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ON DISTRIBUTIONS IN DIFFERENTIAL SPACES

In this paper we present an introduction to the theory of distributions in differential spaces in the sense of Sikorski. In Section 1 we collect the basic definitions for the theory of distributions in differential spaces, analogous to the case of differentiable manifolds. Next, we prove some simple properties of these distributions. We give also several examples which show differences between distributions in differentiable manifolds and distributions in differential spaces.

Sections 2 and 3 are devoted to constructing of some special type of distributions in differential spaces. In particular, the construction of the Hamiltonian distribution is presented.

1. Let (M, C) be a differential space in the sense of Sikorski [5].

Definition 1.1 A function D which assigns to each point $p \in M$ a linear subspace D_p of T_pM is called the *distribution* on a differential space (M, C).

The dimension of D_p is called the *dimension of the distribution* D at the point p. Denote by $\mathfrak{X}(D)$ the set of all smooth vector fields X on (M, C) such that, for any $p \in M$, $X(p) \in D_p$. One can easily show that $\mathfrak{X}(D)$ is a C-submodule of the C-module $\mathfrak{X}(M)$.

A vector field $X \in \mathfrak{X}(D)$ is said to be a D-*vector field* on (M, C), and a smooth vector field X defined on a subset U of M is said to be a D-*vector field* on U if $X(p) \in D_p$, for any

$p \in U$. D-vector fields on U are also called local D-vector fields.

Definition 1.2 A distribution D on a differential space (M, C) is said to be *regular* if, for any point $p \in M$, there exists a neighborhood U of p and D-vector fields $X_1,\dots,X_n$ on U such that $D_q = \mathrm{Lin}\,(X_1(q),\dots,X_n(q))$, for any $q \in U$ and some $n \in \mathbb{N}$.

A distribution D on (M, C) is said to be *completely regular* if D is regular and there exists $n \in \mathbb{N}$ such that $\dim D_p = n$, for any $p \in M$.

The number n is called the *dimension of the completely regular distribution* D on (M, C).

Example 1.1 Let (M, C) be a differential space. A function D which assigns to each point $p \in M$ a linear space $D_p = T_pM$ is a distribution on (M, C). The distribution D on (M, C), defined above, is called the *maximal distribution*. Of course, in this case $\mathfrak{X}(D) = \mathfrak{X}(M)$.

Example 1.2 Let (M, C) be a differential space such that

$$M = \{(x,y) \in \mathbb{R}^2 : xy = 0\} \quad \text{and } C = C^\infty(\mathbb{R}^2)_M.$$

Now, observe that $\dim T_{(x,y)} = 1$ when $(x,y) \in M$ and $x \neq 0$ or $y \neq 0$, and $\dim T_{(0,0)}M = 2$. Moreover, one can prove that any smooth vector field X on (M, C) is such that $X(0,0) = 0$. Hence the maximal distribution D on (M, C) is not regular.

Example 1.3 Let (M, C) be as in Example 1.2 and let D be a distribution on (M, C) defined in the following way:

$$D_{(x,y)} = \begin{cases} T_{(x,y)}M & \text{when } x \neq 0 \text{ and } y = 0 \\ \{0\} & \text{when } x = 0 \text{ and } y \in \mathbb{R}. \end{cases}$$

Now, let us observe that $\mathfrak{X}(D) = \{\pi_1 \cdot \alpha \frac{\partial}{\partial x} : \alpha \in C\}$, where $\pi_1(x,y) = x$. Hence, D is a regular distribution on (M, C) but it is not a completely regular distribution.

Example 1.4 Let (M, C) and (N, B) be differential spaces. Consider the Cartesian product $(M\times N, C\times B)$ of these differential spaces, and let D be a distribution on $(M\times N, C\times B)$ defined by the formula

$$D_{(p,q)} = i_{q*}(T_pM),$$

for any $(p, q) \in M\times N$, where $i_q\colon M \longrightarrow M\times N$, $i_q(p) = (p, q)$, is the natural embedding of (M, C) into $(M\times N, C\times B)$.

In this case, as it is easy to observe,

$$\mathfrak{X}(D) = \{X \in \mathfrak{X}(M\times N)\colon X(\beta\circ pr_N) = 0,\ \beta \in B\},$$

where $pr_N\colon M\times N \longrightarrow N$ is the natural projection.

Hence, we see that the distribution D on $(M\times N, C\times B)$, defined above, is regular iff the maximal distribution on (M, C) is regular. Moreover, D is completely regular iff (M, C) is a differential space of constant differential dimension.

