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GEOMETRICAL PROPERTIES OF GLUING OF DIFFERENTIAL SPACES

This work is a continuation of [17]. The aim of this paper is to study geometric properties of gluing of differential spaces.

1. Basic notions

Let (M,C) be a differential space [19]. Let $\mathfrak{X}(M)$ be the C-module of all smooth vector fields tangent to (M,C). For a subset $K\subset M$ we denote by ι_K the embedding of the differential subspace (K,C_K) into (M,C).

A vector field $X\in\mathfrak{X}(M)$ is called *tangent to a subset* $K\subset M$ [17] if, for any point $p\in K$, there exists a vector $v\in T_pK$ such that $X(p)=(\iota_K)_{*p}v$. Let $\mathfrak{X}^K(M)$ be the C-module of all smooth vector fields tangent to K (see[17]). One can prove that $X\in\mathfrak{X}^K(M)$ if and only if $\forall f\in C\ f|K=0 \Rightarrow Xf|K=0$.

In the sequel, for $X\in\mathfrak{X}^K(M)$, we will denote by $X|K$ the restriction of X to a subspace (K,C_K).

It is easy to prove

Lemma 1.1. Let (M,C) be a differential space and $A,B\subset M$ subsets satisfying the following condition:

(1.1) $(\iota_{A\cap B})_{*p}T_p(A\cap B) = (\iota_A)_{*p}T_pA \cap (\iota_B)_{*p}T_pB$ for any $p\in A\cap B$.

Then $\mathfrak{X}^A(M)\cap\mathfrak{X}^B(M)\subset\mathfrak{X}^{A\cap B}(M)$.

Now we prove

Lemma 1.2. For any subset $A\subset M$ $\mathfrak{X}^A(M)=\mathfrak{X}^{clA}(M)$, where clA is the closure of A in M.

Proof. Of course, $\mathfrak{X}^A(M)\subset\mathfrak{X}^{clA}(M)$. It is enough to prove the iclusion $\mathfrak{X}^{clA}(M)\subset\mathfrak{X}^{A}(M)$. Let $X\in\mathfrak{X}^{clA}(M)$. Let $f\in C$ be an arbitrary function such that $f|A=0$. Of course, $f|clA=0$. Hence $Xf|clA=0$. Thus $Xf|A=0$. This proves that $X\in\mathfrak{X}^{A}(M)$.

Now let (M_1,C_1) and (M_2,C_2) be differential spaces and $h:\Delta_1 \longrightarrow \Delta_2$ a diffeomorphism of differential subspaces (Δ_1,C_{Δ_1}) and (Δ_2,C_{Δ_2}). Let ρ_h be an equivalence relation on the disjoint sum $(N,D)=(M_1\amalg M_2,C_1\amalg C_2)$ identifying a point $p\in\Delta_1$ with $h(p)\in\Delta_2$. The quotient space $(N/\rho_h,D/\rho_h)$ is called the gluing of differential spaces (M_1,C_1) and (M_2,C_2) and it will be denoted in the sequel by $(M_1\cup_h M_2,C_1\cup_h C_2)$ [17]. For any $f_1\in C_1$ and $f_2\in C_2$ such that $f_1|\Delta_1=f_2\circ h$ we denote by $f_1\cup_h f_2$ the function from $C_1\cup_h C_2$, corresponding to the function $f_1\amalg f_2\in C_1\amalg C_2$ [17].

Let $\pi_{\rho_h}:M_1\amalg M_2 \longrightarrow M_1\cup_h M_2$ be a canonical mapping and let us put

$$\hat{\iota}_1:=\pi_{\rho_h}|M_1,\quad \hat{\iota}_2:=\pi_{\rho_h}|M_2.$$

Let us put

$$\hat{M}_j=\hat{\iota}_j(M_j),\quad \hat{C}_j=(C_1\cup_h C_2)_{\hat{M}_j}\quad \text{for } j=1,2$$

and $\Delta=\pi_{\rho_h}(\Delta_1)$.

Clearly $\Delta=\pi_{\rho_h}(\Delta_2)=\hat{M}_1\cap\hat{M}_2$.

One can prove that [17]

$$(1.2)\qquad (\iota_\Delta)_{*p}(T_p\Delta)=(\iota_{\hat{M}_1})_{*p}(T_p\hat{M}_1)\cap(\iota_{\hat{M}_2})_{*p}(T_p\hat{M}_2)\quad\text{for any } p\in\Delta.$$

Now we prove

Proposition 1.3. Let (M_1,C_1) and (M_2,C_2) be disjoint differential spaces and $h:\Delta_1 \longrightarrow \Delta_2$ the gluing diffeomorphism. Assume that Δ_1 is a closed boundary set in M_1, and Δ_2 is a closed boundary set in M_2. Then $\mathfrak{X}(M_1\cup_h M_2)=\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$.

Proof. Of course, Δ is a closed boundary set. It is clear that $\hat{M}_i$ is closed ($\pi_{\rho_h}^{-1}(\hat{M}_1)=M_1\cup\Delta_2$ and $\pi_{\rho_h}^{-1}(\hat{M}_2)=M_2\cup\Delta_1$). It is easy to see that $\hat{M}_1\setminus\Delta$ and $\hat{M}_2\setminus\Delta$ are open subsets. Let us notice that $\hat{M}_i\setminus\Delta$ is dense in $\hat{M}_i$ for $i=1,2$. In fact

$$cl(\hat{M}_i\setminus\Delta)=cl(\hat{M}_i\cap(\hat{M}_1\cup\hat{M}_2\setminus\Delta))=cl\hat{M}_i\cap cl(\hat{M}_1\cup\hat{M}_2\setminus\Delta)=\hat{M}_i\cap(\hat{M}_1\cup\hat{M}_2)=\hat{M}_i.$$

Thus $cl_{\hat{M}_i}(\hat{M}_i\setminus\Delta)=\hat{M}_i\cap cl(\hat{M}_i\setminus\Delta)=\hat{M}_i$.

Now, let $X\in\mathfrak{X}(M_1\cup_h M_2)$. Since $\hat{M}_i\setminus\Delta$ is open, then $X\in\mathfrak{X}^{\hat{M}_i\setminus\Delta}(M_1\cup_h M_2)$ for i=1,2. From Lemma 1.2 it follows that $X\in\mathfrak{X}^{\hat{M}_i}(M_1\cup_h M_2)$, i=1,2. From Lemma 1.1 it follows that $X\in\mathfrak{X}^{\hat{M}_1\cap\hat{M}_2}(M_1\cup_h M_2)=\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$ q.e.d.

Now we prove a very useful lemma.

Lemma 1.4. Let (M,C) be an arbitrary differential space. Then

(i) for any open set U and an arbitrary point $p\in U$, there exists an open subset $V\subset U$ containing p such that $clV\subset U$.

(ii) every "bump" function $\phi:U\longrightarrow[0,1]$, $\phi\in C$, such that $supp\phi\subset V$, is extendible by 0 to a global function $\bar{\phi}:M\longrightarrow[0,1]$.

Proof. (i) Let $\psi:M\longrightarrow[0.1]$ be a "bump" function such that $supp\psi\subset U$. There exists an open set $V\in\tau_C$ containing p such that $\psi|V=1$ [21]. Of course, $clV\subset cl(\phi^{-1}(\{1\}))=\phi^{-1}(\{1\})\subset U$.

(ii) It is easy to see that $\{U,M\setminus clV\}$ is an open covering of M. Of course, $\phi|U\cap(M\setminus clV)=0$. There exists a smooth function $\bar{\phi}:M\longrightarrow[0,1]$ such that $\bar{\phi}|U=\phi$ and $\bar{\phi}|(M\setminus clV)=0$.

Now we prove

Proposition 1.5. Let Δ_1 be closed in (M_1,C_1) and Δ_2 be closed in (M_2,C_2). Assume that a gluing diffeomorphism $h:\Delta_1\longrightarrow\Delta_2$ satisfies the following condition:

(1.3) $$h(Int\Delta_1)=Int\Delta_2.$$

Then

(i) $Int\Delta=\pi_{\rho_h}(Int\Delta_1)$ and $Fr\Delta=\pi_{\rho_h}(Fr\Delta_1)$,

(ii) $\mathfrak{X}(M_1\cup_h M_2)\subset\mathfrak{X}^{\hat{M}_1}(M_1\cup_h M_2)\cap\mathfrak{X}^{\hat{M}_2}(M_1\cup_h M_2)$,

(iii) $\mathfrak{X}(M_1\cup_h M_2)=\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$.

