DEMONSTRATIO MATHEMATICA
Vol. XXIV No 3-4 1991

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INDUCING AND COINDUCING IN GENERAL DIFFERENTIAL SPACES

0. Introduction

The concept of an analytical premanifold (see [4]) is a slight modification of the R. Sikorski's concept of a differential space. Therefore in the present paper analytical premanifolds and complex premanifolds (see [5]) as well will be called general differential spaces. An analytical premanifold will be also called an $\mathbb{R}$-differential space ($\mathbb{R}$-d.s.). For any indexed set of mappings, a $\mathbb{K}$-d.s. ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) induced by this indexed set is defined and characterized by a universal property. Similarly, a $\mathbb{K}$-d.s. coinduced by any indexed set of mappings is defined and universally characterized. It is proved that the topology of the induced (coinduced) $\mathbb{K}$-d.s. is the induced (coinduced) topology by the indexed set of the mappings. In particular, the topology of the quotient $\mathbb{K}$-d.s. $M/\equiv$, where M is a $\mathbb{K}$-d.s. and $\equiv$ is any equivalence relation on the set of all points of M, is equal to the quotient topology $(\mathrm{top}M)/\equiv$, where $\mathrm{top}M$ is the topology of M. It leads to quite a different situation than in the theory of R. Sikorski's differential spaces (see [1]), where the topology of the quotient differential space may be essentially weaker than the quotient topology even in the case of a division of a differentiable manifold by a foliation.

For any function f (treated here as a set of ordered pairs) its domain is denoted by D_f. The f-counterimage of any set T will be denoted by $f^{-1}T$. For any set $S \subset D_f$ the f-image of S is denoted by fS. For any functions f and g we have the well defined function $g\circ f$ in such a way that $D_{g\circ f} = f^{-1}D_g$ and $(g\circ f)(x) = g(f(x))$ for $x \in D_{g\circ f}$. For any functions g, $f_1,\dots,f_n$ we have the function f with the domain D_f of the shape $D_{f_1}\cap\dots\cap D_{f_n}$ and $f(x) = (f_1(x),\dots,f_n(x))$ for any $x \in D_f$, and the function $g\circ f$ being denoted by $g(f_1,\dots,f_n)$.

We recall the concept of a $\mathbb{K}$-d.s. ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). For any set M of functions with values in $\mathbb{K}$ the set $\bigcup_{a\in M} D_a$ is denoted by $\underline{M}$ (see [4] and [5]). The smallest topology on $\underline{M}$ such that all the sets $a^{-1}A$, where $a \in \underline{M}$ and A is open in $\mathbb{K}$, are open is denoted by topM. For any $S \subset \underline{M}$ the set of all functions b with values in $\mathbb{K}$ and satisfying the condition: for any $p \in D_b$ there exist $U \in$ topM and $a \in M$ for which $p \in U\cap S \subset D_b$, $U \subset D_a$ and $b|U\cap S = a|U\cap S$, is denoted by M_S.

For a topology X on a set $\underline{X}$ and for any $S \subset \underline{X}$, the topology induced from X to S we denote by $X|S$. We have then $\mathrm{top}M_S = (\mathrm{top}M)|S$. The set of all functions $g(a_o,\dots,a_n)$, where $a_o,\dots,a_n \in M$ and g is any function with values in $\mathbb{K}$ and analytic on the set D_g open in $\mathbb{K}^{n+1}$, n is any in $\mathbb{N}$, is denoted by anM. A set M of functions with values in $\mathbb{K}$ satisfying the condition $M = M_{\underline{M}} = \mathrm{an}M$ is said to be a $\mathbb{K}$-d.s. For any set G of functions with values in $\mathbb{K}$, the set $(\mathrm{an}G)_{\underline{G}}$ being the smallest $\mathbb{K}$-d.s. containing G is called the $\mathbb{K}$-d.s. generated by G.