One can prove

Proposition 1.1 A distribution D on a differential space (M, C) is completely regular iff $\mathfrak{X}(D)$ is a differential module.

Similarly as in the theory of differentiable manifolds, we may equivalently define a distribution D on a differential space (M, C) as a subset D of the tangent bundle TM of (M, C) such that

$$D_p = D \cap T_pM$$

is a linear subspace of T_pM for any $p \in M$.

In the category of differential spaces the following proposition holds.

Proposition 1.2 Any distribution D on a differential space (M, C) is a vector subbundle of TM.

Definition 1.3 A distribution D on a differential space (M, C) is said to be *involutive* if $[X, Y] \in \mathfrak{X}(D)$ for any $X, Y \in \mathfrak{X}(D)$.

Hence we have

Corollary 1.3 The maximal distribution on any differential space is involutive.

One can show that all distributions considered in the above examples are involutive distributions.

Lemma 1.4 If D is an involutive distribution on a differential space (M, C) then, for an arbitrary smooth 1-form ω on (M, C) such that $\omega(X) = 0$, for $X \in \mathfrak{X}(D)$, we have

$$d\omega(X, Y) = 0$$

for any $X, Y \in \mathfrak{X}(D)$.

Proof. By assumption, for any $X, Y \in \mathfrak{X}(D)$, we have

$$d\omega(X, Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\} = 0.$$

One can prove

Lemma 1.5 Let D be a regular distribution on a differential space (M, C) of D_o- class and of constant differential dimension [9]. Then D is an involutive distribution on (M, C) if, for an arbitrary smooth 1-form ω on (M, C) such that $D \subset \ker\omega$,

$$d\omega(X, Y) = 0$$

for any $X, Y \in \mathfrak{X}(D)$.

From Lemmas 1.4 and 1.5 it follows

Corollary 1.6 Let D be a regular distribution on (M, C) of D_o- class and of constant differential dimension. Then the following conditions are equivalent:

(1) D is an involutive distribution on (M, C),

(2) For an arbitrary smooth 1-form ω on (M, C), if $D \subset \ker\omega$ then $d\omega(X, Y) = 0$, for arbitrary local D-vector fields X, Y.

Definition 1.4 A distribution D on a differential space (M, C) is said to be *integrable* if, for each point $p \in M$, there exists a differential subspace (N, B) of (M, C) such that $p \in N$ and $D_q = (i_N)_*(T_qN)$ for any $q \in N$, where i_N is the natural embedding.

A differential subspace (N, B), satisfying the conditions of Definition 1.4 is called the *integral differential subspace* of D at a point p (or the *integral* of D, for short).

Let us observe that the distribution D on (M×N, C×B), described in Example 1.4, is an integrable distribution. Indeed, for any $(p, q) \in M\times N$, $i_q(M)$ with the differential structure induced from (M×N, C×B) is an integral differential subspace of D, because $(p, q) \in i_q(M)$ and $D_{(\bar{p},q)} = T_{(\bar{p},q)}(i_q(M))$, for any $(\bar{p}, q) \in i_q(M)$.

Evidently, every maximal distribution on a differential space (M, C) is integrable.

From the above considerations it follows

Corollary 1.7 Regularity and the property of being an involutive distribution on a differential space are not necessary conditions for a distribution to be integrable.

We shall prove

Proposition 1.8 Let D be an integrable distribution on a differential space (M, C). Then D is involutive.

Proof. By Definition 1.4, for each point $p \in M$, there exists a differential subspace (N, B) of (M, C) such that $p \in N$ and $D_q = (i_N)_{*q}(T_qN)$, for $q \in N$. Let X, Y be D-vector fields on (M, C). It is obvious that X, Y are tangent to N, i.e. $X(q), Y(q) \in (i_N)_{*q}(T_qN)$, for any $q \in N$.

One can prove that a vector field $Z \in \mathfrak{X}(M)$ is tangent to N if and only if Z satisfies the following condition:

$$(*) \qquad \forall\, f \in C \quad (f_{|N} = 0 \Longrightarrow Zf_{|N} = 0).$$

It is evident that the commutator [X, Y] satisfies the condition (*), where X, Y are D-vector fields on (M, C). Thus [X, Y] is tangent to N. Hence $[X, Y](p) \in (i_N)_{*p}(T_pN) = D_p$, for an arbitrary point $p \in M$. Therefore D is involutive.

<u>2.</u> Let (M, C) be differential space and let $\underset{\circ}{C}$ be a differential substructure of the differential structure C on M.

