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**FACTORIZATION OF A COLLINEATION OF THE THREE-DIMENSIONAL
PROJECTIVE SPACE ONTO TWO NORMAL CYCLIC COLLINEATIONS**

This paper is a complement of works [1] and [2]. So, we shall use conventions and notations used in these papers. In particular:

$P_n(F)$ is a symbol of the n -dimensional projective space over the field F . If F is arbitrary we write simply P_n .

Characteristic of a projective transformation $f: P_n \rightarrow P_n$ is the smallest integer m such, that any point of the space P_n lies in a m -dimensional subspace of the space P_n invariant under the transformation f . We denote this integer as $\text{char } f$.

Subspace of the space P_n , each point of which is invariant under f , we shall call fundamental.

Transformation $f: X \rightarrow X$ is called k -cyclic, when f^k is the identity.

Collineation $f: P_n \rightarrow P_n$ is called *normal cyclic*, when f is a $(n+1)$ -cyclic transformation and $\text{char } f = n$.

The notation $LI^k(a_1, \dots, a_m)$ means, that any k of points a_1, \dots, a_m are linearly independent.

The concept of a normal cyclic collineation is a natural generalization of the concept of an involution in P_1 . The well-known theorem says, that any projective transformation of line is a composition of two involutions. Here arises a question, if this theorem can be generalized to an arbitrary dimension of the projective space. It is, if the following theorem is true: an arbitrary projective transformation

$f: P_n(F) \rightarrow P_n(F)$ is a composition of two (eventually k) normal cyclic collineations.

The partial solution of this problem was given in [1] and [2]. Namely, the above mentioned theorem is true for : $k=2$, $n=2$, F - an infinite field; $k=2$, $n=3$, F - an infinite algebraically closed field; $k=3$, n - arbitrary, F - a field consisting of at least $4n+4$ elements.

In this paper we shall prove the theorem for $k=2$, $n=3$, F - an arbitrary field consisting of at least 16 elements. That is, we shall prove the following theorem:

Theorem 1. If a field F consists of at least 16 elements, then any nonsingular collineation of the projective space $P_3(F)$ is a composition of two normal cyclic collineations.

For collineations with the characteristic 3 the above statement is a special case of the theorem 1[1]. This theorem assumes, that F is an infinite field, but on the strength of the initial considerations from [2] it is sufficient to assume, that $|F| \geq 16$.

The unique collineation with the characteristic 0 is the identity which is equal to ff^{-1} , where f is an arbitrary normal cyclic collineation.

So we are to prove Theorem 1 for collineations with the characteristic 1 or 2.

Let us start with the collineations with the characteristic 1. We shall classify the projective transformations with the respect of the quantity of invariant points under them and give the factorization for each class of collineations.

Lemma 1. If a collineation $f: P_3 \rightarrow P_3$ with the characteristic 1, has a plane Π invariant under f and there are three fixed noncollinear invariant points on the plane Π , then Π is fundamental or it contains a fundamental line.

Proof. Three fixed invariant points P_1, P_2, P_3 of the plane Π give three invariant lines L_1, L_2, L_3 . Take another point P on the plane Π such, that it doesn't belong to any of

lines L_1, L_2, L_3 . Since $\text{charf}=1$, hence there must be an invariant line passing through the point P . If that line doesn't lie on the plane Π , then P is invariant and $LI^3(P_1, P_2, P_3, P)$, what means that Π is fundamental. If the line lies on the plane Π , then it intersects at least one of the lines L_1, L_2, L_3 in a point different from the points P_1, P_2, P_3 and then one of these lines contains three distinct invariant points, consequently it is fundamental. \square .

Lemma 2. If $f: P_3 \rightarrow P_3$ is a collineation such that, $\text{char } f = 1$ and there exists at least one invariant point, then there exists a fundamental plane or line.

