } x_1 = s \text{ and } \underline{L} \text{ is distributive} \\ &= u && \text{since } u <_L s. \end{aligned}$$

Thus $s1\bar{x}_1u \neq s$ and consequently

(5.5.8) $$1\bar{x}_1u \geq s.$$

Now we prove that for k = 1, 2,..., n-1:

$$1\bar{x}_k\bar{x}_{k-1}\ldots\bar{x}_1u \geq s \quad \text{implies} \quad 1\bar{x}_{k+1}\bar{x}_k\ldots\bar{x}_1u \geq s.$$

Suppose on the contrary that

(5.5.9) $$1\bar{x}_{k+1}\bar{x}_k\ldots\bar{x}_1u \geq s.$$

Then by Remark 1.7 and since $s = x_1 \leq_L x_{k+1}$ we have

(5.5.10) $$x_{k+1}1\bar{x}_{k+1}\bar{x}_k\ldots\bar{x}_1u \geq x_{k+1}s = s.$$

Note that by Definition 2.2, (3.3) and distributivity of $\underline{L}$,

(5.5.11) $$\begin{aligned} x_{k+1}1\bar{x}_{k+1}\bar{x}_k\ldots\bar{x}_1u &= x_{k+1}\cdot\textstyle\sum(v\in(z_{k+1}] \mid v\cdot x_{k+1} = x_{k+1}\bar{x}_k\ldots\bar{x}_1u) \\ &= x_{k+1}\bar{x}_k\ldots\bar{x}_1u\ . \end{aligned}$$

By Remark 1.7, since $1 \geq x_{k+1}$,

(5.5.12) $$1\bar{x}_k\bar{x}_{k-1}\ldots\bar{x}_1u \geq x_{k+1}\bar{x}_k\bar{x}_{k-1}\ldots\bar{x}_1u.$$

From (5.5.10) - (5.5.12) it follows that

$$1\bar{x}_k\bar{x}_{k-1}\ldots\bar{x}_1u \geq s$$

what contradicts the induction hypothesis.

Therefore (5.5.7) holds. In particular, for k = n,

$$1\bar{x}_n\bar{x}_{n-1}\ldots\bar{x}_1u \geq s.$$

To complete the proof that $\alpha z < c^\rho$, note that by Remark 4.8

$$(1\bar{x}_n\bar{x}_{n-1}\ldots\bar{x}_1u)^\rho \geq s^\rho.$$

Thus $$1^\rho\,\bar{x}_n^\rho\,\bar{x}_{n-1}^\rho\ldots\bar{x}_1^\rho u^\rho \geq s^\rho.$$

Since $z = u^{\rho}$, $y = s^{\rho}$ and by definition of α, we have

$$\alpha z = 1^{\rho}\,\bar{x}_n^{\rho}\,\bar{x}_{n-1}^{\rho}\ldots\bar{x}_1^{\rho} z \geq y.$$

By Remark 4.8 c^{ρ} is the only coatom in the lattice $(1^{\rho}T(B(L,R)^{\rho},\leq)$. Hence $y\leq c^{\rho}$ and consequently $\alpha z \geq c^{\rho}$, i.e. $\alpha z<c^{\rho}$, which completes the proof. ∎

Theorem 5.6. Let $\underline{L}$ be a finite distributive lattice with exactly one coatom. Let R be a subset of $\leq_L$ satisfying the condition 5.5(*). Then the left normal bandoid $\underline{B}(L,R)^{\rho}$ is subdirectly irreducible. The monolith of $\underline{B}(L,R)^{\rho}$ is a principal congruence $\Theta(a, b)$ for some a, b in $\underline{B}(L,R)^{\rho}$ with b<a.

Proof. Let c be the only coatom of $\underline{L}$. Set

$$a := 1^{\rho} \qquad \text{and} \qquad b := c^{\rho}.$$

By Remark 4.8, b<a. So the condition 5.2(i) holds. By Lemmas 5.3 and 5.4 respectively the conditions 5.2(ii) and 5.2(iii) are satisfied as well. It remains to show that 5.2(iv) holds.

