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# FACTORIZATION OF A COLLINEATION OF THE THREE-DIMENSIONAL PROJECTIVE SPACE ONTO TWO NORMAL CYCLIC COLLINEATIONS

This paper is a complement of works [1] and [2]. So, we shall use conventions and notations used in these papers. In particular:

$P_n(F)$ is a symbol of the n-dimensional projective space over the field F. If F is arbitrary we write simply $P_n$.

*Characteristic* of a projective transformation $f: P_n \longrightarrow P_n$ is the smallest integer m such, that any point of the space $P_n$ lies in a m-dimensional subspace of the space $P_n$ invariant under the transformation f. We denote this integer as char f.

Subspace of the space $P_n$, each point of which is invariant under f, we shall call *fundamental*.

Transformation $f: X \longrightarrow X$ is called k-cyclic, when $f^k$ is the identity.

Collineation $f: P_n \longrightarrow P_n$ is called *normal cyclic*, when f is a (n+1)-cyclic transformation and char f = n.

The notation $LI^k(a_1,\dots,a_m)$ means, that any k of points $a_1,\dots,a_m$ are linearly independent.

The concept of a normal cyclic collineation is a natural generalization of the concept of an involution in $P_1$. The well-known theorem says, that any projective transformation of line is a composition of two involutions. Here arises a question, if this theorem can be generalized to an arbitrary dimension of the projective space. It is, if the following theorem is true: an arbitrary projective transformation f: $P_n(F) \longrightarrow P_n(F)$ is a composition of two (eventually k) normal cyclic collineations.

The partial sollution of this problem was given in [1] and [2]. Namely, the above mentioned theorem is true for : k=2, n=2, F - an infinite field; k=2, n=3, F - an infinite algebraically closed field; k=3, n - arbitrary, F - a field consisting of at least 4n+4 elements.

In this paper we shall prove the theorem for k=2, n=3, F - an arbitrary field consisting of at least 16 elements. That is, we shall prove the following theorem:

**Theorem 1.** If a field F consists of at least 16 elements, then any nonsingular colineation of the projective space $P_3(F)$ is a composition of two normal cyclic collineations.

For collineations with the caracteristic 3 the above statement is a special case of the theorem 1[1]. This theorem assumes, that F is an infinite field, but on the strength of the initial considerations from [2] it is suffitient to assume, that $|F| \geq 16$.

The unique collineation with the characteristic 0 is the identity which is equal to $ff^{-1}$, where f is an arbitrary normal cyclic collineation.

So we are to prove Theorem 1 for collineations with the characteristic 1 or 2.

Let us start with the collineations with the characteristic 1. We shall classify the projective transformations with the respect of the quantity of invariant points under them and give the factorization for each class of collineations.

**Lemma 1.** If a collineation f: $P_3 \longrightarrow P_3$ with the characteristic 1, has a plane $\Pi$ invariant under f and there are three fixed noncollinear invariant points on the plane $\Pi$, then $\Pi$ is fundamental or it contains a fundamental line.

**Proof.** Three fixed invariant points $P_1$, $P_2$, $P_3$ of the plane $\Pi$ give three invariant lines $L_1$, $L_2$, $L_3$. Take another point P on the plane $\Pi$ such, that it doesn't belong to any of lines $L_1$, $L_2$, $L_3$. Since charf=1, hence there must be an invariant line passing through the point P. If that line doesn't lie on the plane $\Pi$, then P is invariant and $LI^3(P_1,P_2,P_3,P)$, what means that $\Pi$ is fundamental. If the line lies on the plane $\Pi$, then it intersects at least one of the lines $L_1$, $L_2$, $L_3$ in a point different from the points $P_1$, $P_2$, $P_3$ and then one of these lines contains three distinct invariant points, consequently it is fundamental. □.

**Lemma 2.** If $f: P_3 \longrightarrow P_3$ is a collineation such that, char f = 1 and there exists at least one invariant point, then there exists a fundamental plane or line.