Proof. (i) Of course, $\pi_{\rho_h}^{-1}(Int\Delta)\subset Int\Delta_1\cup Int\Delta_2$. Thus $Int\Delta\subset\pi_{\rho_h}(Int\Delta_1\cup Int\Delta_2)=\pi_{\rho_h}(Int\Delta_1)$. $\pi_{\rho_h}(Int\Delta_1)=\pi_{\rho_h}(Int\Delta_1\cup Int\Delta_2)$

is open subset of Δ. Thus $\pi_{\rho_h}(\mathrm{Int}\Delta_1)=\mathrm{Int}\Delta$. Of course Δ is closed. Thus $\mathrm{Fr}\Delta=\Delta\setminus\mathrm{Int}\Delta=\pi_{\rho_h}(\Delta_1\setminus\mathrm{Int}\Delta_1)=\pi_{\rho_h}(\mathrm{Fr}\Delta_1)$.

(ii) Let $X\in\mathfrak{X}(M_1\cup_h M_2)$. Since $\pi_{\rho_h}(\hat{M}_i\setminus\mathrm{Fr}\Delta)=$ $=(M_i\setminus\Delta_i)\cup\mathrm{Int}\Delta_1\cup\mathrm{Int}\Delta_2$, $\hat{M}_i\setminus\mathrm{Fr}\Delta$ is open for $i=1,2$. Thus $X\in\mathfrak{X}^{\hat{M}_i\setminus\mathrm{Fr}\Delta}(M_1\cup_h M_2)$ for $i=1,2$.

Analogously as in the proof of Proposition 1.3 one can see that $\hat{M}_i\setminus\mathrm{Fr}\Delta$ is dense in the subspace $\hat{M}_i$ for $i=1,2$. From Lemma 1.2 it follows that $X\in\mathfrak{X}^{\hat{M}_1\cap\hat{M}_2}(M_1\cup_h M_2)=\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$. Thus $\mathfrak{X}(M_1\cup_h M_2)\subset\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$. The inclusion $\mathfrak{X}^{\Delta}(M_1\cup_h M_2)\subset\mathfrak{X}(M_1\cup_h M_2)$ is obvious. This finishes the proof of (ii).

Remark. Assumption 1.3 in Proposition 1.5 is essential. If we glue a one element space $\{0\}$ with a real line $\mathbb{R}$ we obtain $\mathbb{R}$ as a glued space. (i) and (ii) are not true.

Definition 1.1. A gluing diffeomorphism $h:\Delta_1 \longrightarrow \Delta_2$ is called locally extendible to a diffeomorphism if for any point $p\in\Delta_1$ there exists an open neighbourhood $U\in\tau_{C_1}$ of p and a diffeomorphism $\bar{h}_p:U \longrightarrow h(U)$ onto a open subset $h(U)\in\tau_{C_2}$ such that $\bar{h}_p|U\cap\Delta_1=h|U\cap\Delta_1$.

Now we prove

Proposition 1.6. Let Δ_1 and Δ_2 be closed in (M_1,C_1) and (M_2,C_2), respectively. Assume that a gluing diffeomorphism $h:\Delta_1 \longrightarrow \Delta_2$ is locally extendible to a diffeomorphism.

Then

(i) $\tau_{C_1\cup_h C_2}=\tau_{C_1\amalg C_2}/\rho_h$,

(ii) $\hat{\iota}_1:M_1 \longrightarrow \hat{M}_1$ and $\hat{\iota}_2:M_2 \longrightarrow \hat{M}_2$ are diffeomorphisms,

(iii) $h(\mathrm{Int}\Delta_1)=\mathrm{Int}\Delta_2$,

(iv) $\mathfrak{X}(M_1\cup_h M_2)=\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$,

(v) The $C_1\cup_h C_2$-module $\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$ is isomorphic to the $C_1\cup_h C_2$-module

$$\mathfrak{X}_h(M_1,M_2)=\{(X_1,X_2)\in\mathfrak{X}^{\Delta_1}(M_1)\times\mathfrak{X}^{\Delta_2}(M_2):h_*(X_1|\Delta_1)=X_2|\Delta_2\}.$$

Proof. (i) Let $U_1\in\tau_{C_1}$ and $U_2\in\tau_{C_2}$ be sets such that $h(U_1\cap\Delta_1)=U_2\cap\Delta_2$. For any point $p\in U_1\cap\Delta_1$ there exist open subsets $V_1\in\tau_{C_1}$, $p\in V_1$, $V_2\in\tau_{C_2}$ and a diffeomorphism $\bar{h}_p:V_1\longrightarrow V_2$ such that $\bar{h}_p|V_1\cap\Delta_1=h|V_2\cap\Delta_2$. It follows from Lemma 1.4 that there are open subsets $W_1\subset V_1, W_2\subset V_2$ such that $clW_1\subset V_1$ and $clW_2\subset V_2$, and $\bar{h}_p(W_1)=W_2$, $p\in W_1$. Let $\phi_1:V_1\longrightarrow[0,1]$ be a bump function such that $supp\phi_1\subset W_1$ and $\phi_1(p)=1$. It is clear that $\phi_1\circ\bar{h}_p^{-1}$ is a bump function such that supp $\phi_1\circ\bar{h}_p^{-1}\subset W_2$. Now, it follows from Lemma 1.4, that ϕ_1 is extendible by 0 to the function $\bar{\phi}_1\in C_1$ and the function $\phi_2\circ\bar{h}^{-1}$ is extendible by 0 to the function $\bar{\phi}_2\in C_2$, and $\bar{\phi}_1|\Delta_1=\bar{\phi}_2\circ h$. $\bar{\phi}_1\amalg\bar{\phi}_2$ is ρ_h-consistent and satisfies $(\bar{\phi}_1\amalg\bar{\phi}_2)(p)=1$. This proves (i).

(ii) Let $\psi_1:\hat{M}_1\longrightarrow M_1$ be the inverse mapping to $\hat{\iota}_1$. Let $\alpha\in C_1$ be an arbitrary function. We will show that the composition $\alpha\circ\psi_1$ is smooth. If $[p]\notin\Delta$ then there exists a neighbourhood $U\in\tau_{C_1}$ of p, such that $U\cap\Delta_1=\emptyset$ and a function $\alpha_1\in C_1$ such that $\alpha_1|U=\alpha|U$, $\alpha_1|M_1\setminus U=0$. It is clear that $\alpha_1\amalg 0_{M_2}$ is consistent with ρ_h. It is easy to show that $\alpha\circ\psi_1|\pi_{\rho_h}(U)\cap\hat{M}_1=\alpha_1\cup_h 0_{M_2}|\pi_{\rho_h}(U)\cap\hat{M}_1$. If $\hat{\iota}_1(p)=[p]\in\Delta$, then there are open sets $V_1\in\tau_{C_1}$, $V_2\in\tau_{C_2}$, $p\in V_1$ and a diffeomorphism $\bar{h}_p:V_1\longrightarrow V_2$ such that $\bar{h}_p|V_1\cap\Delta_1=h|V_1\cap\Delta_1$. Let W_1, W_2 and $\bar{\phi}_1\amalg\bar{\phi}_2$ be as in (i). By construction $supp(\bar{\phi}_1\amalg\bar{\phi}_2)\subset W_1\cup W_2$. There is an open subset $B\subset W_1$ such that $\bar{\phi}_1|B=1$ (see [21]). Let us denote by $\alpha\circ\bar{h}_p^{-1}\cdot\phi_2$ the extension by 0 of the function $\alpha\circ\bar{h}_p^{-1}\cdot(\bar{\phi}_2|V_2)$. It easy to see that $\alpha\cdot\bar{\phi}_1\amalg\alpha\circ\bar{h}_p^{-1}\cdot\bar{\phi}_2$ is ρ_h-consistent and $\alpha\circ\psi_1|\pi_{\rho_h}(B)\cap\hat{M}_1= =\alpha\bar{\phi}_1\cup_h\alpha\circ\bar{h}_p^{-1}\cdot\bar{\phi}_2|\pi_{\rho_h}(B)\cap\hat{M}_1$. Analogously one can prove the smoothness of the inverse mapping to $\hat{\iota}_2$. This finishes the proof of (ii).

(iii) For any point $p\in Int\Delta_1$, there exists an open neighbourhood U_p of p such that $U_p\subset Int\Delta_1$, $\bar{h}_p(U_p)\in\tau_{C_2}$. Of course, $Int\Delta_1=\bigcup_{p\in Int\Delta_1}U_p$, $Int\Delta_2=\bigcup_{p\in Int\Delta_1}h(U_p)$. Now, it is evident that

$$h(\mathrm{Int}\Delta_1)=h(\bigcup_{p\in \mathrm{Int}\Delta_1} U_p)=\bigcup_{p\in \mathrm{Int}\Delta_1} h(U_p)=\mathrm{Int}\Delta_2.$$

(iv) is a consequence of (iii) and Proposition 1.5.