Let $y \in JI(1^{\rho}T(B(L,R)^{\rho}))\backslash\{0^{\rho}\}$ and $x<y$. If $y=1^{\rho}=a$, then for α the identity mapping on $B(L,R)^{\rho}$, we have $\alpha y=a$ and $\alpha x\leq b$, since by Remark 4.8, b is the only coatom in the lattice $(1^{\rho}T(B(L,R)^{\rho},\leq)$. Hence by Lemma 5.25 the conditon 5.2(iv) holds. By Lemma 5.2 the proof is complete. ∎

Remark 5.7. The bandoids described in Examples 4.10 and 4.12 are subdirectly irreducible. ∎

6. More subdirectly irreducible left normal bandoids

In this section we prove that under certain conditions some subalgebras of the subdirectly irreducible bandoids considered in the previous section are subdirectly irreducible as well.

Let $\underline{L}$ be a finite distributive lattice and let R be a subset of $\leq_L$ containing the element (1, 1).

Lemma 6.1. The element 1^{ρ} is maximal in $(B(L,R)^{\rho},\leq)$.

Proof. Suppose that $1^\rho \le x_{uw}^\rho$ for some $x_{uw}^\rho \in B(L,R)^\rho$. Hence, by Remark 1.6, $1^\rho T(B(L,R)^\rho) \le x_{uw}^\rho T(B(L,R)^\rho)$. But by Remark 4.8

$$(1^\rho T(B(L,R),\cdot) \cong ((1],\cdot) \text{ and } (x_{uw}^\rho T(B(L,R)),\cdot) \cong ((x],\cdot).$$

Since $\underline{L}$ is finite it follows that $x=1$. Consequently $u=w=1$ and $x_{uw}^\rho = 1^\rho$. ■

Lemma 6.2. The set $B(L,R)^\rho\backslash\{1^\rho\}$ is a subuniverse of $\underline{B}(L,R)^\rho$.

Proof. Suppose on the contrary that for $u_{xy}^\rho, w_{zt}^\rho \in B(L,R)^\rho\backslash\{1^\rho\}$ we have $u_{xy}^\rho \cdot w_{zt}^\rho = 1^\rho$. Then, by Remark 1.6, $1^\rho \le u_{xy}^\rho$, a contradiction to Lemma 6.1 . ■

The subalgebra $B(L,R)^\rho\backslash\{1^\rho\}$ of $\underline{B}(L,R)^\rho$ will be denoted by $\underline{B}^1(L,R)^\rho$.

Lemma 6.3. Let $\underline{L}$ and R satisfy the hypothesis of Theorem 5.6 and let $\underline{x},\underline{y}$ be elements of $B^1(L,R)^\rho$ such that $\underline{x}<\underline{y}$. Then $c^\rho\underline{x} \ne c^\rho\underline{y}$, where c denotes the only coatom of $\underline{L}$.

Proof. Let $\underline{y}=y_{uw}^\rho$. By Remark 4.8 we may assume that $\underline{x}=x_{uw}^\rho$ for some $x<y$. First we prove that $1^\rho\underline{x}<c^\rho$, i.e. $(1x_{uw})^\rho<c^\rho$. By Remark 4.8 the last inequality is equivalent to $1x_{uw}<c$ and consequently, by Definition 2.2 and (3.3), to $\sum(v\in(w] \mid vu=x)<c$. Suppose on the contrary that $\sum(v\in(w] \mid vu=x)=c$ or $\sum(v\in(w] \mid vu=x)= =1$. By distributivity of $\underline{L}$ this implies $cu=x$ or $u=1u=x$. Since c is the only coatom of $\underline{L}$, $cu=c$ or $cu=u$. Therefore we have $u=x$ or $c=x$. But by assumption $x<y\le u$. So $u\ne x$ and consequently $x=c$ and $y=u=w=1$. This means that $\underline{y}=1^\rho$, contradicting the fact that $\underline{y}$ is an element of $B^1(L, R)$. This contradiction shows that $1^\rho\underline{x}<c^\rho$.