Definition 2.1 A vector $v \in T_pM$ is said to be a $\underset{\circ}{C}$-*vector* on (M, C) at a point $p \in M$ if $v(\alpha) = 0$, for any $\alpha \in \underset{\circ}{C}$.

Of course, if $\underset{\circ}{C} = \mathbb{R}$ then each vector $v \in T_pM$ is a $\underset{\circ}{C}$-vector on (M, C). In turn, if $\underset{\circ}{C} = C$ then only the zero vector is a $\underset{\circ}{C}$-vector on (M, C).

Let us denote by $\underset{\circ}{C}_p$ the set of all $\underset{\circ}{C}$-vectors on (M, C) at the point $p \in M$. One can easily prove

Lemma 2.2 For every point $p \in M$, $\underset{\circ}{C}_p$ is a linear subspace of T_pM.

Hence we get

Corollary 2.3 A function which assigns to each point $p \in M$ a linear subspace $\underset{\circ}{C}_p$ of T_pM is a distribution on (M, C), called a *distribution of* $\underset{\circ}{C}$*-vectors* on (M, C).

Analogously as in the Section 1, by $\mathfrak{X}(\underset{\circ}{C})$ we denote the set of all vector fields $X \in \mathfrak{X}(M)$ such that $X(p) \in \underset{\circ}{C}_p$ for any $p \in M$.

One can prove

Lemma 2.4 Let X be a smooth vector field on (M, C). Then $X \in \mathfrak{X}(\underset{\circ}{C})$ if and only if $X(\alpha) = 0$, for any $\alpha \in \underset{\circ}{C}$.

Lemma 2.5 The set $\mathfrak{X}(\underset{\circ}{C})$ is a C-submodule of the C-module $\mathfrak{X}(M)$ as well as a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$.

Hence we get

Corollary 2.6 For an arbitrary differential substructure $\underset{\circ}{C}$ of a differential structure C on M, the distribution of $\underset{\circ}{C}$-vectors on (M, C) is involutive.

In particular we have

Corollary 2.7 If (M, C) is a differentiable manifold then each distribution of $\underset{\circ}{C}$-vectors is integrable in the category of differential spaces.

Now, let (M, C) be a differential space and let $\underset{\circ}{\mathfrak{X}}(M)$ be a C-submodule of $\mathfrak{X}(M)$. Let us put

$$\overset{\circ}{T}_pM = \{v \in T_pM\colon\ v = X(p) \text{ for } X \in \underset{\circ}{\mathfrak{X}}(M)\}.$$

It is easy to prove

Lemma 2.8 The set $\overset{\circ}{T}_pM$ is a linear subspace of the tangent space T_pM to (M, C) at $p \in M$, for any $p \in M$.

Consequently, we get

Corollary 2.9 The function $\tilde{\underset{\circ}{\mathfrak{X}}}$ which assigns to each point $p \in M$ a linear subspace $\overset{\circ}{T}_pM$ of T_pM is a distribution on (M, C).

Evidently, in this case we have

$$\tilde{\underset{\circ}{\mathfrak{X}}}_p = \overset{\circ}{T}_pM \quad \text{and} \quad \mathfrak{X}(\tilde{\underset{\circ}{\mathfrak{X}}}) \supset \underset{\circ}{\mathfrak{X}}(M),$$

where $\mathfrak{X}(\tilde{\underset{\circ}{\mathfrak{X}}}) = \{X \in \mathfrak{X}(M)\colon X(p) \in \overset{\circ}{T}_pM \text{ for } p \in M\}$. Of course, $\mathfrak{X}(\underset{\circ}{\mathfrak{X}})$ is a C-submodule of the C-module $\mathfrak{X}(M)$.

Lemma 2.10 A distribution $\tilde{\underset{\circ}{\mathfrak{X}}}$ on a differential space (M, C), determined by a C-submodule $\underset{\circ}{\mathfrak{X}}(M)$ of $\mathfrak{X}(M)$, is regular iff $\dim T_pM < \infty$, for any $p \in M$. Moreover, $\tilde{\underset{\circ}{\mathfrak{X}}}$ is an involutive distribution iff $\mathfrak{X}(\tilde{\underset{\circ}{\mathfrak{X}}})$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Next, let us put

$$C_{\underset{\circ}{\mathfrak{X}}} := \{\alpha \in C\colon X(\alpha) = 0 \quad \text{for } X \in \underset{\circ}{\mathfrak{X}}(M)\}.$$

Lemma 2.11 The set $C_{\underset{\circ}{\mathfrak{X}}}$, defined above, is a differential substructure of the differential structure C on M.