(v) Let $X\in\mathfrak{X}(M_1\cup_h M_2)$. By Proposition 1.5 (ii), $X\in\mathfrak{X}^{\hat{M}_1}(M_1\cup_h M_2)$ and $X\in\mathfrak{X}^{\hat{M}_2}(M_1\cup_h M_2)$. Let us put $X_i=(\hat{\iota}_i^{-1})_*(X|\hat{M}_i)$, $i=1,2$. It is clear that $X_1\in\mathfrak{X}^{\Delta_1}(M_1)$ and $X_2\in\mathfrak{X}^{\Delta_2}(M_2)$. Now, let $H:\mathfrak{X}_h(M_1,M_2)\longrightarrow\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$ be defined by

$$H(Y_1,Y_2)=Y_1\cup_h Y_2, \tag{1.4}$$

where $Y_1\cup_h Y_2$ is the vector field satisfying $Y_1\cup_h Y_2|\hat{M}_j=(\hat{\iota}_j)_*Y_j$, for $j=1,2$ [17]. Of course, H is an isomorphism.

The following equalities hold:

$$f_1\cup_h f_2\cdot Y_1\cup_h Y_2=f_1Y_1\cup_h f_2Y_2, \tag{1.5}$$

$$(Y_1\cup_h Y_2)(f_1\cup_h f_2)=Y_1f_1\cup_h Y_2f_2, \tag{1.6}$$

$$[X_1\cup_h X_2,Y_1\cup_h Y_2]=[X_1,Y_1]\cup_h[X_2,Y_2] \tag{1.7}$$

for any $(X_1,X_2),(Y_1,Y_2)\in\mathfrak{X}_h(M_1,M_2)$ and $(f_1,f_2)\in C_1\times C_2$ such that $f_1|\Delta_1=f_2\circ h$.

One can prove

Lemma 1.7. If Δ_1,Δ_2 are submanifolds of differential spaces (M_1,C_1) and (M_2,C_2) of class $\mathcal{D}_o$ [24], then every vector $w\in T\Delta$ is extendible to a smooth vector field.

Proof. Let $w\in T_p\Delta$, $p=[p_1]=[p_2]$. Let $X\in\mathfrak{X}(\Delta)$ be a vector field such that $w=X(p)$. There are vector fields $Y_1\in\mathfrak{X}^{\Delta_1}(M_1)$ and $Y_2\in\mathfrak{X}^{\Delta_2}(M_2)$, $(\hat{\iota}_1)_*(Y_1|\Delta_1)=X$ and $(\hat{\iota}_2)_*(Y_2|\Delta_2)=X$. It is easy to see that $Y_1\cup_h Y_2$ is a vector field such that $w=(Y_1\cup_h Y_2)(p)$.

Proposition 1.8. Let (M_1,C_1) and (M_2,C_2) be differential spaces of class $\mathcal{D}_o$ and $h:\Delta_1\longrightarrow\Delta_2$ be a gluing diffeomorphism between closed boundary subspaces.

Then, for any $p\in\Delta$, the subspace $T_p^{ex}(M_1\cup_h M_2)$ of all extendible vectors is equal to $(\iota_\Delta)_*(T_p^{ex}\Delta)$.

Proof. Let $w\in T_p^{ex}(M_1\cup_h M_2)$, $p\in\Delta$. There exists a vector field $X\in\mathfrak{X}(M_1\cup_h M_2)$ such that $w=X(p)$. By Proposition 1.3 $X\in\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$. Let $Y\in\mathfrak{X}(\Delta)$ be a vector field such that $(i_\Delta)_*Y(q)=X(q)$ for $q\in\Delta$. Thus $w=X(p)=(i_\Delta)_{*p}Y(p)\in(\iota_\Delta)_*(T_p^{ex}\Delta)$. Conversely, let

$w\in(i_\Delta)_*(T_p^{ex}\Delta)$. There is a vector field $Y\in\mathcal{X}(\Delta)$ such that $w=(i_\Delta)_{*p}Y(p)$. Since (M_1,C_1) and (M_2,C_2) are spaces of class $\mathcal{D}_o$ and Δ_1,Δ_2 are closed, there exist vector fields $X_1\in\mathcal{X}^{\Delta_1}(M_1)$, $X_2\in\mathcal{X}^{\Delta_2}(M_2)$ such that $(\hat{i}_1)_*(X_1|\Delta_1)=Y$, $(\hat{i}_2)_*(X_2|\Delta_2)=Y$. Now it is clear that $Y=X_1\cup_h X_2|\Delta$. Thus

$$w=(i_\Delta)_{*p}Y(p)=(X_1\cup_h X_2)(p)\in T_p^{ex}(M_1\cup_h M_2).$$

This finishes the proof.

Now we prove

Lemma 1.9. Let (M_1,C_1) and (M_2,C_2) be differential spaces and $h:\Delta_1\longrightarrow\Delta_2$ be a gluing diffeomorphism. If $c:[a,b]\longrightarrow M_1\cup_h M_2$ is a smooth curve such that $c(a)\in\hat{M}_1$, $c(b)\in\hat{M}_2$, then $c'(t)\in\bar{T}\Delta:=(\iota_\Delta)_*T\Delta$, for any $t\in c^{-1}(\Delta)$.

Proof. Let $t_0\in c^{-1}(\Delta)$. Of course, $c|[a,t_0]$ is a smooth curve in $\hat{M}_1$. Thus $c'(t_0)\in T_p\hat{M}_1$, where $p=c(t_0)$. Analogously, $c|[t_0,b]$ is a smooth curve in $\hat{M}_2$ and $c'(t_0)\in T_p\hat{M}_2$. Hence $c'(t_0)\in T_p\hat{M}_1\cap T_p\hat{M}_2=T_p\Delta$.

Definition 1.2. Curves $c_1:(a,t_0+\varepsilon)\longrightarrow M_1$ and $c_2:(t_0-\varepsilon,b)\longrightarrow M_2$, where $t_o\in(a,b)$, $\varepsilon>0$, are said to be ε-ρ_h-consistent if $c_1(-\varepsilon,\varepsilon)\subset\Delta_1$, $c_2(-\varepsilon,\varepsilon)\subset\Delta_2$ and $h(c_1(t))=c_2(t)$ for $t\in(-\varepsilon,\varepsilon)$.

Of course, for any ε-ρ_h-consistent curves $c_1:(a,t_0+\varepsilon)\longrightarrow M_1$ and $c_2:(t_0-\varepsilon,b)\longrightarrow M_2$, the mapping $c_1\cup_h c_2:(a,b)\longrightarrow M_1\cup_h M_2$ given by the formula

$$(c_1\cup_h c_2)(t)=\begin{cases}\pi_{\rho_h}(c_1(t)) & \text{for } t\in(a,t_0]\\ \pi_{\rho_h}(c_2(t)) & \text{for } t\in(t_0,b)\end{cases}$$

is a smooth curve.

It is easy to prove

Proposition 1.10. Let $X=X_1\cup_h X_2\in\mathcal{X}(M_1\cup_h M_2)$ be an arbitrary vector field on the glued space $(M_1\cup_h M_2,C_1\cup_h C_2)$. If $c_1:(a,t_0+\varepsilon)\longrightarrow M_1$ is an integral curve of X_1, $(\varepsilon>0,\ t_o\in(a,b))$, $c_2:(t_0-\varepsilon,b)\longrightarrow M_2$ is an integral curve of X_2 and c_1, c_2 are ε-ρ_h-consistent curves, then $c_1\cup_h c_2=\pi_{\rho_h}\circ(c_1\amalg c_2):(a,b)\longrightarrow M_1\cup_h M_2$ is an integral curve of X. If $c_1:(a,t_0]\longrightarrow M_1$ is an integral curve of X_1 and $c_2:[t_0,b)\longrightarrow M_2$ is an integral curve of X_2 and $h(c_1(t_o))=c_2(t_o)\in\Delta_2$, then $c_1\cup_h c_2$ is a piecewise smooth integral curve of X.

Now we will define a special case of a gluing diffeomorphism which produces a special kind of singularity so called of the edge type.

Definition 1.3. A gluing diffeomorphism $h:\Delta_1 \longrightarrow \Delta_2$ of disjoint differential spaces (M_1,C_1) and (M_2,C_2) is said to be the edge type if there exist differential spaces $(B_1,\mathcal{B}_1)$, $(B_2,\mathcal{B}_2)$ and $(Z,\mathcal{Z})$ such that the following conditions are satisfied :

(1) for any point $p\in\Delta_1$, there exist open neighbourhoods $W_1\in\tau_{C_1}$ of p and $W_2\in\tau_{C_2}$ of h(p) , open sets $U_1\in\tau_{\mathcal{B}_1}$, $U_2\in\tau_{\mathcal{B}_2}$, $T\in\tau_{\mathcal{Z}}$, points $b_1\in B_1, b_2\in B_2$ and diffeomorphisms $\phi_1:W_1 \longrightarrow U_1\times T$, $\phi_2:W_2 \longrightarrow U_2\times T$,

(2) $\Delta_1\cap W_1=\phi_1^{-1}(\{b_1\}\times T)$, $\Delta_2\cap W_2=\phi_2^{-1}(\{b_2\}\times T)$

(3) $\phi_2\circ h|\Delta_1\cap W_1=h_0\circ\phi_1|\Delta_1\cap W_1$, where $h_0:\{b_1\}\times T \longrightarrow \{b_2\}\times T$ is the diffeomorphism given by:

$$h_0(b_1,t)=(b_2,t) \quad \text{for } t\in T.$$

Now one can prove

Proposition 1.11. Let (M_1,C_1), (M_2,C_2) be disjoint differential spaces and $h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism of the edge type between closed spaces.