Consequently, by Proposition 1.8,

$$c^\rho\underline{x} = c^\rho 1^\rho \underline{x} = 1^\rho\underline{x}. \tag{6.3.1}$$

We will show that $c^\rho\underline{x} = 1^\rho\underline{y}$. If $c^\rho\underline{y} \ge c^\rho$, then obviously $c^\rho\underline{x} \ne c^\rho\underline{y}$. Recall that by Remark 4.8 c^ρ is the only coatom in the lattice $(1^\rho T(B(L,R)^\rho, \le)$. So if not $c^\rho\underline{y}\ge c^\rho$ then $c^\rho\underline{y}<c^\rho$.

Let $c^\rho\underline{y} < c^\rho$. Since c^ρ is the only coatom in the lattice

$(1^\rho T(B(L,R)^\rho,\le)$, we have that $(1^\rho \underline{y})c^\rho = 1^\rho \underline{y}$ or $(1^\rho \underline{y})c^\rho = c^\rho$. But by Proposition 1.8 and Remark 1.1, $c^\rho \underline{y} = c^\rho(1^\rho \underline{y}) = (1^\rho \underline{y})c^\rho$. As a consequence, since $c^\rho \underline{y} < c^\rho$, we obtain

$$c^\rho \underline{y} = (1^\rho \underline{y})c^\rho = 1^\rho \underline{y}. \tag{6.3.2}$$

By Lemma 5.3, $1^\rho \underline{x} \ne 1^\rho \underline{y}$. Hence, using (6.3.1) and (6.3.2), we conclude that $c^\rho \underline{x} \ne c^\rho \underline{y}$. ∎

Lemma 6.4. Let $\underline{L}$ and R satisfy the hypothesis of Theorem 5.6 and let c be the only coatom of $\underline{L}$. Assume additionally that

$$(x,1) \in R \ \text{ if and only if } \ x \in \{1,c\}.$$

Moreover let $\underline{x}$, $\underline{y}$ be elements of $B^1(L, R)$ satisfying 5.1(ii) and such that $\{\underline{x}, \underline{y}\} \ne \{c^\rho, c^\rho_{c1}\}$.

Then

$$c^\rho \underline{x} \ne c^\rho \underline{y}.$$

Proof. Let $\underline{x} = x^\rho_{uw}$ and $\underline{y} = y^\rho_{zt}$. First we show that $x,y \notin \{1,c\}$. If $x=1$, then since $x \le u \le w$, it follows that $u=w=1$ and $\underline{x} = 1^\rho$, a contradiction. Therefore $x \ne 1$. One can show analogously that $y \ne 1$. Now assume that $x=c$. Note that, since $\underline{x}$, $\underline{y}$ satisfy 5.1(ii), the orbits $(\underline{x}T(B(L,R)^\rho),\cdot)$ and $(\underline{y}T(B(L,R)^\rho),\cdot)$ are isomorphic. Hence, by Remark 4.8 we obtain

$$((y],\cdot) \cong (\underline{y}T(B(L,R)^\rho),\cdot) \cong (\underline{x}T(B(L,R)^\rho),\cdot) \cong ((x],\cdot) = ((c],\cdot).$$