Let M and N be $\mathbb{K}$-d.s. and f be a function with $D_f = \underline{M}$ and $f\underline{M} \subset \underline{N}$, i. e. $f\colon \underline{M} \longrightarrow \underline{N}$. We say that f maps smoothly M into N what we write in the form

$$f\colon M \longrightarrow N \tag{0.1}$$

iff for any $b \in N$ we have $b\circ f \in M$. If N is generated by G then we have (0.1) iff for any $b \in G$ the function $b\circ f$ belongs to M.

For any set S the set of all functions with values in $\mathbb{K}$ and domains contained in S will be denoted by discr(S, $\mathbb{K}$) and called the discreet $\mathbb{K}$-d.s. on S.

1. Inducing of $\mathbb{K}$-d.s.

Let $f = (f_i;\ i \in I)$ and $N = (N_i;\ i \in I)$, where f_i are functions, N_i are $\mathbb{K}$-d.s. and $f_i D_{f_i} \subset \underline{N_i}$ for $i \in I$. For any function b with values in $\mathbb{K}$ and the domain D_b contained in $\bigcup_j \underline{N_j}$, assume $f_i^*(b) = b \circ f_i$. We have then $f_i^* N_i = \{b \circ f_i;\ b \in N_i\}$. This yields

$$\underline{\bigcup_i f_i^* N_i} = \bigcup_i D_{f_i}. \tag{1.1}$$

The $\mathbb{K}$-d.s. f^*N of the form $(\mathrm{an} \bigcup_i f_i^* N_i)_{\bigcup_j D_{f_j}}$ will be called the induced $\mathbb{K}$-d.s. from the indexed set N of K-d.s. by the indexed set f of functions.

The following theorem gives a universal characterization of the induced $\mathbb{K}$-d.s. f^*N.

Theorem 1.0 f^*N is the exactly one $\mathbb{K}$-d.s. fulfilling the condition

(i) for any $\mathbb{K}$-d.s. L and any h we have $h: L \longrightarrow M$ iff

$h: \underline{L} \longrightarrow \bigcup_j D_{f_j}$, $h^{-1} D_{f_i} \in \mathrm{top} L$ and

$f_i \circ h: L_{h^{-1} D_{f_i}} \longrightarrow N_i$ for $i \in I$.

For this M the equality $\underline{M} = \bigcup_j D_{f_j}$ holds.

By a direct verification we get the following lemmas for any $\mathbb{K}$-d.s. L and M.

Lemma 1.1 If $a \in M$ and $D_a \subset S \subset \underline{M}$ then $a \in M_S$.

Lemma 1.2 If $h: L \longrightarrow M$ and $S \subset \underline{M}$ then

$$h|h^{-1}S: L_{h^{-1}S} \longrightarrow M_S.$$

Lemma 1.3 If $S \in \mathrm{top} L$ then $L_S \subset L$.

Proof. (of Th. 1.0.) First we proove that (i) yields

$$\underline{M} = \bigcup_i D_{f_i}. \tag{1.2}$$

Let $j \in I$. We set $L = \mathrm{discr}(D_{f_j}, \mathbb{K})$ and $h = \mathrm{id}_{\underline{L}}$. We have then

$h\colon \underline{L} \longrightarrow \bigcup_i D_{f_i}$ and, for any $i \in I$, $h^{-1}D_{f_i} = D_{f_j} \cap D_{f_j}$.

Thus $$L_{h^{-1}D_{f_i}} = \mathrm{discr}(D_{f_j} \cap D_{f_i}, \mathbb{K})$$

and $$f_i \circ h = f_i | D_{f_j} \cap D_{f_i} \colon L_{h^{-1}D_{f_i}} \longrightarrow N_i .$$

Moreover, $h^{-1}D_{f_i} \in \mathrm{topdiscr}(D_{f_j}, \mathbb{K}) = \mathrm{top}L$. According to (i) we have : $L \longrightarrow M$. In other words, $h\colon \mathrm{discr}(D_{f_j}, \mathbb{K}) \longrightarrow M$. This yields $D_{f_j} \subset \underline{M}$ for $j \in I$. Now, we take $h = \mathrm{id}_{\underline{M}}$. We have then $h\colon M \longrightarrow M$. From (i) it follows that $h\colon \underline{M} \longrightarrow \bigcup_i D_{f_i}$. This yields $\underline{M} \subset \bigcup_i D_{f_i}$, and we get (1.2).