Then

(i) $\tau_{C_1\cup_h C_2}=\tau_{C_1\amalg C_2}/\rho_h$,

(ii) the mappings $\hat{\iota}_1:M_1 \longrightarrow \hat{M}_1$ and $\hat{\iota}_2:M_2 \longrightarrow \hat{M}_2$ are embeddings,

(iii) $T_p(M_1\cup_h M_2)=T_{p_1}M_1\oplus T_{p_2}M_2$ for $p\in\Delta$, $p=[p_1]=[p_2]$,

(iv) $\mathfrak{X}(M_1\cup_h M_2)=\mathfrak{X}^{\Delta}(M_1\cup_h M_2)$,

(v) $\bar{T}\Delta:=(\iota_\Delta)_*T\Delta$ is the set of all vectors tangent to the glued space at singular points extendible to smooth vector field.

Proof is similar to the proof of Proposition 3.2 in [17].

Example 1.1. Let M_1 and M_2 be differential manifolds of dimension n_1 and n_2 respectively, let Δ_1 and Δ_2 be k-dimensional submanifolds ($k<\min\{n_1,n_2\}$), and let

$h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism. Then h is a gluing diffeomorphism of the edge type. In fact, for any point $p_1\in\Delta_1$ there exist charts $\phi_1:W_1 \longrightarrow U_1\times T\subset\mathbb{R}^{n_1}$ on M_1 and $\phi_2:W_2 \longrightarrow U_2\times T\subset\mathbb{R}^{n_2}$ on M_2 such that W_1 is an open neighbourhood of p_1, W_2 is an open neighbourhood of $h(p_1)$, $U_1\subset\mathbb{R}^{n_1-k}$, $U_2\subset\mathbb{R}^{n_2-k}$, $T\subset\mathbb{R}^k$ are open subsets, and $\phi_1(\Delta_1\cap W_1)=\{0\}\times T$, $\phi_2(\Delta_2\cap W_2)=\{0\}\times T$ (see [6] for example). It obvious that if we put $(B_1,\mathcal{B}_1)=(\mathbb{R}^{n_1-k},C^\infty(\mathbb{R}^{n_1-k}))$, $(B_2,\mathcal{B}_2)=(\mathbb{R}^{n_2-k},C^\infty(\mathbb{R}^{n_2-k}))$, $(Z,\mathcal{Z})=(\mathbb{R}^k,C^\infty(\mathbb{R}^k))$, the conditions of Definition 1.2 are satisfied.

2. Gluing of tangent bundles, forms and connections

Let (M_1,C_1) and (M_2,C_2) be differential spaces, and $h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism. Of course, $h_*:T\Delta_1 \longrightarrow T\Delta_2$ is a diffeomorphism of differential spaces, which induces a diffeomorphism $\bar{h}_*:\bar{T}\Delta_1 \longrightarrow \bar{T}\Delta_2$ of the spaces $\bar{T}\Delta_1:=(\iota_{\Delta_1})_*T\Delta_1$ and $\bar{T}\Delta_2:=(\iota_{\Delta_2})_*T\Delta_2$ such that the following diagram

$$\begin{array}{ccc} T\Delta_1 & \xrightarrow{\;h_*\;} & T\Delta_2 \\ {\scriptstyle(\iota_{\Delta_1})_*}\downarrow & & \downarrow{\scriptstyle(\iota_{\Delta_2})_*} \\ \bar{T}\Delta_1 & \xrightarrow{\;\bar{h}_*\;} & \bar{T}\Delta_2 \end{array}$$

commutes.

Now we may consider $\bar{h}_*:\bar{T}\Delta_1 \longrightarrow \bar{T}\Delta_2$ as a gluing diffeomorphism of tangent differential spaces (TM_1,TC_1) and (TM_2,TC_2) [7]. Let $\Phi:TM_1\cup_{\bar{h}_*}TM_2 \longrightarrow T(M_1\cup_h M_2)$ be a mapping given by :

(2.1) $$\Phi(\pi_{\rho_{\bar{h}_*}}(v))=(\pi_{\rho_h})_*v \text{ for any } v\in TM_1\sqcup TM_2.$$

It is easy to verify the correctness of (2.1). The smoothness of ϕ is a consequence of the following equalities:

(2.2) $$d(f_1\cup_h f_2)\circ\Phi=df_1\cup_{\bar{h}_*}df_2,$$

(2.3) $$f_1\cup_h f_2\circ\pi\circ\Phi=f_1\circ\pi_1 \cup_{\bar{h}_*} f_2\circ\pi_2$$

for any $f_1 \in C_1$, $f_2 \in C_2$, $f_2 \circ h = f_1 | \Delta_1$, where $\pi : T(M_1 \cup_h M_2) \longrightarrow M_1 \cup_h M_2$, $\pi_1 : TM_1 \longrightarrow M_1$, $\pi_2 : TM_2 \longrightarrow M_2$ are the canonical projections.

Let $\tilde{\pi} : TM_1 \cup_{\bar{h}_*} TM_2 \longrightarrow M_1 \cup_h M_2$ be defined by:

(2.4) $\tilde{\pi}(\pi_{\rho_{\bar{h}_*}}(v)) = \pi_{\rho_h}(\pi(v))$ for any $v \in TM_1 \amalg TM_2$.

It is easy to verify

(2.5) $f_1 \cup_h f_2 \circ \tilde{\pi} = f_1 \circ \pi_1 \cup_{\bar{h}_*} f_2 \circ \pi_2$ for any $f_1 \cup_h f_2 \in C_1 \cup_h C_2$.

Thus $\tilde{\pi}$ is a smooth projection. The following diagram

$$\begin{array}{ccc} TM_1 \cup_{\bar{h}_*} TM_2 & \xrightarrow{\Phi} & T(M_1 \cup_h M_2) \\ & {\scriptstyle\tilde{\pi}} \searrow \quad \swarrow {\scriptstyle\pi} & \\ & M_1 \cup_h M_2 & \end{array}$$

commutes. Of course, for any point $p \in \Delta_1$ the fiber $\tilde{\pi}^{-1}([p]) = \{\pi_{\rho_{\bar{h}_*}}(v) : v \in T_pM_1 \cup T_{h(p)}M_2\}$ corresponds to the subset $\Phi(\tilde{\pi}^{-1}([p]) = (\pi_{\rho_h})_{*p} T_pM_1 \cup (\pi_{\rho_h})_{*h(p)} T_{h(p)}M_2$. It is easy to show that Φ is a bijection of $TM_1 \cup_{\bar{h}_*} TM_2$ onto $\tilde{T}(M_1 \cup_h M_2) :=$ $= (\pi_{\rho_h})_* TM_1 \cup (\pi_{\rho_h})_* TM_2$.

Now we prove

Lemma 2.1. Let (M_1, C_1) and (M_2, C_2) be disjoint differential spaces and let $p_i \in M_i$, $i=1,2$, be arbitrary points. Let $*: \{p_1\} \longrightarrow \{p_2\}$ be the natural gluing diffeomorphism of one-element subspaces.

Then the bijection $\Phi : TM_1 \cup_* TM_2 \longrightarrow \tilde{T}(M_1 \cup_* M_2)$ is a diffeomorphism onto its image $\tilde{T}(M_1 \cup_* M_2)$. Moreover, the fiber $\tilde{\pi}^{-1}([p_1])$ is diffeomorphic to the glued space $T_{p_1}M_1 \cup_* T_{p_2}M_2$ obtained by identifying zero vectors.

Proof. Of course, $\bar{T}\Delta_1 = \{0\}$, $T\bar{\Delta}_2 = \{0\}$. It follows from Proposition 3.1 in [17] that the differential space $(TM_1 \cup_* TM_2, TC_1 \cup_* TC_2)$ is generated by the set

$$\{\widehat{df_1} : f_1 \in C_1\} \cup \{\widehat{df_2} : f_2 \in C_2\} \cup \{\widehat{f_1 \circ \pi_1} : f_1 \in C_1\} \cup \{\widehat{f_2 \circ \pi_2} : f_2 \in C_2\},$$

where the symbol $\hat{F}$ means the constant extension of a given function F to the glued space (see [16], [17]). Let Ψ be the inverse mapping to Φ. It is easy to verify the following equalities:

$$(2.6)\qquad \widehat{df_i}\circ\Psi = d\hat{f}_i|\tilde{T}(M_1\cup_* M_2),$$

$$(2.7)\qquad \widehat{f_i\circ\pi_i}\circ\Psi = \hat{f}_i\circ\pi_i\,|\tilde{T}(M_1\cup_* M_2),$$

for any $f_i\in C_i$, i=1,2. Thus Ψ is smooth.