**Proof.** Since the collineation f has an invariant point, therefore there must be an invariant plane $\Pi$. Take a point R not on the plane $\Pi$. An invariant line passing through the point R intersects the plane $\Pi$ in an invariant point. So there is at least one invariant point on the plane $\Pi$. Let us consider three possible cases:

Case one : There is exactly one invariant point P on the plane $\Pi$. Then each invariant line passes through the point P and so we have a bundle of invariant planes and by the duality principle there exists a fundamental plane there.

Case two : There are exactly two distinct invariant points $P_1,P_2$ on the plane $\Pi$. Then all invariant lines passing through the points of $\Pi$ different from $P_1,P_2$ must be included in $\Pi$. Since the plane $\Pi$ has exactly two invariant points, then those lines must perform a pencil. So with the respect to the principle of duality, there must exist a fundamental line on the plane $\Pi$ against to the assumption, that there are exactly two invariant points there. Therefore the case two never holds.

Case three : There are at least three invariant points on the plane $\Pi$. If the points are collinear, then they form the fundamental line. In the other case according to Lemma 1 the plane $\Pi$ is fundamental or it contains a fundamental line. □.

If a collineation has a fundamental plane $\Pi$, then by the duality principle there exists a bundle of invariant planes there. The center P of it is an invariant point. If the point P belongs to plane Π, then the transformation f is an elation. In the other case f is a homology. So, there are only two types of collineations, witch have a fundamental plane. Both of them have the characteristic 1 and satisfy assumptions of the following lemma:

**Lemma 3.** Suppose, that $f: P_n(F) \longrightarrow P_n(F)$ is a collineation such that: char f = m, $H_1,\dots,H_p$ are all fundamental subspaces under f, the dimensions of which are $k_1,\dots,k_p$ ($k_1>0, k_2\geq 0,\dots,k_p\geq 0$). Let the numbers $k_1,\dots,k_p$ satisfy:

$$k_1+\dots+k_p = n-m.$$

If the field F has at least 4n+4 elements, then the transformation f is a composition of two normal cyclic collineations.

**Proof.** Lemma is an immediate corollary from Theorem 2[1] and initial considerations of paper [2]. □.

Therefore any collineation with the characteristic 1, witch has a fundamental plane is a composition of two normal cyclic collineations.

A collineation having a fundamental line and, at the same time having no fundamental plane, is called axis collineation. Notice that by the duality principle such collineation possesses a pencil of invariant planes and its axis is an invariant line.

**Lemma 4.** If a collineation $f: P_3 \longrightarrow P_3$ is an axis collineation and char f = 1, then an axis of a pencil of invariant planes is a fundamental line.

**Proof.** Let L denote an axis of a pencil of a invariant planes. Assume first, that each plane of this pencil contains no three noncollinear invariant points. Let Π be one of these planes. Then there exists a point $P_1$ on the plane Π, witch is not united. Let $L_1$ denote an invariant line passing through the point $P_1$. The point $P_1$ is not invariant, hence the line $L_1$ is included in the plane Π and intersects the axis L in a point $R_1$. Assume, that another invariant line $L_2$ included in the plane Π cuts the axis L in a point $R_2$ different from the point $R_1$. The common point P of the lines $L_1$ and $L_2$ is of course invariant and doesn't belong to the axis L. The points $R_1, R_2$ and P are three non collinear invariant points on the plane $\Pi$, what contradicts our assumptions. Therefore all invariant lines on the plane $\Pi$ form a pencil with the center on the axis L. As there are no three non collinear united points on the plane $\Pi$, so the restriction of the collineation f to this plane is an elation. For the plane $\Pi$ was arbitrarily chosen, we have the same situation on each plane from the pencil L. Moreover the fundamental line in each plane must be the axis L, because in the other case the fundamental lines would generate a fundamental plane or space against to the assumption, that f was an axis collineation.