Proof. Since the collineation f has an invariant point, therefore there must be an invariant plane Π . Take a point R not on the plane Π . An invariant line passing through the point R intersects the plane Π in an invariant point. So there is at least one invariant point on the plane Π . Let us consider three possible cases:

Case one : There is exactly one invariant point P on the plane Π . Then each invariant line passes through the point P and so we have a bundle of invariant planes and by the duality principle there exists a fundamental plane there.

Case two : There are exactly two distinct invariant points P_1, P_2 on the plane Π . Then all invariant lines passing through the points of Π different from P_1, P_2 must be included in Π . Since the plane Π has exactly two invariant points, then those lines must perform a pencil. So with the respect to the principle of duality, there must exist a fundamental line on the plane Π against to the assumption, that there are exactly two invariant points there. Therefore the case two never holds.

Case three : There are at least three invariant points on the plane Π . If the points are collinear, then they form the fundamental line. In the other case according to Lemma 1 the plane Π is fundamental or it contains a fundamental line. \square .

If a collineation has a fundamental plane Π , then by the duality principle there exists a bundle of invariant planes

there. The center P of it is an invariant point. If the point P belongs to plane Π , then the transformation f is an elation. In the other case f is a homology. So, there are only two types of collineations, which have a fundamental plane. Both of them have the characteristic 1 and satisfy assumptions of the following lemma:

Lemma 3. Suppose, that $f: P_n(F) \rightarrow P_n(F)$ is a collineation such that: $\text{char } f = m$, H_1, \dots, H_p are all fundamental subspaces under f , the dimensions of which are k_1, \dots, k_p ($k_1 > 0, k_2 \geq 0, \dots, k_p \geq 0$). Let the numbers k_1, \dots, k_p satisfy:

$$k_1 + \dots + k_p = n-m.$$

If the field F has at least $4n+4$ elements, then the transformation f is a composition of two normal cyclic collineations.

Proof. Lemma is an immediate corollary from Theorem 2[1] and initial considerations of paper [2]. \square .

Therefore any collineation with the characteristic 1, which has a fundamental plane is a composition of two normal cyclic collineations.

A collineation having a fundamental line and, at the same time having no fundamental plane, is called axis collineation. Notice that by the duality principle such collineation possesses a pencil of invariant planes and its axis is an invariant line.

Lemma 4. If a collineation $f: P_3 \rightarrow P_3$ is an axis collineation and $\text{char } f = 1$, then an axis of a pencil of invariant planes is a fundamental line.

Proof. Let L denote an axis of a pencil of invariant planes. Assume first, that each plane of this pencil contains no three noncollinear invariant points. Let Π be one of these planes. Then there exists a point P_1 on the plane Π , which is not united. Let L_1 denote an invariant line passing through the point P_1 . The point P_1 is not invariant, hence the line L_1 is included in the plane Π and intersects the axis L in a point R_1 . Assume, that another invariant line L_2 included in the plane Π cuts the axis L in a point R_2 different from the

point R_1 . The common point P of the lines L_1 and L_2 is of course invariant and doesn't belong to the axis L . The points R_1, R_2 and P are three non collinear invariant points on the plane Π , what contradicts our assumptions. Therefore all invariant lines on the plane Π form a pencil with the center on the axis L . As there are no three non collinear united points on the plane Π , so the restriction of the collineation f to this plane is an elation. For the plane Π was arbitrarily chosen, we have the same situation on each plane from the pencil L . Moreover the fundamental line in each plane must be the axis L , because in the other case the fundamental lines would generate a fundamental plane or space against to the assumption, that f was an axis collineation.

Assume now, that on each plane from the pencil L there are three non collinear united points. Let Π denote one of these planes. Then according to Lemma 1 Π is a fundamental plane or it contains a fundamental line. The first case is contradictory to the assumption, that f is an axis collineation. So the second case must be hold and we have a homology on the plane Π (and consequently on each plane from the pencil L). As previously it can be proofed, that the fundamental line on each plane from this pencil must coincide with the axis L .