Proposition 2.2. Let (M_1,C_1) and (M_2,C_2) be disjoint differential spaces and let $p_i\in M_i$, i=1,2, be arbitrary points. For an arbitrary differential space $(Z,\mathcal{Z})$, let $h:\{p_1\}\times Z \longrightarrow \{p_2\}\times Z$ be the diffeomorphism defined by

$$(2.8)\qquad h(p,z)=(p,z) \text{ for } z\in Z.$$

Then the bijection $\Phi^Z:T(M_1\times Z)\cup_{\bar{h}_*}T(M_2\times Z) \longrightarrow \tilde{T}(M_1\times Z\cup_h M_2\times Z)$ defined anlogously to (2.1) is a diffeomorphism. Moreover, every fiber $\tilde{\pi}^{-1}([p_1,z])$ is diffeomorphic to the glued space

$$T_{(p_1,z)}(M_1\times Z)\cup_{\bar{h}_{*(p_1,z)}}T_{(p_2,z)}(M_2\times Z).$$

Proof. Let $\pi_{M_1}:M_1\times Z \longrightarrow M_1$, $\pi_{M_2}:M_2\times Z \longrightarrow M_2$, $\overset{1}{\pi}_Z:M_1\times Z \longrightarrow Z$, $\overset{2}{\pi}_Z:M_2\times Z \longrightarrow Z$ be the natural projection. It is evident that the diffeomorphisms

$$((\pi_{M_1})_*,(\overset{1}{\pi}_Z)_*):T(M_1\times Z) \longrightarrow TM_1\times TZ$$

and

$$((\pi_{M_2})_*,(\overset{2}{\pi}_Z)_*):T(M_2\times Z) \longrightarrow TM_2\times TZ$$

induce, for the spaces $TM_1\times TZ$ and $TM_2\times TZ$, the gluing diffeomorphism $h_Z:\{0_{p_1}\}\times TZ\to\{0_{p_2}\}\times TZ$ defined by

$$(2.9)\qquad h_Z(0_{p_1},w)=(0_{p_2},w) \text{ for } w\in TZ,$$

where $0_{p_1}\in TM_1$ is the zero vector from the fiber $T_{p_1}M_1$, and $0_{p_2}\in TM_2$ is the zero vector from the fiber $T_{p_2}M_2$. h_Z corresponds to the gluing diffeomorphism $\bar{h}_*$. One can see that the natural mapping $((\pi_{M_1})_*,(\overset{1}{\pi}_Z)_*)\cup_{\bar{h}_*}((\pi_{M_2})_*,(\overset{2}{\pi}_Z)_*)$ from

$T(M_1\times Z) \cup_{\bar{h}_*} T(M_2\times Z)$ onto $TM_1\times TZ \cup_{h_z} TM_2\times TZ$ is a diffeomorphism. By Proposition 3.2 in [17], there exists the natural diffeomorphism between $TM_1\times TZ\cup_{h_z} TM_2\times TZ$ and $(TM_1\cup_* TM_2)\times TZ$. Therefore, $T(M_1\times Z)\cup_{\bar{h}_*} T(M_2\times Z)$ is diffeomorphic to $(TM_1\cup_* TM_2)\times TZ$. On the other hand, since in view of Proposition 3.2 in [17] $M_1\times Z\cup_h M_2\times Z$ is diffeomorphic to $(M_1\cup_* M_2)\times Z$, the space $T(M_1\times Z\cup_h M_2\times Z)$ is naturally diffeomorphic to $T(M_1\cup_* M_2)\times TZ$.

It is easy to see that the following diagram

$$\begin{array}{ccc} T(M_1\times Z)\cup_{\bar{h}_*} T(M_2\times Z) & \xrightarrow{\Phi^Z} & T(M_1\times Z\cup_h M_2\times Z) \\ \text{diffeo}\ \cong \downarrow & & \downarrow \text{diffeo}\ \cong \\ (TM_1\cup_* TM_2)\times TZ & \xrightarrow{\Phi\times id} & T(M_1\cup_* M_2)\times TZ \end{array}$$

commutes, where $\Phi: TM_1\cup_* TM_2 \longrightarrow \tilde{T}(M_1\cup_* M_2)$ is the bijection from Lemma 2.1. Since, by Lemma 2.1, Φ is a diffeomorphism onto its image, $\Phi\times id$ is a diffeomorphism onto its image. Now is clear that Φ^Z is a diffeomorphism onto its image. Of course

$$\tilde{T}(M_1\times Z\cup_h M_2\times Z)\cong \tilde{T}(M_1\cup_* M_2)\times TZ.$$

From Proposition 2.2 it follows

Corollary 2.3. Let (M_1,C_1), (M_2,C_2) be disjoint differential spaces and $h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism of the edge type between closed spaces. Then the bijection $\Phi: TM_1\cup_{\bar{h}_*} TM_2 \longrightarrow \tilde{T}(M_1\cup_h M_2)$ defined by (2.1) is a diffeomorphism. Moreover, for any point $[p]\in\Delta$, where $p\in\Delta_1$, every fiber $\tilde{\pi}^{-1}([p])$ is diffeomorphic to the glued space $T_pM_1\cup_{\bar{h}_{*p}} T_{h(p)}M_2$, where $\bar{h}_{*p}:=\bar{h}_*|\bar{T}_p\Delta_1$.

Now we describe a gluing of global forms.

Let $h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism of differential spaces (M_1,C_1) and (M_2,C_2).

Definition 2.1. Two global k-forms

$$\omega_1:\mathfrak{X}(M_1)\times\ldots\times\mathfrak{X}(M_1) \longrightarrow C_1 \quad \text{and} \quad \omega_2:\mathfrak{X}(M_2)\times\ldots\times\mathfrak{X}(M_2) \longrightarrow C_2$$

are called h-*consistent* if the following condition is satisfied:

for any $X_1,\dots,X_k \in \mathfrak{X}^{\Delta_1}(M_1)$ and $Y_1,\dots,Y_k \in \mathfrak{X}^{\Delta_2}(M_2)$, if $h_*(X_i|\Delta_1)=Y_i|\Delta_2$, for $i=1,2,\dots k$, then

$$\omega_1(X_1,\dots,X_k)|\Delta_1=\omega_2(Y_1,\dots,Y_k)\circ h.$$

For arbitrary h-consistent k-forms $\omega_1 \in \Omega^k(M_1)$ and $\omega_2 \in \Omega^k(M_2)$ let ω be k-form on $M_1 \cup_h M_2$ defined by:

$$\omega(X_1 \cup_h Y_1,\dots,X_k \cup_h Y_k)=\omega_1(X_1,\dots,X_k)\cup_h \omega_2(Y_1,\dots,Y_k) \tag{2.10}$$

for any h-consistent vector fields $X_1,\dots,X_k \in \mathfrak{X}(M_1)$ and $Y_1,\dots,Y_k \in \mathfrak{X}(M_2)$.

In the sequel, k-form ω defined by (2.10) will be denoted by $\omega_1 \cup_h \omega_2$.

Now one can easily prove

Lemma 2.4. For any pointwise k-forms $\omega_1 \in A^k(M_1)$ and $\omega_2 \in A^k(M_2)$ such that $\iota_{\Delta_1}^* \omega_1 = h^* \omega_2$, the global k-forms $\bar{\omega}_1$ and $\bar{\omega}_2$, which are images of ω_1 and ω_2 respectively by the natural homomorphism [9], are h-consistent.

Proof. It is evident that the conditio n $\iota_{\Delta_1}^* \omega_1 = h^* \omega_2$ is equivalent to the following condition:

$$\omega_1((\iota_{\Delta_1})_* u_1,\dots,(\iota_{\Delta_1})_* u_k)=\omega_2(h_* u_1,\dots,h_* u_k) \tag{2.11}$$

for any $(u_1,\dots,u_k) \in T^k\Delta$.

