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ON THE DIMENSION OF THE TANGENT DIFFERENTIAL SPACE

In this paper we investigate the dimension of the tangent space to the tangent differential space ([1]). In section 2 we show some relations between singularities of a differential space of class $\mathcal{D}_o$ ([9],[10]) and singularities of its tangent differential space.

1. Main results

Let (M,C) be a differential space ([6],[7],[8]), $p \in M$, $v \in T_pM$ and let π: TM $\longrightarrow$ M be the natural projection of the tangent bundle TM on M ([1]).

Proposition 1.1. The kernel of the tangent mapping $(d\pi)_v$ is isomorphic to $T_v(T_pM)$.

Proof. Let ι: $T_pM \longrightarrow TM$ be a imbedding, for any $w \in T_pM$, $\iota(w) = w$. The tangent mapping $(d\iota)_v$: $T_v(T_pM) \longrightarrow T_v(TM)$ is a monomorphism, because ι is one to one. Then $T_v(T_pM)$ is isomorphic to the image of the tangent mapping $(d\iota)_v$. It will be proved that the image of the tangent mapping $(d\iota)_v$ is equal to the kernel of the tangent mapping $(d\pi)_v$ ($im(d\iota)_v = ker(d\pi)_v$).

Let $\alpha \in C$ and $w \in T_v(T_pM)$. Then

$$(d\pi)_v((d\iota)_v w)\alpha = ((d\iota)_v w)(\alpha\circ\pi) = w(\alpha\circ\pi\circ\iota) = w(\alpha(p)) = 0.$$

Now it is obvious that $im(d\iota)_v \subseteq ker(d\pi)_v$.

Now, we prove the inclusion $ker(d\pi)_v \subseteq im(d\iota)_v$. Let $u \in ker(d\pi)_v$. Then, for any $\alpha \in C$, $u(\alpha\circ\pi) = 0$.

The set $\{d_p\alpha: \alpha \in C\}$ generates differential structure on T_pM, where $d_p\alpha$ is a function given by

$$(d_p\alpha)v = v(\alpha), \text{ for any } v \in T_pM.$$

Let us define a mapping $w_0: \{d_p\alpha: \alpha \in C\} \longrightarrow \mathbb{R}$ as follows

$$w_0(d_p\alpha) = u(d\alpha), \text{ for any } \alpha \in C .$$

We will check the correctness of the above definition. Let us assume $d_p\alpha = 0$. Then there exist functions $\phi_1,\dots,\phi_n \in C$, $\omega \in C^\infty(\mathbb{R}^n)$ and a neighbourhood U of the point p in M such that

$$\alpha_{|U} = \omega\circ(\phi_1,\dots,\phi_n)_{|U}, \quad \partial_i\omega(\phi_1(p),\dots,\phi_n(p)) = 0,$$

for any $i = 1,\dots n$ ([3]).
One can see that

$$w_0(d_p\alpha) = u(d\alpha) = u(d(\alpha_{|U})) = u(d(\omega\circ(\phi_1,\dots,\phi_n)_{|U})) =$$
$$= u(\sum_{i=1}^n \partial_i\omega((\phi_1,\dots,\phi_n)\circ\pi_{|\pi^{-1}(U)})d(\phi_{i|U})) =$$
$$= \sum_{i=1}^n \partial_i\omega(\phi_1(p),\dots\phi_n(p))u(d(\phi_{i|U}) +$$
$$+ \sum_{i=1}^n(\sum_{j=1}^n \partial_{ji}^2\omega(\phi_1(p),\dots,\phi_n(p))w(\phi_j\circ\pi))d\phi_i(v) = 0 .$$

Let $\alpha,\beta \in C$ and $d_p\alpha = d_p\beta$. Now, one can easily see that $w_0(d_p\alpha) = w_0(d_p\beta)$, which proves the correctness of the definition of w_0. Moreover, the mapping w_0 is linear, because the mapping u is linear.