Since $\underline{L}$ is finite and c is the only coatom of $\underline{L}$ it follows that $y=c$. Consequently $\underline{x} = c^\rho_{c1}$ or $\underline{x} = c^\rho$, and $\underline{y} = c^\rho_{c1}$ or $\underline{y} = c^\rho$. But by the assumption that $\underline{x}$, $\underline{y}$ satisfy 5.1(ii) we have that $\underline{x} \ne \underline{y}$. Therefore $\{\underline{x},\underline{y}\} = \{c^\rho, c^\rho_{c1}\}$, a contradiction. It follows that $x \ne c$. Similarly we show that $y \ne c$. So we have

$$x,y \notin \{1.c\}. \tag{6.4.1}$$

To prove that $c^\rho \underline{x} \ne c^\rho \underline{y}$, we first show that $1^\rho \underline{x} \le c^\rho$. Suppose on the contrary that $1^\rho \underline{x} \le c^\rho$. Since by Remark 4.8 c^ρ is the only coatom of the lattice $(1^\rho T(B(L,R)),\le)$, it follows that $1^\rho \underline{x} = 1^\rho$. This is equivalent to $(1x_{uw})^\rho = 1^\rho$. Thus, by Lemma 4.4 and (B2) we have that $1x_{uw} = 1$. Since $x \le u$, $1u_{uw} = 1$ as well. Using Lemma 4.2 we get $x_{uw} = u_{uw}$, i.e. $x=u$. Note that

$1u_{uw} = \phi_{uw11}u_{uw} = \sum(v \in (w] \mid vu=u) = w$. It follows that $w=1$.

Consequently $\underline{x}=x^{\rho}_{uw}=x^{\rho}_{x1}$. Since $(x.1)\in R$ iff $x\in\{1,c\}$, we conclude that $x\in\{1,c\}$, contradicting (6.4.1). This contradiction shows that $1^{\rho}\underline{x} \le c^{\rho}$. Hence by Proposition 1.8, we have $c^{\rho}\underline{x} = c^{\rho}1^{\rho}\underline{x} = 1^{\rho}\underline{x}$. Analogously we show that $c^{\rho}\underline{y}=1^{\rho}\underline{y}$.

By Lemma 5.4, $1^{\rho}\underline{x}\neq 1^{\rho}\underline{y}$. Therefore $c^{\rho}\underline{x}\neq c^{\rho}\underline{y}$. ∎

Lemma 6.5. Let $\underline{L}$ and R satisfy the hypothesis of Theorem 5.6 and let c be the only coatom of $\underline{L}$.

Let $t \in JI(1^{\rho}T(B(L,R)^{\rho}))\setminus\{1^{\rho}\}$ and $s<t$. Then there exists $\beta \in L(B^1(L, R)^{\rho})$ such that $\beta t = c^{\rho}$ and $\beta s<c^{\rho}$.

Proof. By Lemma 5.5 there exists $\alpha\in L(B(L, R)^{\rho})$ such that $\alpha t=1^{\rho}$ and $\alpha s<c^{\rho}$. If α is the identity mapping on $B(L,R)^{\rho}$, then it suffices to put $\beta:= L(c^{\rho})$. Let $\alpha=L(z_1)\circ L(z_2)\circ\ldots\circ L(z_n)$. Then $z_1z_2\ldots z_nt = 1^{\rho}$ and $z_1z_2\ldots z_ns < c^{\rho}$. Let $z_{i_1},z_{i_2},\ldots,z_{i_k}$ be a subsequence of the sequence $z_1,z_2,\ldots,z_n$ obtained by dropping all elements equal to 1^{ρ}.

Using Propositions 1.8 and 1.9 we get the following:

$$c^{\rho} = c^{\rho}1^{\rho} = c^{\rho}z_1z_2\ldots z_nt = c^{\rho}1^{\rho}z_1z_2\ldots z_nt = c^{\rho}1^{\rho}z_{i_1}z_{i_2}\ldots z_{i_k}t$$
$$= c^{\rho}z_{i_1}z_{i_2}\ldots z_{i_k}$$

and

$$c^{\rho} > z_1z_2\ldots z_ns = c^{\rho}z_1z_2\ldots z_ns = c^{\rho}1^{\rho}z_1z_2\ldots z_ns =$$
$$= c^{\rho}1^{\rho}z_{i_1}z_{i_2}\ldots z_{i_k}s = c^{\rho}z_{i_1}z_{i_2}\ldots z_{i_k}s .$$