Let $X_1,\dots,X_k \in \mathfrak{X}(M_1)$ and $Y_1,\dots,Y_k \in \mathfrak{X}(M_2)$ be arbitrary vector fields such that $h_*(X_i|\Delta_1)=Y_i|\Delta_2$, for $i=1,2,\dots k$. It is easy to see (using (2.11)) that

$$\omega_1((\iota_{\Delta_1})_* (X_1|\Delta_1)(p),\dots,(\iota_{\Delta_1})_*(X_k|\Delta_1)(p))=$$
$$=\omega_2(h_*(X_1|\Delta_1)(p),\dots,h_*(X_k|\Delta_1)(p))$$

for $p \in \Delta_1$, or equivalently

$$\omega_1(X_1(p),\dots,X_k(p))=\omega_2(Y_1(h(p)),\dots,Y_k(h(p))) \text{ for } p \in \Delta_1.$$

Hence

$$\bar{\omega}_1(X_1,\dots,X_k)(p)=\bar{\omega}_2(Y_1,\dots,Y_k)(h(p)) \text{ for any } p \in \Delta_1.$$

Thus $\bar{\omega}_1(X_1,\dots,X_k)|\Delta_1=\bar{\omega}_2(Y_1,\dots,Y_k)\circ h$. This finishes the proof.

Corollary 2.5. Every k-forms $\bar{\omega}_1 \in \Omega^k(M_1)$ and $\bar{\omega}_2 \in \Omega^k(M_2)$ such that $i_{\Delta_1}^{*}\omega_1=h^{*}\omega_2$ may be glued together to the global k-form $\bar{\omega}_1\cup_h\bar{\omega}_2\in\Omega^k(M_1\cup_h M_2)$.

Now one can easily prove

Proposition 2.6. Let $\Omega^k{}_h(M_1,M_2)$ be the set of all h-consistent pairs $(\omega_1,\omega_2)\in\Omega^k(M_1)\times\Omega^k(M_2)$. Assume that Δ_1 and Δ_2 are closed boundary subsets in M_1 and M_2, respectively. Then the mapping $\square:\Omega^k{}_h(M_1,M_2)\longrightarrow\Omega^k(M_1\cup_h M_2)$, given by

$$\square(\omega_1,\omega_2)=\omega_1\cup_h\omega_2 \tag{2.12}$$

is a monomorphism of $C_1\cup_h C_2$-modules.

Definition 2.2. Global (k,1) tensor fields $\omega_1:\mathfrak{X}(M_1)\times\ldots\times\mathfrak{X}(M_1)\longrightarrow\mathfrak{X}(M_1)$ and $\omega_2:\mathfrak{X}(M_2)\times\ldots\times\mathfrak{X}(M_2)\longrightarrow\mathfrak{X}(M_2)$ are said to be h-*consistent* if for any $X_1,\ldots,X_k\in\mathfrak{X}^{\Delta_1}(M_1)$ and $Y_1,\ldots,Y_k\in\mathfrak{X}^{\Delta_2}(M_2)$ satisfying $h_*(X_i|\Delta_1)=Y_i|\Delta_2$ for $i=1,\ldots,k$, the following conditions are satisfied:

$$\omega_1(X_1,\ldots,X_k)\in\mathfrak{X}^{\Delta_1}(M_1),\ \omega_2(Y_1,\ldots,Y_k)\in\mathfrak{X}^{\Delta_2}(M_2)\ \text{and}$$

$$h_*(\omega_1(X_1,\ldots,X_k)|\Delta_1)=\omega_2(Y_1,\ldots,Y_k)|\Delta_2.$$

For any h-consistent tensor fields ω_1 and ω_2 of the type (k,1) let ω be the tensor field on $M_1\cup_h M_2$ defined by

$$\omega(X_1\cup_h Y_1,\ldots,X_k\cup_h Y_k)=\omega_1(X_1,\ldots,X_k)\cup_h\omega_2(Y_1,\ldots,Y_k). \tag{2.13}$$

In the sequel the tensor field ω defined by (2.13) will be denoted by $\omega_1\cup_h\omega_2$.

Analogously to (2.12), for the space satisfying the assumptions of Proposition 2.6, the mapping $(\omega_1,\omega_2)\longmapsto\omega_1\cup_h\omega_2$ is a monomorphism of $C_1\cup_h C_2$ module of all pairs of h-consistent tensor fields of type (k,1) into $C_1\cup_h C_2$ module of tensor fields of type (k,1) on the glued space $M_1\cup_h M_2$.

Definition 2.3. Linear connection $\overset{1}{\nabla}$ in (M_1,C_1) and $\overset{2}{\nabla}$ in (M_2,C_2) are called h-consistent if for any $X_1,X_2\in\mathfrak{X}^{\Delta_1}(M_1)$ and $Y_1,Y_2\in\mathfrak{X}^{\Delta_2}(M_2)$ such that $h_*(X_i|\Delta_1)=Y_i|\Delta_2$ for $i=1,2$, the following conditions are satisfied :

$$\overset{1}{\nabla}_{X_1}X_2\in\mathfrak{X}^{\Delta_1}(M_1),\ \overset{2}{\nabla}_{Y_1}Y_2\in\mathfrak{X}^{\Delta_2}(M_2)\ \text{and}\ h_*(\overset{1}{\nabla}_{X_1}X_2|\Delta_1)=\overset{2}{\nabla}_{Y_1}Y_2|\Delta_2.$$

Now it is easy to prove

Proposition 2.7. Let (M_1,C_1), (M_2,C_2) be disjoint differential spaces and $h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism. For any linear connection $\overset{1}{\nabla}$ in (M_1,C_1) and $\overset{2}{\nabla}$ in (M_2,C_2) if they are h-consistent, then the mapping $\nabla:\mathcal{X}(M_1\cup_h M_2)\times\mathcal{X}(M_1\cup_h M_2) \longrightarrow \mathcal{X}(M_1\cup_h M_2)$ defined by

$$\nabla_X Y=\overset{1}{\nabla}_{X_1}Y_1\cup_h\overset{2}{\nabla}_{X_2}Y_2 \tag{2.14}$$

for $X,Y\in\mathcal{X}(M_1\cup_h M_2)$, where $X=X_1\cup_h X_2$, $Y=Y_1\cup_h Y_2$, is a linear connection in the glued space $(M_1\cup_h M_2,C_1\cup_h C_2)$.

Proof is a simple consequence of (1.5)-(1.7).

In the sequel the connection ∇ given by (2.14) corresponding to $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ will be denoted by $\overset{1}{\nabla}\cup_h\overset{2}{\nabla}$.

Definition 2.4. Let (M_1,C_1), (M_2,C_2) be differential spaces of constant differential dimension, let ∇_1 be a linear connection in M_1 and ∇_2 be a linear connection in M_2.

The product connection $\nabla_1\times\nabla_2$ ([6]) in the Cartesian product $M_1\times M_2$ is defined by

$$(\nabla_1\times\nabla_2)_w Y=(i_q)_*(K_1\circ pr_{M_1**}Y_*w)+(i_p)_*(K_2\circ pr_{M_2**}Y_*w) \tag{2.15}$$

for any $w\in T_{(p,q)}(M_1\times M_2)$, $Y\in\mathcal{X}(M_1\times M_2)$, $(p,q)\in M_1\times M_2$, where

$$pr_{M_1}:M_1\times M_2 \longrightarrow M_1,\ pr_{M_2}:M_1\times M_2 \longrightarrow M_2$$

are the natural projections,

$$i_q:M_1 \longrightarrow M_1\times M_2,\ i_p:M_2 \longrightarrow M_1\times M_2$$

are the natural imbedings,

$$K_1:TTM_1 \longrightarrow TM_1\ ,\ K_2:TTM_2 \longrightarrow TM_2$$

are the connection mappings corresponding to ∇_1 and ∇_2 respectively [1], [6].

Now we prove

Proposition 2.8. Let (M_1,C_1), (M_2,C_2), $(Z,\mathcal{Z})$ be differential spaces of constant differential dimension and $(p_1,p_2)\in M_1\times M_2$ an arbitrary point . Let $h:\Delta_1 \longrightarrow \Delta_2$ be a gluing diffeomorphism of $M_1\times Z$ and $M_2\times Z$ defined by (2.8), where $\Delta_1=\{p_1\}\times Z$, $\Delta_1=\{p_2\}\times Z$. Then

(i) For arbitrary connections ∇_1 in M_1, ∇_2 in M_2, ∇_3 in Z, the product connections $\nabla_1\times\nabla_3$ and $\nabla_1\times\nabla_3$ are h-consistent.

(ii) There exists the linear connection $\nabla=\nabla_1\times\nabla_3\cup_h\nabla_2\times\nabla_3$ in the glued space $M_1\times Z\cup_h M_2\times Z$ corresponding to $\nabla_1\times\nabla_3$ and $\nabla_2\times\nabla_3$, by (2.14).

Proof. (i) Let $X_1, Y_1 \in \mathfrak{X}^{\Delta_1}(M_1\times Z)$ and $X_2, Y_2 \in \mathfrak{X}^{\Delta_2}(M_2\times Z)$ such that $h_*(X_1|\Delta_1)=X_2|\Delta_2$ and $h_*(Y_1|\Delta_1)=Y_2|\Delta_2$. There exist vector fields A, $B\in\mathfrak{X}(Z)$ such that $X_i|\Delta_i=(i_{p_i})_*A$ and $Y_i|\Delta_i=(i_{p_i})_*B$, for i=1,2.