Assume now, that on each plane from the pencil L there are three non collinear united points. Let $\Pi$ denote one of these planes. Then according to Lemma 1 $\Pi$ is a fundamental plane or it contains a fundamental line. The first case is contradictory to the assumption, that f is an axis collineation. So the second case must be hold and we have a homology on the plane $\Pi$ (and consequently on each plane from the pencil L). As previously it can be proofed, that the fundamental line on each plane from this pencil must coincide with the axis L.

Note that it is impossible, that a restriction of the transformation f to one plane is an elation and a restriction of f to another plane is a homology at the same time. To prove this fact assume conversely, that on a plane $\Pi_1$ we have an elation with a center $P_1$ and on a plane $\Pi_2$ - a homology with a center $P_2$. The fundamental lines of considered elation and homology must both, as previously, coincide with the axis L, because in the other case they would set a fundamental plane (if they would have a common point) or the plane $\Pi_1$ would be fundamental (if they wouldn't have any common point). Moreover the point $P_2$ doesn't lie on the axis L, and so each plane containing a line from the pencil $P_1$ of the invariant lines on the plane $\Pi_1$ and the point $P_2$ must be invariant. Hence we have a pencil $P_1P_2$ of invariant planes different from the pencil L.

These two pencils give in the point $P_1$ a bundle of invariant lines and consequently - a bundle of invariant planes. But this is impossible, for f is an axis collineation. □.

An axis collineation, the restrictions of which to all invariant planes are homologies, satisfies the assumptions of lemma 3, hence it is a composition of two normal cyclic collineations.

An axis collineation, the restrictions of which to all invariant planes are elations, is a composition of two normal cyclic collineations (Theorem 4[1]).

**Lemma 5.** If a collineation f: $P_3 \longrightarrow P_3$ with the characteristic 1, has no united points, then f is a composition of two normal cyclic collineations.

**Proof.** Take into account three different invariant lines $L_1$, $L_2$, $L_3$. As the collineation f has no united points, hence these lines are mutually skew. Let points $P_1$, $P_2$, $P_3$ belong to the line $L_1$; points $P_4$, $P_5$, $P_6$ - to the line $L_2$; points $P_7$, $P_8$ - to the line $L_3$. Moreover let $f(P_1)=P_2$, $f(P_2)=P_3$, $f(P_4)=P_5$, $f(P_5)=P_6$, $f(P_7)=P_8$. We shall denote this using only indexes of points by symbol :

$$f: \begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 2 & 3 & 5 & 6 & 8 \end{matrix} .$$

If the collineation f is an involution, then $P_1=P_3$, $P_4=P_6$ and f is a composition of transformations g and h defined in the following way :

$$g: \begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 4 & 5 & 2 & 1 & x \end{matrix} ,$$

$$h: \begin{matrix} 4 & 5 & 2 & 1 & x \\ 2 & 3 & 5 & 6 & 8 \end{matrix} ,$$

where x is such that $LI^4(4,5,2,1,x)$. Collineations g and h are well defined and they are normal cyclic on the strength of the lemma 1[1].

If the collineation f is not an involution, then we can choose such coordinate system and such points $P_1,\dots,P_8$, that they have the following coordinates:

1 : (1,0,0,0)
2 : (0,1,0,0)
3 : (1,a,0,0)
4 : (0,0,1,0)
5 : (0,0,0,1)
6 : (0,0,b,1)
7 : (1,1,1,1)
8 : (c,d,e,f)

and $b\neq 0$. There are some conditions which the numbers a,b,c,d,e,f must satisfy. These conditions come from linear independence of some quadruples of points. For instance, from $LI^4(2,3,5,6,8)$ we obtain :

$$ac-d\neq 0,\ -c\neq 0,\ bf-e\neq 0,\ e\neq 0.$$

Then the collineation f is a composition of transformations g and h defined in the following way :

$$g: \begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 5 & 1 & 2 & 4 & 9 \end{matrix},$$

$$h: \begin{matrix} 5 & 1 & 2 & 4 & 9 \\ 2 & 3 & 5 & 6 & 8 \end{matrix},$$

where the point $P_9$ has coordinates $(-c,\frac{bf-e}{b},e,ac-d)$. The sufficient conditions for $LI^4(5,1,2,4,9)$ are :

$$-c\neq 0,\ bf-e\neq 0,\ e\neq 0,\ ac-d\neq 0.$$

As we already noticed these are satisfied. Hence the collineations g and h are well defined. Moreover the collineation g is normal cyclic according to Lemma 1[1].