Note that it is impossible, that a restriction of the transformation f to one plane is an elation and a restriction of f to another plane is a homology at the same time. To prove this fact assume conversely, that on a plane Π_1 we have an elation with a center P_1 and on a plane Π_2 - a homology with a center P_2 . The fundamental lines of considered elation and homology must both, as previously, coincide with the axis L , because in the other case they would set a fundamental plane (if they would have a common point) or the plane Π_1 would be fundamental (if they wouldn't have any common point). Moreover the point P_2 doesn't lie on the axis L , and so each plane containing a line from the pencil P_1 of the invariant lines on the plane Π_1 and the point P_2 must be invariant. Hence we have a pencil P_1P_2 of invariant planes different from the pencil L .

These two pencils give in the point P_1 a bundle of invariant lines and consequently - a bundle of invariant planes. But this is impossible, for f is an axis collineation. \square .

An axis collineation, the restrictions of which to all invariant planes are homologies, satisfies the assumptions of lemma 3, hence it is a composition of two normal cyclic collineations.

An axis collineation, the restrictions of which to all invariant planes are elations, is a composition of two normal cyclic collineations (Theorem 4[1]).

Lemma 5. If a collineation $f: P_3 \rightarrow P_3$ with the characteristic 1, has no united points, then f is a composition of two normal cyclic collineations.

Proof. Take into account three different invariant lines L_1, L_2, L_3 . As the collineation f has no united points, hence these lines are mutually skew. Let points P_1, P_2, P_3 belong to the line L_1 ; points P_4, P_5, P_6 - to the line L_2 ; points P_7, P_8 - to the line L_3 . Moreover let $f(P_1)=P_2, f(P_2)=P_3, f(P_4)=P_5, f(P_5)=P_6, f(P_7)=P_8$. We shall denote this using only indexes of points by symbol :

$$f: \begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 2 & 3 & 5 & 6 & 8 \end{matrix} .$$

If the collineation f is an involution, then $P_1=P_3, P_4=P_6$ and f is a composition of transformations g and h defined in the following way :

$$g: \begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 4 & 5 & 2 & 1 & x \end{matrix} ,$$

$$h: \begin{matrix} 4 & 5 & 2 & 1 & x \\ 2 & 3 & 5 & 6 & 8 \end{matrix} ,$$

where x is such that $LI^4(4,5,2,1,x)$. Collineations g and h are well defined and they are normal cyclic on the strength of the lemma 1[1].

If the collineation f is not an involution, then we can choose such coordinate system and such points P_1, \dots, P_8 , that they have the following coordinates:

1 : (1,0,0,0)
 2 : (0,1,0,0)
 3 : (1,a,0,0)
 4 : (0,0,1,0)
 5 : (0,0,0,1)
 6 : (0,0,b,1)
 7 : (1,1,1,1)
 8 : (c,d,e,f)

and $b \neq 0$. There are some conditions which the numbers a, b, c, d, e, f must satisfy. These conditions come from linear independence of some quadruples of points. For instance, from $LI^4(2,3,5,6,8)$ we obtain :

$$ac-d \neq 0, -c \neq 0, bf-e \neq 0, e \neq 0.$$

Then the collineation f is a composition of transformations g and h defined in the following way :

$$g: \begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 5 & 1 & 2 & 4 & 9 \end{matrix},$$

$$h: \begin{matrix} 5 & 1 & 2 & 4 & 9 \\ 2 & 3 & 5 & 6 & 8 \end{matrix},$$

where the point P_9 has coordinates $(-c, \frac{bf-e}{b}, e, ac-d)$. The sufficient conditions for $LI^4(5,1,2,4,9)$ are :

$$-c \neq 0, bf-e \neq 0, e \neq 0, ac-d \neq 0.$$

As we already noticed these are satisfied. Hence the collineations g and h are well defined. Moreover the collineation g is normal cyclic according to Lemma 1[1].