Now, for a fixed $z\in Z$, let us put $w_i=X_i(p_i,z)$, i=1,2. Of course, $w_i=((i_{p_i})_*A)(z)$.

It is easy to verify the following equalities :

(2.16) $$\pi_{M_i **}Y_{i*}w_i=0 \text{ for } i=1,2,$$

(2.17) $$\overset{i}{\pi}_{Z\,**}Y_{i*}w_i=B_{*z}(A(z)) \text{ for } i=1,2.$$

Hence from (2.15) we obtain:

$$(\nabla_i\times\nabla_3)_{w_i}Y_i=(i_{(i,z)})_*K_3\circ\overset{i}{\pi}_{Z**}Y_{i*}w_i=i_{(i,z)}K_3(B_{*z}(A(z))),$$

for $z\in Z$, i=1,2, where K_3 is the connection mapping corresponding to ∇_3, $i_{(i,z)}:Z\longrightarrow M_i\times Z$ is the embedding

$$i_{(i,z)}(z)=(p_i,z),\ i=1,2.$$

Hence it is evident that $(\nabla_i\times\nabla_3)_{w_i}Y_i$ is tangent to Δ_i, for i=1,2, and

$$h_*((\nabla_1\times\nabla_3)_{X_1}Y_1|\Delta_1)=(\nabla_2\times\nabla_3)_{X_2}Y_2|\Delta_2.$$

(ii) is a consequence of (i) and Proposition 2.7.

Proposition 2.9. Let (M_1,C_1), (M_2,C_2), $(Z,\mathcal{Z})$ be differential spaces of constant differential dimension and $(p_1,p_2)\in M_1\times M_2$ an arbitrary point. Let $h:\Delta_1\longrightarrow\Delta_2$ be a gluing diffeomorphism of $M_1\times Z$ and $M_2\times Z$ defined by (2.8).

Then

(i) For any semi-Riemannian metrics g_1 on M_1, g_2 on M_2, g_3 on Z, the semi-Riemannian metrics $\eta_1=\pi_{M_1}^{*}g_1+\overset{1}{\pi}{}_Z^{*}g_3$ and $\eta_2=\pi_{M_2}^{*}g_2+\overset{2}{\pi}{}_Z^{*}g_3$ are h-consistent.

(ii) The Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ corresponding to metrics η_1 and η_2, respectively, are h-consistent.

(iii) The glued connection $\nabla = \overset{1}{\nabla} \cup_h \overset{2}{\nabla}$, with respect to the glued metric $\eta = \eta_1 \cup_h \eta_2$, satisfies the well-known condition:

$$Z\eta(X,Y) = \eta(\nabla_Z X, Y) + \eta(X, \nabla_Z Y),$$

$$\nabla_X Y = \nabla_Y X + [X,Y].$$

Sketch of the proof. (i) It is easy to observe that $i_{p_1}{}^* \eta_1 = g_3$ and $i_{p_2}{}^* \eta_2 = g_3$, where $i_{p_i} : Z \longrightarrow M_i \times Z$, i=1,2, are the natural embeddings. Thus it is clear that $i_{\Delta_1}{}^* \eta_1 = h^* \eta_2$.

(ii) Let ∇_1, ∇_2, ∇_3 be the Levi-Civita connection corresponding to the metrics g_1, g_2, g_3, respectively. By using local vector basis in the Cartesian product, the definition of the Levi-Civita connection and (2.15) one can verify that $\overset{i}{\nabla} = \nabla_i \times \nabla_3$ for i=1,2.

Now, from Proposition 2.8 it follows that $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are h-consistent.

The proof of (iii) is a simple verification. (iii) is a simple consequence of the similar properties of the Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ and the equalities (1.5)-(1.7).

3. Some comments on applications

a) Gluing of Robertson-Walker spacetimes.

Let S be a connected three dimensional Riemannian manifold of constant curvature k=-1,0 or 1 with the metric tensor g. Let I_1 and I_2 be intervals in $\mathbb{R}$ and let $f_1>0$, $f_2>0$ be smooth functions on I_1 and I_2, respectively. Consider the Robertson-Walker spacetimes $I_1 \times_{f_1} S$ and $I_2 \times_{f_2} S$ [8]. By definition the Robertson-Walker spacetime $I_i \times_{f_i} S$, i=1,2, is the product space $I_i \times S$ furnished with the metric tensor

(3.1) $$\eta_i = -d\pi_{I_i} \otimes d\pi_{I_i} + (f_i \circ \pi_{I_i})^2 \pi_S^* g, \text{ for } i=1,2.$$

Let $t_1 \in I_1$ and $t_2 \in I_2$ be arbitrary points. Let $h: \Delta_1 \longrightarrow \Delta_2$ be the gluing diffeomorphism given by

(3.2) $$h(t_1,p)=(t_2,p) \text{ for } p\in S,$$
where $\Delta_1=\{(t_1,p):p\in S\}$ and $\Delta_2=\{(t_2,p):p\in S\}$.
Consider the glued space $I_1\times S\cup_h I_2\times S$. It is evident, that if the scale functions f_1 and f_2 satisfy the condition $f_1(t_1)=f_2(t_2)$, then the metrics η_1 and η_2 are h-consistent. In view of Proposition 2.6 one can glue η_1 and η_2. The glued 2-form $\eta=\eta_1\cup_h\eta_2$ is a semi-Riemannian metric on $I_1\times S\cup_h I_2\times S$ of the signature (- - + + +) in Δ.

Let ∇_i, i=1,2, be the Levi-Civita connection of $I_i\times S$ corresponding to η_i.

Now we prove

Proposition 3.1. Let $I_i\times_{f_i} S$, i=1,2, be Robertson-Walker spacetimes with $f_1(t_1)=f_2(t_2)$, for some $t_1\in I_1$, $t_2\in I_2$.

Then

(i) If $f_1'(t_1)=f_2'(t_2)=0$, the Levi-Civita connections are h-consistent,

(ii) the glued connection $\nabla=\nabla_1\cup_h\nabla_2$ is the Levi-Civita connection corresponding to η and the Riemann curvature tensors R of ∇ and R_i of ∇_i satisfy $R=R_1\cup_h R_2$,

(iii) if additionally $f_1''(t_1)=f_2''(t_2)$, then the Ricci curvature of η is the gluing of respective curvatures of η_1 and η_2.

Sketch of the proof. (i) Let $X_1,Y_1\in\mathfrak{X}^{\Delta_1}(I_1\times S)$ and $X_2,Y_2\in\mathfrak{X}^{\Delta_2}(I_2\times S)$ be vector fields such that X_1 is h-consistent with X_2 and Y_1 is h-consistent with Y_2. There exist vector fields $A,B\in\mathfrak{X}(S)$ such that $X_i|\Delta_i=(i_{t_i})_*A$ and $Y_i|\Delta_i=(i_{t_i})_*B$, for i=1,2, where $i_{t_i}:S\longrightarrow I_i\times S$ is the natural embedding. Let $\bar{A},\bar{B}$ be the lifts of A and B to $I_1\times S$. Consider Δ_1 as a semi-Riemannian submanifold of $I_1\times S$. From Lemma 1.4 in [8] it follows that

$$(\nabla_{1X_1}Y_1)(t_1,p)=(\nabla_{1\bar{A}}\bar{B})(t_1,p),$$

for every $p\in S$. By Corollary 8.12 in [8], the normal component of $\nabla_{1\bar{A}}\bar{B}$ has the following form:

$$\text{nor}\nabla_{1\bar{A}}\bar{B}=\eta_1(A,B)\cdot\frac{f'}{f}\cdot\partial_t,$$

where ∂_t is the lift of $\frac{\partial}{\partial t}$ on I_1 to $I_1\times S$.

Now it is clear that if $f'(t_1)=0$, then $\text{nor}\nabla_{1\bar{A}}\bar{B}(t_1,p)=0$, for $p\in S$. Thus

$$(\nabla_{1X_1}Y_1)(t_1,p)=\tan(\nabla_{1\bar{A}}\bar{B})(t_1,p)=(\nabla'_{\bar{A}|\Delta_1}\bar{B}|\Delta_1)(t_1,p),$$

where $\tan(\nabla_{1\bar{A}}\bar{B})$ denotes the tangent component of $(\nabla_{1\bar{A}}\bar{B})$ (see [8]) and ∇' is the Levi-Civita connection of the submanifold Δ. Since the embedding $i_{t_1}:S\longrightarrow\Delta_1$ is a homothety, i_{t_1} preserves the Levi-Civita connections ([8]). Thus $\nabla'_{\bar{A}|\Delta_1}\bar{B}|\Delta_1=(i_{t_1})_*(\nabla_{0A}B)$, where ∇_0 is the Levi-Civita connection of S. Therefore $(\nabla_{1X_1}Y_1)|\Delta_1=(i_{t_1})(\nabla_{0A}B)$. Analogously one can prove the equality $(\nabla_{2X_2}Y_2)|\Delta_2=$ $=(i_{t_2})_*(\nabla_{0A}B)$. This shows that ∇_1 and ∇_2 are h-consistent.