Let us assume that a $\mathbb{K}$-d.s. M_1 also fulfils (i). Hence it follows that $\underline{M_1} = \bigcup_i D_{f_i}$. Taking $L = M_1$ and $h = \mathrm{id}_{\underline{L}}$, according to (i) fulfilled by M_1, we get $h\colon M_1 \longrightarrow M$. Hence $M \subset M_1$. Similarly, $M_1 \longrightarrow M$. It is to be proved that the $\mathbb{K}$-d.s. M of the form f^*N fulfils (i). To do this let us take any $\mathbb{K}$-d.s. L and $h\colon L \longrightarrow M$. By definition of f^*N from 1.1 it follows that $f_i\colon M_{D_{f_i}} \longrightarrow N_i$. By 1.2 we get

$$h | h^{-1}D_{f_i} \colon L_{h^{-1}D_{f_i}} \longrightarrow M_{D_{f_i}} .$$

This yields

$$h\colon \underline{L} \longrightarrow \bigcup_i D_{f_i}, \tag{1.3}$$

$$h^{-1}D_{f_i} \in \mathrm{top}L \text{ and } f_i \circ h\colon L_{h^{-1}D_{f_i}} \longrightarrow N_i \tag{1.4}$$

for $i \in I$.

Now, assume (1.3) and (1.4). Let us take $b \in \bigcup_i f_i^*N_i$. Then $b = a \circ f_i$, where $a \in N_i$ and $i \in I$. According to (1.3) we get $b \circ h = a \circ f_i \circ h \in L$. Thus, $h\colon L \longrightarrow M$. Q.E.D.

The Cartesian product $\mathbb{P}_i N_i$ of an arbitrary indexed set $(N_i; i \in I)$ of $\mathbb{K}$-d.s. is said to be the induced $\mathbb{K}$-d.s. from this indexed set by $(f_i;\ i \in I)$, where $f_i(p) = p(i)$ for $i \in \mathbb{P}_j N_j$. According to 1.0, the product $\mathbb{P}_i N_i$ is exactly one

$\mathbb{K}$-d.s. such that for any $\mathbb{K}$-d.s. L and any function h we have $h: L \longrightarrow M$ iff $h: \underline{L} \longrightarrow \mathbb{P}_j N_j$, $h^{-1}\mathbb{P}_j N_j \in \mathrm{top}L$ and $h_i: L_{h^{-1}\mathbb{P}_j N_j} \longrightarrow N_i$, where $h_i(q) = h(q)(i)$ for $q \in \underline{L}$ and $i \in I$.

Theorem 1.4 The topology $\mathrm{top} f^* N$ is the smallest topology X on the set $\bigcup_i D_{f_i}$ such that

(ii) $D_f \in X$ and $f_i: X|D_f \longrightarrow \mathrm{top}N_i$ for $i \in I$.

Proof. Let us set $M = f^*N$. By (i) we get $D_{f_i} \in \mathrm{top}M$ and $f: M_{D_{f_i}} \longrightarrow N_i$. Then $f_i: \mathrm{top}M|D_{f_i} \longrightarrow \mathrm{top}N_i$ for $i \in I$. Now, assume that a topology X on the set $\bigcup_i D_{f_i}$ fulfills (ii). Take any $a \in f^* N_i$ and any set A open in $\mathbb{K}$. Then $a = b \circ f_i$, where $b \in N_i$. Thus $a^{-1}A \in X|D_{f_i} \subset X$. This yields $a^{-1} \in X$. Hence it follows that for any $a \in \bigcup_i f_i^* N_i$ and any A open in $\mathbb{K}$ we have $a^{-1}A \in X$. Therefore $\mathrm{top}M \subset X$. Q.E.D.