Now, we will prove that w_0 can be extended to a vector $w \in T_v(T_pM)$. Let $\omega \in C^\infty(\mathbb{R}^n)$ and $\phi_1,\dots,\phi_n \in C$ such that $\omega\circ(d_p\phi_1,\dots,d_p\phi_n) = 0$.
Let us assume that $d_p\phi_1,\dots,d_p\phi_n$ are linear independent. Then there exist vectors $v_1,\dots,v_n \in T_pM$ such that

$$v_i\phi_j = \delta_{ij}, \text{ for any } i,j = 1,\dots,n.$$

For any $a_1,\dots,a_n \in \mathbb{R}$,

$$(\omega\circ(d_p\phi_1,\ldots,d_p\phi_n))(v + \sum_{i=1}^{n} a_i v_i) = \omega(v\phi_1 + a_1,\ldots,v\phi_n + a_n).$$

Now it is obvious that $\omega(v\phi_1+a_1,\ldots,v\phi_n+a_n) = 0$ for any $a_1,\ldots,a_n \in \mathbb{R}$. Then $\partial_i\omega(v\phi_1,\ldots,v\phi_n) = 0$, for any $i = 1,\ldots,n$. Let us assume that, for $k\le n$, $d_p\phi_1,\ldots,d_p\phi_k$ are linear independent and $d_p\phi_{k+1},\ldots,d_p\phi_n$ are their linear combinations

$$d_p\phi_i = \sum_{j=1}^{k} b_{ij} d_p\phi_j,$$

where $b_{ij} \in \mathbb{R}$,for any $i = k + 1,\ldots,n$, $j = 1,\ldots,k$.

Now, let us define a function $\theta \in C^\infty(\mathbb{R}^k)$ as follows

$$\theta(x_1,\ldots,x_k) = \omega(x_1,\ldots,x_k,\sum_{j=1}^{k} b_{k+1,j}\, x_j,\ldots,\sum_{j=1}^{k} b_{nj}\, x_j),$$

for any $(x_1,\ldots,x_k) \in \mathbb{R}^k$.

It is obvious that $\theta(d_p\phi_1,\ldots,d_p\phi_k) = 0$. Since $d_p\phi_1,\ldots,d_p\phi_k$ are linear independent, $\partial_i\theta(v\phi_1,\ldots,v\phi_k) = 0$, for any $i = 1,\ldots,k$. On the other hand, for any $i = 1,\ldots,k$,

$$\partial_i\theta(x_1,\ldots,x_k) = \partial_i\omega(x_1,\ldots,x_k,\sum_{j=1}^{k} b_{k+1,j}x_j,\ldots,\sum_{j=1}^{k} b_{nj}x_j) +$$

$$+\sum_{l=k+1}^{n} b_{li}\,\partial_l\omega(x_1,\ldots,x_k,\sum_{j=1}^{k} b_{k+1,j}\, x_j,\ldots,\sum_{j=1}^{k} b_{nj}\, x_j),$$

$$\partial_i\omega(v\phi_1,\ldots,v\phi_n) =$$

$$= -\sum_{l=k+1}^{n} b_{li}\,\partial_l\omega(v\phi_1,\ldots,v\phi_n).$$

Then

$$\sum_{i=1}^{n} \partial_i\omega(v\phi_1,\ldots,v\phi_n)w_0(d_p\phi_i) =$$

$$= \sum_{i=1}^{k}(-\sum_{l=k+1}^{n} b_{li}\,\partial_l\omega(v\phi_1,\ldots,v\phi_n))w_0(d_p\phi_i) +$$

$$+\sum_{i=k+1}^{n}\partial_i\omega(v\phi_1,\ldots,v\phi_n)w_0(\sum_{j=1}^{k} b_{ij}d_p\phi_j) =$$

$$= -\sum_{i=1}^{k}(\sum_{l=k+1}^{n} b_{li}\,\partial_l\omega(v\phi_1,\ldots,v\phi_n))w_0(d_p\phi_i) +$$

$$+\sum_{j=1}^{k}\Big(\sum_{i=k+1}^{n} b_{ij}\partial_i\omega(v\phi_1,\dots,v\phi_n)\Big)w_0(d_p\phi_j) = 0.$$