Proof. Let $\alpha_1,\ldots,\alpha_n \in C_{\underset{\circ}{\mathfrak{X}}}$, $\omega \in C^\infty(\mathbb{R}^n)$ and $X \in \underset{\circ}{\mathfrak{X}}(M)$. Then we have

$$X(\omega\circ(\alpha_1,\ldots,\alpha_n)) = \sum_{i=1}^{n} \omega'_{|i}(\alpha_1,\ldots,\alpha_n)X(\alpha_i) = 0.$$

Hence $\omega\circ(\alpha_1,\ldots,\alpha_n) \in C_{\underset{\circ}{\mathfrak{X}}}$. Next, let $\alpha \in C$ be such that, for any $p \in M$, there is an open neighborhood U of p and $\beta \in C_{\underset{\circ}{\mathfrak{X}}}$ such that $\beta_{|U} = \alpha_{|U}$. Then, for any $X \in \underset{\circ}{\mathfrak{X}}(M)$ we have

$$X_{|U}(\alpha_{|U}) = X(\alpha)_{|U} = X_{|U}(\beta_{|U}) = X(\beta)_{|U} = 0.$$

Hence $\alpha \in C_{\underset{\circ}{\mathfrak{X}}}$.

Of course, the differential substructure $C_{\underset{\circ}{\mathfrak{X}}}$ of C determines also a distribution of $C_{\underset{\circ}{\mathfrak{X}}}$-vectors on (M, C). Obviously, by Corollary 2.6, the distribution of $C_{\underset{\circ}{\mathfrak{X}}}$-vectors on (M, C) is involutive.

3. Let (M, C) be a differential space.

Definition 3.1 A skew-symmetric 2-linear mapping $\{\cdot,\cdot\} : C\times C \longrightarrow C$ satisfying the conditions

(i) $\{\alpha\cdot\beta, \gamma\} = \alpha\cdot\{\beta, \gamma\} + \beta\cdot\{\alpha, \gamma\}$,

(ii) $\{\{\alpha, \beta\}, \gamma\} + \{\{\gamma, \alpha\}, \beta\} + \{\{\beta, \gamma\}, \alpha\} = 0$,

for any $\alpha, \beta, \gamma \in C$ is called the *Poisson structure* on a differential space (M, C).

The system $((M, C), \{\cdot,\cdot\})$ will be called the *Poisson differential space*. From Definition 3.1 it follows immediately that $(C, \{\cdot,\cdot\})$ is a Lie algebra.

Next, from (i) it follows that, for every $\alpha \in C$, the mapping $X_\alpha := \{\cdot, \alpha\} : C \longrightarrow C$ is a smooth vector field on (M, C), called the *Hamiltonian vector field* and α is called the *hamiltonian* of the vector field X_α on the Poisson differential space $((M, C), \{\cdot,\cdot\})$.

Let $((M, C), \{\cdot,\cdot\})$ be a Poisson differential space. A function $\alpha \in C$ is said to be a *Casimir function* if $\{\alpha, \beta\} = 0$, for any $\beta \in C$.

Denote by C_C the set of all Casimir functions on the Poisson differential space $((M, C), \{\cdot,\cdot\})$. Analogously as in Lemma 2.10 we prove

Lemma 3.1 The set C_C of all Casimir functions on a Poisson differential space $((M, C), \{\cdot,\cdot\})$ is a differential substructure of the differential structure C on M.

It is easy to verify that C_C is an ideal of the Lie algebra $(C, \{\cdot,\cdot\})$. The set of all Hamiltonian vector fields on $((M, C), \{\cdot,\cdot\})$ is usually denoted by $\mathcal{H}(M)$.

One can prove

Lemma 3.2 The set $\mathcal{H}(M)$ of all Hamiltonian vector fields on a Poisson differential space $((M, C), \{\cdot,\cdot\})$ is a module over C_C. Moreover, $\mathcal{H}(M)$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$.

It is easy to verify that

$$[X_\alpha, X_\beta] = X_{\{\alpha,\beta\}}$$

for any $\alpha, \beta \in C$.

From Definition 3.1 and the definition of Casimir function it follows

Corollary 3.3 Every Casimir function α is a first integral of any Hamiltonian vector field X_β on a Poisson differential space $((M, C), \{\cdot,\cdot\})$.

Observe that, by Lemma 3.1, the differential substructure C_C on $((M, C), \{\cdot,\cdot\})$ determines on (M, C) a distribution H of all C_C-vectors.