The smallest topology X on the set $\bigcup_i D_{f_i}$ satisfying the condition

(iii) $D_{f_i} \in X$ and $f_i: X|D_{f_i} \longrightarrow Y_i$ for $i \in I$

is the induced topology from the indexed set $(Y_i;\ i \in I)$ of topologies by f of the form $(f_i;\ i \in I)$. According to 1.4, the topology of the induced $\mathbb{K}$-d.s. from $(N_i;\ i \in I)$ by f is the induced topology from $(\mathrm{top}N_i;\ i \in I)$ by f. In particular, $\mathrm{top}\mathbb{P}_i N_i = \mathbb{P}_i \mathrm{top}N_i$.

2. Coinducing of $\mathbb{K}$-d.s.

Let us consider the indexed set M of the form $(M_i;\ i \in I)$, where the $\mathbb{K}$-d.s. M_i satisfies $\underline{M_i} = D_{f_i}$ for $i \in I$. For any function b with values in $\mathbb{K}$ and $D_b \subset \bigcup_i f_i D_{f_i}$ we set $f_j^*(b) = b \circ f_j$ for $j \in I$. It is easy to check that

$$\bigcap_i f_i^{*-1} M_i = \mathrm{an} \bigcap_i f_i^{*-1} M_i .$$

Let us set

$$f_* M = \Big(\bigcap_i f_i^{*-1} M_i\Big)_{\bigcup_j D_{f_j}} .$$

The $\mathbb{K}$-d.s. f_*M will be called the coinduced $\mathbb{K}$-d.s. from M by f.

Theorem 2.0 f_*M is the exactly one $\mathbb{K}$-d.s. N fulfilling the condition

(iv) for any $\mathbb{K}$-d.s. P and any h we have $h: N \longrightarrow P$ iff

(2.1) $h: \bigcup_i f_i \underline{M_i} \longrightarrow P$ and $h \circ f_i : M_i \longrightarrow P$ for $i \in I$.

Proof. Let us set $N = f_*M$. We have then $f_i : M_i \longrightarrow N$. Assuming that $h: N \longrightarrow P$ we get (2.1). Now, let us assume (2.1) and take $b \in P$. Then $b \circ h \circ f_i \in M_i$ for $i \in I$. Thus, $b \circ h \in \bigcap_i f_i^{*-1} M_i$, i. e. $b \circ h \in f_*M = N$. Therefore $h: N \longrightarrow P$.

Let us take any $\mathbb{K}$-d.s. N and N_1 fulfilling (iv). Setting $P = N$ and $h = id_{\underline{N}}$ we get $\underline{N} = \bigcup_i f_i \underline{M_i}$ and $f_i : M_i \longrightarrow N$ for $i \in I$. Similarly, $N_1 = \bigcup_i f_i \underline{M_i}$ and $f_i : M_i \longrightarrow N_1$ for $i \in I$. By (iv) we have $h: N \longrightarrow N_1$ and $h: N_1 \longrightarrow N$. Therefore $N = N_1$. Q.E.D.

Theorem 2.1 The topology $\mathrm{top} f_*M$ is the largest topology Y on the set $\bigcup_i f_i \underline{M_i}$ such that

(v) $f_i : \mathrm{top} M_i \longrightarrow Y$ for $i \in I$.

Proof. We have $f_i : M_i \longrightarrow f_*M$. Thus $f_i : \mathrm{top} M_i \longrightarrow \mathrm{top} f_*M$ for $i \in I$. Assume that a topology Y on $\bigcup_i f_i \underline{M_i}$ fulfills (v). Let us take any set $V \in Y$. Then $f_i^{-1} V \in \mathrm{top} M_i$. Hence it follows that the function $0_{f_i^{-1}V}$ with the domain equal to $f_i^{-1}V$ and the only value 0 belongs to M_i. We remark that $0_V \circ f_i = 0_{f_i^{-1}V}$ and $V \subset \bigcup_j f_j \underline{M_j}$. Thus, $0_V \in f_i^{*-1} M_i$ for $i \in I$. Hence it follows that $0_V \in f_*M$. Q.E.D.