The matrix of the collineation h is following :

$$H = \begin{bmatrix} -b & 0 & 0 & 0 \\ -ab & 0 & 0 & -b \\ 0 & 0 & b & 0 \\ 0 & b & 1 & 0 \end{bmatrix}$$

By simple calculation one can check, that $H^4 = b^4 I$, where I is the identity matrix. These means, that h is also a normal cyclic collineation. \square .

Finally, in view of the lemmas 2,3,4,5 we find that if $|F| \geq 16$, then any collineation of $P^3(F)$ with the characteristic

1 is a composition of two normal cyclic collinations.

Now we shall deal with collineations with the characteristic 2. If $\text{char } f = 2$, then each point of the space lies on an invariant plane.

Lemma 6. If a collineation $f: P_3 \rightarrow P_3$ has the characteristic 2, then there exists a pencil of invariant planes.

Proof. Assume oppositely, that three invariant planes intersect each other along three distinct lines L_1, L_2, L_3 . Then these lines have one common point P . Let R be an arbitrary point beyond from these three planes. Consider an invariant plane Π containing the point R . If the plane Π contains one of the lines L_1, L_2, L_3 , then this line belongs to three invariant planes, hence it is an axis of a pencil of invariant planes.

If the plane Π contains the point P , but includes none of the lines L_1, L_2, L_3 , then four considered planes intersect along six distinct lines, four of which are such, that any three of them are linearly independent. So they generate a bundle of invariant lines in the point P , what is contradictory to the assumption, that $\text{char } f = 2$.

If the plane Π doesn't contain the point P , then four considered planes form a simplex. Take another invariant plane Φ . This plane cannot intersect the above-mentioned simplex along four distinct lines, because in this case it would be fundamental against to the assumption, that $\text{char } f = 2$. Therefore the plane Φ intersects this simplex along three lines. Hence one of these lines belongs to three invariant planes and consequently is an axis of a pencil of invariant planes.

□.

From the above lemma and the duality principle it follows that a transformation with the characteristic 2 has a fundamental line. But an axis of a pencil of invariant planes cannot be fundamental, because then there would be a homology or elation on each plane from the pencil, so the characteristic of f would be 1.

Lemma 7. A collineation $f: P_3 \rightarrow P_3$ with the characteristic 2 is a composition of two normal cyclic collineations.

Proof. Since $\text{char } f = 2$, then there exist an invariant plane Π and points P_1, P_2, P_3, P_4 on it such that $f(P_1) = P_2$, $f(P_2) = P_3$, $f(P_3) = P_4$ and $\text{LI}^3(P_1, P_2, P_3, P_4)$. Let P_5 and P_6 be two points from a fundamental line and beyond from the plane Π . Then $\text{LI}^4(1, 2, 3, 5, 6)$ and $\text{LI}^4(2, 3, 4, 5, 6)$. So the collineation f is well defined by the assigning :

$$f: \begin{matrix} 1 & 2 & 3 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 \end{matrix} .$$

According to Lemma 1[1] transformations g and h defined in the following way :

$$g: \begin{matrix} 1 & 2 & 3 & 5 & 6 \\ 5 & 2 & 1 & 6 & 3 \end{matrix} ,$$

$$h: \begin{matrix} 5 & 2 & 1 & 6 & 3 \\ 2 & 3 & 4 & 5 & 6 \end{matrix}$$

are normal cyclic collineations and their composition gives f .

□.

The above lemma ends the proof of Theorem 1.

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Received April 25, 1991.

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The following individuals have acted as referees of the articles published in this volume:

K. Cegiełka, K. Głazek, R. Godowski, B. Grochowski, K. Halkowska, T. Kepka, J. Kuraś,
E. Orłowska, E. Płonka, J. Płonka, B. Przybylski, B. Pöndlück, P. Pták, J. Rogulski,
A. Romanowska, B. Roszkowska, A. Rutkowski, W. Sasin, J. Smith, A. Strasburger, K. Szymiczek,
W. Waliszewski, B. Wojdyło, Z. Żekanowski.

The editor expresses them his sincere thanks for their advice and collaboration.