DEMONSTRATIO MATHEMATICA
Vol. XXIV No 3-4 1991

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ON SOME PROPERTY OF PASCALIAN HEXAGONS

Let P_2 denote a projective pappian plane (finite or infinite). We say that the sextuple $(a_1,\dots,a_6)$ of distinct points of P_2 forms a hexagon if the lines $a_1a_2,\dots,a_5a_6,a_6a_1$ are all distinct. A hexagon $a_1a_2\dots a_6$ is said to be pascalian if the points $a_1a_2 \cap a_4a_5$, $a_2a_3 \cap a_5a_6$, $a_3a_4 \cap a_6a_1$ colline.

Assume that we are given in P_2 six points $a_1,\dots,a_6$ such that no three of them are collinear. Define

(1) $b_i = a_ia_{i+2} \cap a_{i+1}a_{i+3}$, $i=1,\dots,6$ (we reduce the indices mod 6).

One can check easily that the points b_i are all distinct and the sextuple $(b_1,\dots,b_6)$ is a hexagon. Moreover, only the points b_1,b_3,b_5 or b_2,b_4,b_6 may be collinear among all the point triples from the set $\{b_1,\dots,b_6\}$.

Proposition 1. Suppose there is a point s such that the points s,a_i,a_{i+3} colline for $i=1,2,3$. Then the hexagon $a_1a_2\dots a_6$ is pascalian if and only if the hexagon $b_1b_2\dots b_6$ is pascalian and the points s,b_i,b_{i+3} colline for $i=1,2,3$.

Proof. It is known [1] that the theorem of Pascal holds in a projective plane P if and only if the coordinatizing field of P is commutative, i.e. P is pappian. Suppose the hexagon $a_1a_2\dots a_6$ is pascalian. By the converse of the theorem of Pascal we infer that the points $a_1,\dots,a_6$ are on a proper conic. It implies that for every permutation $(i_1,\dots,i_6)$ of the numbers $1,\dots,6$ the hexagon $a_{i_1},\dots a_{i_6}$ is pascalian.

Hence the points

(2) $b_i = a_i a_{i+2} \cap a_{i+1} a_{i+3}$, $b_{i+3} = a_{i+3} a_{i+5} \cap a_{i+4} a_i$,
$s = a_{i+1} a_{i+4} \cap a_{i+2} a_{i+5}$, $i = 1,2,3$

are collinear.

On the other hand, the points

(3) $a_i = b_{i+3} b_{i+4} \cap b_i b_{i+5}$, $a_{i+3} = b_i b_{i+1} \cap b_{i+2} b_{i+3}$,
$s = b_{i+1} b_{i+4} \cap b_{i+2} b_{i+5}$

are also collinear for $i = 1,2,3$.

Since the lines $b_1b_2, \ldots, b_5b_6$, b_6b_1, b_1b_4, b_2b_5, b_3b_6 are all distinct, the point sextuples $(b_2,b_1,b_6,b_3,b_4,b_5)$, $(b_3,b_2,b_1,b_4,b_5,b_6)$, $(b_4,b_3,b_2,b_5,b_6,b_1)$ are hexagons. It follows from (3) that these hexagons are pascalian.

Notice that if the points b_1,b_3,b_5 are collinear, then the points b_2,b_4,b_6 are collinear too, and conversely. In fact, if e.g. the points b_1,b_3,b_5 colline, then the triangles $a_1a_3a_5$ and $a_2a_4a_6$ are doubly perspective. Hence they are triply perspective and, consequently, the points b_2,b_4,b_6 are on an axis of perspectivity.

We have then to consider the following cases:

1) The points $b_1,\ldots,b_6$ are such that no three of them are collinear. Then they are on a proper conic and hence the hexagon $b_1b_2\ldots b_6$ is pascalian.

2) The points b_1,b_3,b_5 are on a line and, at the same time, the points b_2,b_4,b_6 are on an another line. From the theorem of Pappus it follows that the points $b_1b_2 \cap b_4b_5$, $b_2b_3 \cap b_5b_6$, $b_3b_4 \cap b_6b_1$ colline, i.e. the hexagon $b_1b_2\ldots b_6$ is pascalian.

If now the hexagon $b_1b_2\ldots b_6$ is pascalian and the points s,b_i,b_{i+3} colline for $i = 1,2,3$, then, in view of (2) and (3), we see that the hexagon $a_1a_2\ldots a_6$ is pascalian and the points s,a_i,a_{i+3} colline for $i = 1,2,3$.

Proposition 2. Let $(a_1,\ldots,a_6)$ and $(b_1,\ldots,b_6)$ be two sextuples of points of P_2 related themselves as in (1) and such that no three of the points $a_1,\ldots,a_6$ are on a line. Then the hexagons $a_1a_2\ldots a_6$ and $b_1b_2\ldots b_6$ are both pascalian if and only if there exists such a point s that the points s, a_i, a_{i+3} and s, b_i, b_{i+3} are collinear for $i = 1,2,3$.

Proof. Let both the hexagons $a_1 \ldots a_6$ and $b_1 \ldots b_6$ be pascalian. Let us denote by s the point $a_1a_4 \cap a_2a_5$ and by s^* the point $b_1b_4 \cap b_2b_5$. Since the hexagons $a_1 \ldots a_6$, $b_1 \ldots b_6$ are pascalian, the hexagons $a_1a_4a_6a_2a_5a_3$ and $b_1b_4b_3b_2b_5b_6$ are pascalian. Hence the points s, b_3, b_6 as well as s^*, a_3, a_6 are collinear. Analogously, we show that $s \in b_1b_4$, $s \in b_2b_5$, $s^* \in a_1a_4$, $s^* \in a_2a_5$. Thence $s = s^*$.

If now there exists a point s such that $s \in a_ia_{i+3}$ and $s \in b_ib_{i+3}$ for $i = 1,2,3$, then, in particular, the points $b_2 = a_2a_4 \cap a_3a_5$, $b_5 = a_1a_5 \cap a_2a_6$, $s = a_1a_4 \cap a_3a_6$ are collinear. It means that the hexagon $a_1a_4a_2a_6a_3a_5$ is pascalian. It implies that the points a_i, $i = 1,\ldots,6$ are on a proper conic and hence the hexagon $a_1 \ldots a_6$ is pascalian. By Proposition 1, we infer that the hexagon $b_1b_2 \ldots b_6$ is pascalian.

REFERENCES

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Received April 24, 1991.