(ii) The Levi-Civita properties of the glued connection ∇ are simple consequence of the respective properties of ∇_1 and ∇_2.

Now we will prove that R_1 and R_2 are h-consistent. Let $X_1, Y_1, Z_1\in\mathfrak{X}^{\Delta_1}(I_1\times S)$ and $X_2, Y_2, Z_2\in\mathfrak{X}^{\Delta_2}(I_2\times S)$ be h-consistent. There exist vector fields A, B, $C\in\mathfrak{X}(S)$ such that $X_i|\Delta_i=(i_{t_i})_*A$, $Y_i|\Delta_i=(i_{t_i})_*B$, $Z_i|\Delta_i=(i_{t_i})_*C$, for $i=1,2$.

From Corollary 9.12 in [8] it follows that

$$R_1(X_1,Y_1)Z_1|\Delta_1=\left[\left(\frac{f_1'}{f_1}\right)^2+\frac{k}{f_1{}^2}\right][\eta_1(\bar{A},\bar{C})\bar{B}-\eta_1(\bar{B},\bar{C})\bar{A}]|\Delta_1.$$

Hence

$$R_1(X_1,Y_1)Z_1|\Delta_1=[(f_1'(t_1))^2+k][g(A,C)\bar{B}|\Delta_1-g(B,C)\bar{A}|\Delta_1].$$

Analogously one can see the equality

$$R_2(X_2,Y_2)Z_2|\Delta_2=[(f_2'(t_2))^2+k][g(A,C)(i_{t_i})_*B-g(B,C)(i_{t_i})_*A|\Delta_1].$$

Since $f_1'(t_1)=f_2'(t_2)=0$, it is evident that

$$h_*(R_1(X_1,Y_1)Z_1|\Delta_1)=R_2(X_2,Y_2)Z_2|\Delta_2.$$

(iii) By Corollary 10.12 in [8]

$$\mathrm{Ric}_1(X_1,Y_1)|\Delta_1=\mathrm{Ric}_1(\bar{A},\bar{B})|\Delta_1=$$

$$=\left\{\left[2\left(\frac{f_1'}{f_1}\right)^2+\frac{2k}{f_1^{\,2}}+\frac{f_1''}{f_1}\right]\eta_1(\bar{A},\bar{B})\right\}|\Delta_1=$$

$$=\left[2\left(\frac{f_1'(t_1)}{f_1(t_1)}\right)^2+\frac{2k}{f_1(t_1)^2}+\frac{f_1''(t_1)}{f_1(t_1)}\right]f_1(t_1)^2 g(A,B).$$

Thus $\mathrm{Ric}_1(X_1,Y_1)|\Delta_1=[2k+f_1''(t_1)f_1(t_1)]g(A,B)$.

Analogously, $\mathrm{Ric}_2(X_2,Y_2)|\Delta_2=[2k+f_2''(t_2)f_2(t_2)]g(A,B)$.
Now it is clear that Ric_1 and Ric_2 are h-consistent and $\mathrm{Ric}=\mathrm{Ric}_1\cup_h\mathrm{Ric}_2$.

b) Gluing of Friedman cosmological models.

Recall that a Friedman cosmological model is a Robertson-Walker spacetime such that the scale function f satisfies the Friedman equation $f'^{\,2}+k=\frac{A}{f}$, where $A=\frac{8\pi M}{3}$ and $H=\frac{f'}{f}$ is positive, for some t_0 [8].

Now, let $I_1=(-\infty,0]$ and $I_2=[0,\infty)$. Assume that $f_1:I_1\longrightarrow\mathbb{R}$ and $f_2:I_2\longrightarrow\mathbb{R}$ are continuous functions satisfying the Friedman equation. Consider Friedman models $I_1\times_{f_1}S$ and $I_2\times_{f_2}S$. Of course, the metrics η_1 and η_2, given by (3.1), degenerate on $\Delta_1=\{(0,p):p\in S\}\subset I_1\times S$ and $\Delta_2=\{(0,p):p\in S\}\subset I_2\times S$, respectively. Let $h:\Delta_1\longrightarrow\Delta_2$ be the gluing diffeomorphism given by (3.1), for $t_1=0$ and $t_2=0$. The glued 2-form η is degenerate on the set of singular points Δ. Since $\lim_{t\to0^-}f_1'(t)=\infty$, $\lim_{t\to0^+}f_2'(t)=\infty$, all points of Δ are the curvature singularities in the classification scheme by Ellis and Schmidt [2] as it can be easily seen from Corollary 9.12 and 9.13 in [8].

Let $t_1\in\mathrm{Int}I_1$ and $t_2\in\mathrm{Int}I_2$ be such elements that $f_1(t_1)=f_2(t_2)$. If we glue together the Friedman models $I_1\times_{f_1}S$ and $I_2\times_{f_2}S$ by the gluing diffeomorphism h given by (3.2), we obtain the set Δ of quasi-regular singularities [2]. This is a consequence of the fact that $I_1\times_{f_1}S$ and $I_2\times_{f_2}S$ may be embedded into the glued space by $\hat{\iota}_1$ and $\hat{\iota}_2$. The geometry of the glued space $I_1 \times S \cup_h I_2 \times S$ is determined by the geometry of $I_1 \times_{f_1} S$ and $I_2 \times_{f_2} S$.

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REFERENCES

[1] **K.Cegiełka:** A connection in a differential module, Colloq. Math. 34 (1976), 271-286.

[2] **G.F.R. Ellis, B.G. Schmidt:** Singular space-times, GRG Vol.8, No. 11 (1977), 915-953.

[3] **S.W. Hawking, G.R.R. Ellis:** The Large Scale structure of space-time, Cambridge U.P. 1973.

[4] **M. Heller, P. Multarzyński, W. Sasin:** The algebraic approach to space-time geometry, Acta Cosmol. 16 (1989), 53-85.

[5] **M. Heller, W. Sasin:** Regular singularities in space-time, Acta Cosmol. 17 (1990), 7-18.

[6] **D. Gromoll, W. Klingerberg, W. Meyer:** Riemannsche Geometrie im Grossen, Springer-Verlag, Berlin, 1968.

[7] **A. Kowalczyk:** Tangent differential spaces and smooth forms, Demonstratio Math., 13 (1980) 893-905.

[8] **B. O'Neill:** Semi-riemannian manifolds, Academic Press, New York 1981.

[9] **W. Sasin:** On some exterior algebra of the differential forms over a differential space, Demonstratio Math. 19 (1986), 1063-1075.

[10] **W. Sasin, Z. Żekanowski** :On locally finitely generated differential space, Demonstratio Math. 20 (1987), 477-487.

[11] **W. Sasin:** On equivalence relation on a differential space, Praha, Czechoslovakia CMUC 29, 3 (1988), 529-539.

[12] **W. Sasin:** On locally countably generated differential spaces, Demonstratio Math. 21 (1988), 897-912.

[13] **W. Sasin, Z. Żekanowski:** Some relation between almost symplectic, pseudo-riemannian and almost product structures on differential spaces, Demonstratio Math. 21 (1988), 1139-1152.

[14] **W. Sasin:** The de Rham cohomology of differential spaces, Demonstratio Math. 22 (1989), 249-270.

[15] **W. Sasin:** Infinite Cartesian product of differential groups, Math. Nachr. 149 (1990), 61-70.

[16] **W. Sasin:** The wedge product of differential spaces, Proc. Winter School on Geometry and Physics, Srni, 6-13 January, 1990, Supplemento ai Rendiconti del Circolo Matematicodi Palermo, in press.

[17] **W. Sasin:** Gluing of differential spaces , Demonstratio Math.25 (1992), in press.

[18] **W. Sasin, K. Spallek:** Gluing of differentiable spaces and applications, to be published.

[19] **R. Sikorski:** Abstract covariant derivative, Colloq. Math. 18 (1967), 251-272.

[20] **R. Sikorski:** Differential modules, Colloq. Math. 24 (1971), 45-70.

[21] **R. Sikorski:** Wstęp do geometrii różniczkowej, PWN, Warszawa, 1972.

[22] **K.Spallek:** Differenzierbare Räume, Math. Ann. 180 (1969),269-296.

[23] **S. Sternberg:** Lectures on differential geometry, Prentice Hall, Inc. Englewood Cliffs, N.J. 1964.

[24] **P. Walczak** : On a class of differential spaces satisfying the theorem on diffeomorphism, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 22 (1974), 805-814.

[25] **W. Waliszewski:** On a coregular division of a differential space by an equivalence relation, Colloq. Math. 26 (1972), 281-291.

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