According to 2.1 the topology of the coinduced $\mathbb{K}$-d.s. from $(M_i;\ i \in I)$ by f of the form $(f_i;\ i \in I)$ is the coinduced topology from $(\mathrm{top}M_i;\ i \in I)$ by f (cf.[2]).

A very important case is that I is a one element set and the only function f_i is the quotient mapping of an equivalence relation on the set of all points of a given $\mathbb{K}$-d.s.

Let M be a $\mathbb{K}$-d.s. and $\equiv$ be an equivalence relation on the set $\underline{M}$. We have the set $\underline{M}/\equiv$ of all cosets of the relation $\equiv$ and the natural mapping $f_1\colon \underline{M} \longrightarrow \underline{M}/\equiv$. Here $I = \{1\}$, $M_1 = \underline{M}$, $D_{f_1} = \underline{M}$ and $f_1 D_{f_1} = \underline{M}/\equiv$. According to 2.1 we have $\mathrm{top}(M/\equiv) = (\mathrm{top}M)/\equiv$.

3. Comparison between $\mathbb{R}$-d.s. and R. Sikorskis's differential spaces

R. Sikorski in the paper [1] introduced the concept of a differential space as a system (S, C), where S is a set and C is a set of all real valued functions defined on S. Sikorski assumed that C is closed with respect to localization and with respect to superposition with all C^∞-functions on $\mathbb{R}^n$ for any natural n. Recall the meaning of the localization. Following Sikorski denote the weakest topology on S for which all the functions $a \in C$ are continuous by τ_C. Let C_S denote the set of all $b\colon S \longrightarrow \mathbb{R}$ such that for any $p \in S$ there exist $U \in \tau_C$ and $a \in C$ for which $b|U = a|U$, $p \in U$. Closedness with respect to localization means $C = C_S$. Similarly, denote the set of all $g(a_o,\dots,a_n)$, where $a_o,\dots, a_n \in C$ and g is a real valued C^∞-function on $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$ (cf. [3]) by scC. Closedness with respect to superposition with C^∞-functions means $C = \mathrm{sc}C$. Following Sikorski we say that a function f maps smoothly a differential space (M, C) into (N, D) iff $f\colon \underline{M} \longrightarrow \underline{N}$ and for any $b \in D$ we have $b \circ f \in C$.

For any equivalence relation $\equiv$ on S we have the smooth natural mapping $p\colon \longmapsto \equiv(p)\colon (S, C) \longrightarrow (S/\equiv, C/\equiv)$, where $C/\equiv$ is the set of all functions $b\colon S/\equiv \longrightarrow \mathbb{R}$ such that the function $p \longmapsto b(\equiv(p))$ belongs to C. It is easy to state that $\tau_{C/\equiv} \subset \tau_C/\equiv$. This inclusion cannot be replaced by the equality

$\tau_{C/\equiv} = \tau_C/\equiv$ even in the case when $\equiv$ is given by a foliation of a differentiable manifold.

Example. Let $S = \mathbb{R}^2 - \{(0, 1)\}$ and F be a foliation all the leaves of which are $\mathbb{R}\times\{y\}$, when $y \neq 0$, $(-\infty; 0)\times\{0\}$ and $(0; +\infty)\times\{0\}$ as well. Let $p \equiv q$ iff $p, q \in S$ and for some leaf L of F the points p and q are in L. The set of all C^∞-functions on S is denoted by C. It is easy to check that $\tau_{C/\equiv} \neq \tau_C/\equiv$.

The above example shows a lack of coherence of the topology of the quotient Sikorski's differential space and the topology of leaves in the theory of foliations, while the coherence between the topology of quotient $\mathbb{R}$-d.s. and the topology of leaves is preserved.

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Received May 9, 1991.