Thus, there exists a vector $w \in T_v(T_pM)$, such that, for any $\alpha \in C$,

$$w(d_p\alpha) = w_0(d_p\alpha) = u(d\alpha) .$$

Then, for any $\alpha \in C$,

$$(d\iota)_v w(d\alpha) = w(d\alpha\circ\iota) = w(d_p\alpha) = u(d\alpha)$$

and

$$(d\iota)_v w(\alpha\circ\pi) = w(\alpha\circ\pi\circ\iota) = w(\alpha(p)) = 0,\ u(\alpha\circ\pi) = 0.$$

The set $\{d\alpha\colon \alpha \in C\}\cup\{\alpha\circ\pi\colon \alpha \in C\}$ generates a differential structure on TM. The tangent vectors $u, (d\iota)_v w \in T_v(TM)$ are equal on these sets, then $u = (d\iota)_v w$. Since the vector $u \in \ker(d\pi)_v$, $\ker(d\pi)_v \subseteq \mathrm{im}(d\iota)_v$. Now, in view of this inclusion and the first part of this proof, one can easily see that $\ker(d\pi)_v = \mathrm{im}(d\iota)_v$. On the other hand, the vector spaces $\mathrm{im}(d\iota)_v$ and $T_v(T_pM)$ are isomorphic. Then the vector spaces $\ker(d\pi)_v$ and $T_v(T_pM)$ are isomorphic too.

Now, we will prove

Corollary 1.2. Let (M,C) be a differential space such that $\dim T_qM$ is finite for any $q \in M$. Then

$$\dim T_v(TM) = \dim T_v(T_pM) + \dim(\mathrm{im}(d\pi)_v),$$

for any $p \in M$, $v \in T_pM$.

Proof. The tangent mapping $(d\pi)_v\colon T_v(TM) \longrightarrow T_pM$ is linear, then

$$\dim T_v(TM) = \dim(\ker(d\pi)_v) + \dim(\mathrm{im}(d\pi)_v).$$

Now, in view of Proposition 1.1, one can easily prove this corollary.

Let N be a differential subspace of M.

Proposition 1.3. If a tangent vector $v \in T_pM$ can be extended to a smooth vector field on N then

$$\dim(\mathrm{im}(d\pi)_v) \geq \dim T_pN.$$

Proof. Let a smooth vector field $X \in \mathfrak{X}(N)$ be an extension of the vector $v \in T_pM$ ($X(p) = v$). For any $q \in N$, the tangent space T_qN is a vector subspace of the tangent space T_qM, because N is a differential subspace of M. Let us consider the mapping $\pi|_{TN} : TN \longrightarrow N$.

The vector field X is a section of the tangent bundle TN, then

$$\pi|_{TN} \circ X = \mathrm{id}_N.$$

Let us notice that

$$(d(\pi|_{TN} \circ X))_p = (d(\pi|_{TN}))_v \circ (dX)_p = (d\pi)_v|_{T_v(TN)} \circ (dX)_p.$$

On the other hand, one can see that

$$(\mathrm{id}_N)_p = \mathrm{id}_{T_pN} .$$

Then it is easy to see that

$$(d\pi)_v|_{T_v(TN)} \circ (dX)_p = \mathrm{id}_{T_pN}.$$

Now, one can see that the mapping $(d\pi)_v|_{T_v(TN)}$ is "onto" the tangent space T_pN. It means that $T_pN \subseteq \mathrm{im}(d\pi)_v$.

Corollary 1.4. Let (M,C) be a differential space such that $\dim T_qM$ is finite for any $q \in M$. If a tangent vector $v \in T_pM$ can be extended to a smooth vector field on M then

$$\dim T_v(TM) = \dim T_v(T_pM) + \dim(T_pM).$$

Proof. This is an obvious consequence of Proposition 1.3 and Corollary 1.2.

2. Singular points of the tangent bundle of differential spaces of class D_o

Let (M,C) be a differential space of class $\mathcal{D}_o$. Then one can prove that the tangent differential space (TM,TC) is a differential space of class $\mathcal{D}_o$ [10].

Definition 2.1. A point $p \in M$ is a regular point of (M,C) of class $\mathcal{D}_o$ if there exists a neighbourhood U of this point in M such that, for any $q \in U$, $\dim T_qM = \dim T_pM$.