Similarly as in Section 2 by $\mathfrak{X}(H)$ we denote the set of all C_C-vector fields on $((M, C), \{\cdot,\cdot\})$. Of course, $\mathfrak{X}(H)$ is a

C-submodule of $\mathfrak{X}(M)$ and a Lie subalgebra of $\mathfrak{X}(M)$.

Definition 3.2 The vector fields of $\mathfrak{X}(H)$ are said to be a *generalized Hamiltonian vector fields* on $((M, C), \{\cdot,\cdot\})$.

The distribution on $((M, C), \{\cdot,\cdot\})$ determined by $\mathfrak{X}(H)$ is called the *Hamiltonian distribution.*

From Corollary 2.5 we get

Corollary 3.4 Every Hamiltonian distribution on a Poisson differential space is involutive.

Consequently, we get

Corollary 3.5 Every Hamiltonian distribution on a Poisson differentiable manifold is an integrable distribution.

Once again we come back for a moment to the Lie algebra $\mathfrak{X}(M)$. As is known, the mapping

$$L_X := [X,\cdot]: \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

for any $X \in \mathfrak{X}(M)$, is an endomorphism of the linear space $\mathfrak{X}(M)$, and the set $gl(\mathfrak{X}(M))$ of all such endomorphisms, with the Lie bracket given by

$$[L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$$

for any $X, Y \in \mathfrak{X}(M)$, is a Lie algebra.

One can easily prove

Proposition 3.6 If $\underset{\circ}{C}$ is a differential substructure of the differential structure C on M, then $\mathfrak{X}(\underset{\circ}{C})$ is an L_X-invariant linear subspace of the linear space $\mathfrak{X}(M)$, for any $X \in \mathfrak{X}(\underset{\circ}{C})$.

Hence we get

Corollary 3.7 Let D be a distribution on a differential space (M, C). Then the following assertions are equivalent:

(i) D is involutive,

(ii) $\mathfrak{X}(D)$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$,

(iii) $\mathfrak{X}(D)$ is an L_X-invariant linear subspace of the, linear space $\mathfrak{X}(M)$ over $\mathbb{R}$, for any $X \in \mathfrak{X}(D)$.

Finally, let us consider a Poisson differential space $((M, C), \{\cdot,\cdot\})$. In this case we have at least two Lie algebras: $(C, \{\cdot,\cdot\})$ and $(\mathfrak{X}(M), [\cdot,\cdot])$.

The mappings

$$\rho\colon C \ni \alpha \longmapsto X_\alpha \in \mathcal{H}(M)$$

as well as

$$\sigma\colon \mathfrak{X}(M) \ni X \longmapsto L_X \in gl(\mathfrak{X}(M)),$$

called the *adjoin representations*, are homomorphisms of the Lie algebras $(C, \{\cdot,\cdot\})$ and $(\mathfrak{X}(M), [\cdot,\cdot])$, respectively. Of course, the composition $\sigma\circ\rho$ is a homomorphism of the Lie algebra $(C, \{\cdot,\cdot\})$ into the Lie algebra $(\mathfrak{X}(M), [\cdot,\cdot])$.

Evidently, $\ker(\sigma\circ\rho) = C_C$ is an ideal of C and $\mathrm{Im}(\sigma\circ\rho) = \mathcal{H}^*(M) := gl(\mathcal{H}(M))$ is a Lie subalgebra of the Lie algebra $gl(\mathfrak{X}(M))$.

Evidently, by definition,

$$\mathfrak{X}^*(M) = \{L_{X_\alpha} : \alpha \in C \text{ and } X_\alpha \in \mathcal{H}(M)\}.$$

Moreover, it is easy to prove the relations $L_{X_\alpha} + L_{X_\beta} = L_{X_{\alpha+\beta}}$ and $[L_{X_\alpha}, L_{X_\beta}] = L_{X_{\{\alpha,\beta\}}}$, for any $\alpha, \beta \in C$.

It seems to be justified the following

Definition 3.3 A smooth 1-form $L_{X_\alpha} \in \mathcal{H}^*(M)$ with values in $\mathfrak{X}(M)$ is said to be a *Hamiltonian 1-form* on the Poisson differential space $((M, C), \{\cdot,\cdot\})$.

It is easy to prove

Proposition 3.8 The linear space $\mathcal{H}(M)$ is L_{X_α}-invariant linear subspace of the linear space $\mathfrak{X}(M)$ over $\mathbb{R}$, for any $X_\alpha \in \mathcal{H}(M)$.

And finally,

Proposition 3.9 The Lie algebras: C/C_C, $\mathcal{H}(M)$ and $\mathcal{H}^*(M)$ are isomorphic.

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