Let $\beta := L(c^{\rho})\circ L(z_{i_1})\circ L(z_{i_2})\circ\ldots\circ L(z_{i_k})$. Then obviously $\beta\in L(B^1(L,R)^{\rho})$, $\beta t=c^{\rho}$ and $\beta s<c^{\rho}$. ∎

Lemma 6.6. Let $\underline{L}$ be a finite distributive lattice with exactly one coatom c. Then the elements c^{ρ} and c^{ρ}_{c1} satisfy the condition 5.1(ii).

Proof. Note that

$$c^{\rho}_{c1}c^{\rho} = (c_{c1}c)^{\rho} = (c_{c1}\phi_{11c1}c)^{\rho} = (c_{c1}c_{c1})^{\rho} = c^{\rho}_{c1} \tag{6.6.1}$$

and

$$c^{\rho}c^{\rho}_{c1} = (cc_{c1})^{\rho} = (c\phi_{c111}c_{c1})^{\rho} = \tag{6.6.2}$$

$$= (c1)^{\rho} \qquad \text{by Remark 3.6}$$
$$= c^{\rho}.$$

Therefore $(\{c^{\rho}, c^{\rho}_{c1}\}, \cdot)$ is a left zero semigroup. It remains to show that $c^{\rho}T(B(L,R)^{\rho})\backslash c^{\rho}_{c1}T(B(L,R)^{\rho}) = \{c^{\rho}\}$ and $c^{\rho}_{c1}T(B(L,R)^{\rho})\backslash c^{\rho}T(B(L,R)^{\rho}) = \{c^{\rho}_{c1}\}$.

Suppose on the contrary that $\underline{b} \in c^{\rho}T(B(L,R)^{\rho})\backslash c^{\rho}_{c1}T(B(L,R)^{\rho})$ and $\underline{b} \neq c^{\rho}$. Then $\underline{b} < c^{\rho}$. We want to show that $c^{\rho}_{c1}\underline{b} = \underline{b}$, i.e. $\underline{b} \in c^{\rho}_{c1}T(B(L,R)^{\rho})$, which will give a contradiction. By Remark 4.8 we may assume that $\underline{b} = b^{\rho}$ for some $b <_{L} c$ We have the following:

$$c^{\rho}_{c1}\underline{b} = c^{\rho}_{c1}b^{\rho} = (c_{c1}b)^{\rho} = (c_{c1}\phi_{11c1}b)^{\rho} = (c_{c1}(cb)_{c1})^{\rho} \tag{6.6.3}$$
$$= (ccb)^{\rho}_{c1} = (cb)^{\rho}_{c1} = b^{\rho}_{c1}.$$

To prove that $c^{\rho}_{c1}\underline{b} = \underline{b}$ it remains to show that $b \;\rho\; b_{c1}$.

Let $u \leq_{L} b$. Then since $b <_{L} c$, we also have $u <_{L} c$. Observe that

$$1bu_{c1} = bu_{c1} = b\phi_{c111}u_{c1} = b \cdot \sum(v \mid vc = u) = bu = u$$
$$= \sum(v \mid vc = u) = \phi_{c111}u_{c1} = 1u_{c1}.$$

The fourth and sixth equalities hold since for every $v \in L$, $vc = u$ if and only if $v = c$.