The matrix of the collineation h is following :

$$H = \begin{bmatrix} -b & 0 & 0 & 0 \\ -ab & 0 & 0 & -b \\ 0 & 0 & b & 0 \\ 0 & b & 1 & 0 \end{bmatrix}$$

By simple calculation one can check, that $H^4=b^4I$, where I is the identity matrix. These means, that h is also a normal cyclic collineation. □.

Finally, in view of the lemmas 2,3,4,5 we find that if $|F|\geq 16$, then any collineation of $P^3(F)$ with the characteristic

1 is a composition of two normal cyclic collinations.

Now we shall deal with collineations with the characteristic 2. If char f = 2, then each point of the space lies on an invariant plane.

**Lemma 6.** If a collineation f: $P_3 \longrightarrow P_3$ has the characteristic 2, then there exists a pencil of invariant planes.

**Proof.** Assume oppositely, that three invariant planes intersect each other along three distinct lines $L_1$, $L_2$, $L_3$. Then these lines have one common point P. Let R be an arbitrary point beyond from these three planes. Consider an invariant plane Π containing the point R. If the plane Π contains one of the lines $L_1$, $L_2$, $L_3$, then this line belongs to three invariant planes, hence it is an axis of a pencil of invariant planes.

If the plane Π contains the point P, but includes none of the lines $L_1, L_2, L_3$, then four considered planes intersect along six distinct lines, four of witch are such, that any three of them are linearly independent. So they generate a bundle of invariant lines in the point P, what is contradictory to the assumption, that char f = 2.

If the plane Π doesn't contain the point P, then four considered planes form a simplex. Take another invariant plane Φ. This plane cannot intersect the above-mentioned simplex along four distinct lines, because in this case it would be fundamental against to the assumption, that char f = 2. Therefore the plane Φ intersects this simplex along three lines. Hence one of these lines belongs to three invariant planes and consequently is an axis of a pencil of invariant planes.

□.

From the above lemma and the duality principle it follows that a transformation with the characteristic 2 has a fundamental line. But an axis of a pencil of invariant planes cannot be fundamental, because then there would be a homology or elation on each plane from the pencil, so the characteristic of f would be 1.

**Lemma 7.** A collineation f: $P_3 \longrightarrow P_3$ with the characteristic 2 is a composition of two normal cyclic collineations.

**Proof.** Since char f = 2, then there exist an invariant plane Π and points $P_1, P_2, P_3, P_4$ on it such that $f(P_1)=P_2$, $f(P_2)=P_3$, $f(P_3)=P_4$ and $LI^3(P_1,P_2,P_3,P_4)$. Let $P_5$ and $P_6$ be two points from a fundamental line and beyond from the plane Π. Then $LI^4(1,2,3,5,6)$ and $LI^4(2,3,4,5,6)$. So the collineation f is well defined by the assigning :

$$f: \begin{matrix} 1 & 2 & 3 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 \end{matrix} .$$

According to Lemma 1[1] transformations g and h defined in the following way :

$$g: \begin{matrix} 1 & 2 & 3 & 5 & 6 \\ 5 & 2 & 1 & 6 & 3 \end{matrix} ,$$

$$h: \begin{matrix} 5 & 2 & 1 & 6 & 3 \\ 2 & 3 & 4 & 5 & 6 \end{matrix}$$

are normal cyclic collineations and their composition gives f. □.

The above lemma ends the proof of Theorem 1.

## References


[1] **K. Witczyński:** Projective collineations as products of cyclic collineations, Demonstratio Math. 4 (1979), 1111-1125.

[2] **K. Witczyński:** On generators of the group of projective transformations, Demonstratio Math. 4 (1981), 1053-1075.



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