Remark 2.1. It is easy to prove that if the above condition is satisfied, then (U,C_U) is a differential space of class $\mathcal{D}_o$ of constant differential dimension ([7],[8]).

Definition 2.2. A point $p \in M$ is a singular point of (M,C) of class $\mathcal{D}_o$ if this point is not regular point of (M,C).

Now, we will prove.

Proposition 2.4. Let (M,C) be a differential space of class $\mathcal{D}_o$. The following conditions are equivalent:

(i) the point $p \in M$ is a regular point of (M,C),

(ii) there exists a vector $v \in T_pM$, which can be extended to a smooth vector field on M, such that the vector v is a regular point of (TM,TC),

(iii) every vector $v \in T_pM$ is a regular point of (TM,TC).

Proof. (ii) $\Rightarrow$ (i) Let a vector $v \in T_pM$ satisfies the condition (ii). Then there exists an open neighborhood V of the vector v in TM such that, for any vector $w \in V$,

$$\dim T_v(TM) = \dim T_w(TM).$$

Let X be a smooth vector field on M such that $X(p) = v$. The set $X^{-1}(V)$ is open in TM, because the vector field $X: M \longrightarrow TM$ is a smooth mapping. The point $p \in X^{-1}(V)$, because the vector $v \in V$. Thus the set $X^{-1}(V)$ is a neighbourhood of point p in M. Now, let $q \in X^{-1}(V)$, then

$$\dim T_v(TM) = \dim T_{X(q)}(TM).$$

On the other hand, for any $s \in M$ and $u \in T_sM$, a vector space T_sM is isomorphic to $T_u(T_sM)$. In view of Corollary 1.2, we have

$$\dim T_v(TM) = 2 \cdot \dim T_pM,$$

and analogously,

$$\dim T_{X(q)}(TM) = 2 \cdot \dim T_qM.$$

Then, for any $q \in X^{-1}(V)$,

$$\dim T_pM = \dim T_qM.$$

Therefore the point p is a regular point of M.

(i) ⇒ (iii) The point p is a regular point of M, then, in view of Remark 2.1, there exists a neighbourhood U of the point p in M such that the differential space (U, C_U) is a differential space of class $\mathcal{D}_o$ of constant differential dimension. Then every vector field $v \in \pi^{-1}(U)$ can be extended to a smooth vector field on M ([7],[8]). The set $\pi^{-1}(U)$ is open and $T_pM \subseteq \pi^{-1}(U)$. In view of Corollary 1.2 we have

$$\dim T_w(TM) = 2 \cdot \dim T_pM,$$

for every $w \in \pi^{-1}(U)$. Therefore every vector $w \in T_pM$ is a regular point of TM.

(iii) ⇒ (ii) We should prove that there exists a vector $v \in T_pM$, which can be extended to a smooth vector field. This vector is the zero vector $0 \in T_pM$.

Now one can easily prove.

Corollary 2.5. Let (M,C) be a differential space of class $\mathcal{D}_o$. The following conditions are equivalent:

(i) the point $p \in M$ is a singular point of (M,C),

(ii) every vector $v \in T_pM$, which can be extended to a smooth vector field on M, is a singular point of (TM,TC),

(iii) there exists a vector $v \in T_pM$, which is a singular point of (TM,TC).

If a point $p \in M$ is a singular point of M, then it is possible that there exists a vector $v \in T_pM$, which is a regular point of TM. Let us cosider the following

Example 2.1. Let $M = \left\{ \frac{1}{n} : n \in \mathbb{N}\setminus\{0\} \right\} \cup \{0\}$ and $C = (C^\infty(\mathbb{R}))_M$. Then

$$\dim T_pM = \begin{cases} 0 \text{ for any } p \neq 0, \\ 1 \text{ for any } p = 0, \end{cases}$$

thus $0 \in M$ is a singular point of M. Let $v \in T_pM$, then

$$\dim T_v(TM) = \begin{cases} 1 \text{ for any } v \neq 0, \\ 2 \text{ for any } v = 0. \end{cases}$$

It is easy to see that every $v \in T_pM$, which is not equal to the zero vector $0 \in T_pM$, is a regular point of TM.

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