Moreover

$$1b_{c1}u = 1b_{c1}\phi_{11c1}u = 1b_{c1}(cu)_{c1} = 1(bcu)_{c1} = 1u_{c1} = \phi_{c111}u_{c1}$$
$$= \sum(v \mid vc = u) = u = 1u.$$

By Definition of ρ and Remark 4.8 this completes the proof that

$$b^{\rho} = b^{\rho}_{c1}. \tag{6.6.4}$$

Thus $c^{\rho}_{c1}\underline{b} = \underline{b}$, i.e. $\underline{b} \in c^{\rho}_{c1}T(B(L, R)^{\rho})$, a contradiction. Consequently

$c^{\rho}T(B(L,R)^{\rho})\backslash c^{\rho}_{c1}T(B(L,R)^{\rho}) = \{c^{\rho}\}$. Similarly we show that $c^{\rho}_{c1}T(B(L,R)^{\rho})\backslash c^{\rho}T(B(L,R)^{\rho}) = \{c^{\rho}_{c1}\}$. ∎

Theorem 6.7. Let $\underline{L}$ and R satisfy the hypothesis of Theorem

5.6. Let c be the only coatom of $\underline{L}$ and let

$$(x,1)\in R \quad \text{if and only if} \quad x\in\{1,c\}.$$

Then the bandoid $\underline{B}^1(L, R)^\rho$ is subdirectly irreducible. The monolith of $\underline{B}^1(L,R)^\rho$ is a principal congruence $\Theta(a,b)$ for some a, b satisfying 5.1(ii).

Proof. We will show that $\Theta(c^\rho, c^\rho_{c1})$ is the monolith of $\underline{B}^1(L,R)^\rho$. By Lemma 6.6, c^ρ, c^ρ_{c1} satisfy 5.1(ii). In view of Lemma 5.1 it suffices to show that $(c^\rho, c^\rho_{c1}) \in \Theta(x, y)$ for every $x,y \in B^1(L, R)^\rho$ such that $x<y$ or x,y satisfy 5.1(ii). If $\{x,y\} = \{c^\rho, c^\rho_{c1}\}$ then the proof is obvious. Let $x,y \in B^1(L,R)^\rho$ be such that $x<y$ or x,y satisfy 5.1(ii) and $\{x,y\} \neq \{c^\rho, c^\rho_{c1}\}$. By Lemma 6.3 and Lemma 6.4 we have that $c^\rho x \neq c^\rho y$. Without loss of generality we may assume that there exists $t\in JI(1^\rho T(B(L,R)^\rho))$ such that $t\leq c^\rho x$ and $t\leq c^\rho y$. Then $(c^\rho x)t=t$ and $(c^\rho y)t<t$. By Lemma 6.5 there exists $\beta\in L(B^1(L,R)^\rho)$ such that $\beta t=c^\rho$ and $\beta((c^\rho y)t)<c^\rho$. Set $\underline{b}:=\beta((c^\rho y)t)$. Note that

$$\begin{aligned}\Theta(x, y) &\geq \Theta(c^\rho x, c^\rho y) \geq \Theta((c^\rho x)t, (c^\rho y)t) = \Theta(t, (c^\rho y)t)\\ &\geq \Theta(\beta t, \beta((c^\rho y)t)) = \Theta(c^\rho, \underline{b}).\end{aligned}$$

Therefore

$$(c^\rho, \underline{b}) \in \Theta(x,y). \tag{6.7.1}$$

Now it suffices to show that $(c^\rho_{c1}, \underline{b}) \in \Theta(x,y)$. By Lemma 6.6 we have

$$c^\rho_{c1} c^\rho = c^\rho_{c1} \tag{6.7.2}$$

and

$$c^\rho T(B(L,R)^\rho)\setminus c^\rho_{c1} T(B(L,R)^\rho) = \{c^\rho\}. \tag{6.7.3}$$

Since $\underline{b}<c^\rho$, (6.7.3) implies that $\underline{b}\in c^\rho_{c1} T(B(L,R)^\rho)$, i.e.

$$c^\rho_{c1}\underline{b} = \underline{b}. \tag{6.7.4}$$

From (6.7.1), (6.7.2) and (6.7.4) it follows that

$$(c^\rho_{c1}, \underline{b}) \in \Theta(x,y). \tag{6.7.5}$$

As a consequence of (6.7.1) and (6.7.4) we obtain that $(c^{\rho}, c^{\rho}_{c1}) \in \Theta(x,y)$, which completes the proof. ∎

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