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DIFFERENTIAL SPACES AND SINGULARITIES IN DIFFERENTIAL SPACE-TIMES

In this paper we investigate singularities of space-time using of the theory differential spaces in the sense of Sikorski [12], [13], [14]. If space-time is modeled by a differential space rather than by a differential manifold, space-time singularities can be regarded as points of the differential space in question. The theory of differential spaces opens some possibilities to classify singularities of space-times [1], [3]. In Section 3 we present such a classification. The differential space methods turns out to be a very efficient tool in dealing with the classical singularity problems [3], [6].

In Section 1 we recall necessary definitions and theorems from the theory of differential spaces. In Section 2 we describe some properties of functions and forms of class C^k on a differential space, which are very important in the next sections.

1. Preliminaries

Let M be a non-empty set and C a set of real functions defined on M. Denote by τ_C the weakest topology on M in which all functions from C are continuous. Let scC be the set of all real functions on M of the form $\omega\circ(f_1,\dots,f_n)$, where $\omega \in \varepsilon_n$, $f_1,\dots,f_n \in C$, $n \in \mathbb{N}$ and ε_n is the set of all real C^∞ functions on $\mathbb{R}^n$. For any subset $A \subset M$ we denote by C_A the set of all real functions f on A such that for any point p of A there exist in τ_C an open neighborhood $U \in \tau_C$ of p and a function $g \in C$ such that $f|A \cap U = g|A \cap U$.

The set C is called the differential structure on M iff $C = scC = C_M$. Then the pair (M,C) is said to be the differential space [14], [15]. It is easy to see that C is a linear ring over $\mathbb{R}$.

A differential structure C on M is said to be generated by a set C_0 of real functions on M if $C = (scC_0)_M$. A differential space (M,C) is said to be finitely generated by a set $C_0 = \{f_1,\dots,f_n\}$ if $C = (scC_0)_M$. If (M,C) is a differential space and A is an arbitrary non-empty subset of M, then (A,C_A) is also a differential space, which is called a differential subspace of (M,C).

Let (M,C) and (N,D) be differential spaces. A mapping $F: M \longrightarrow N$ is said to be a smooth mapping of (M,C) into (N,D) if $f \circ F \in C$ for any $f \in D$. Then we write $F: (M,C) \longrightarrow (N,D)$ [15].

We define the notion of a tangent vector to a differential space (M,C) at a point $p \in M$ as a linear mapping $v: C \longrightarrow R$ satisfying the following condition:

$$v(f\cdot g) = f(p)\cdot v(g) + g(p)\cdot v(f) \quad \text{for any } f,g \in C.$$

The set of all tangent vectors to (M,C) at a point $p \in M$ we denote by $T_p(M,C)$ (shortly T_pM) and call the tangent space to (M,C) at p.

If $F: (M,C) \longrightarrow (N,D)$ is a smooth mapping between differential spaces then for each point $p \in M$ the mapping $F_{*p}: T_pM \longrightarrow T_{f(p)}N$ defined by

$$(F_{*p}v)(f) = v(f\circ F) \quad \text{for any } f \in D \text{ and } v \in T_pM,$$

is a linear mapping.

Let $TM := \bigcup_{p\in M} T_pM$ be a disjoint sum of tangent spaces to (M,C). By TC we denote the differential structure on TM [10] generated by the set $\{f\circ\pi : f\in C\} \cup \{df: f\in C\}$, where $\pi: TM \longrightarrow M$ is defined by the formula

$$\pi(v) = p \quad \text{for any } v \in T_pM \text{ and } p \in M,$$

and $df\colon TM \longrightarrow \mathbb{R}$ is the function defined by

$$(df)(v) = v(f) \quad \text{for } v \in TM.$$

A smooth vector field tangent to (M,C) is a mapping $X\colon (M,C) \longrightarrow (TM,TC)$ such that $\pi \circ X = id_M$. Denote by $\mathfrak{X}(M)$ the C-module of all smooth vector fields tangent to (M,C).

A differential space (M,C) is said to be of constant differential dimension n if for any $p \in M$ there exist a neighborhood $U \in \tau_C$ of p and smooth vector fields $X_1,\dots,X_n \in \mathfrak{X}(U)$ such that for any $q \in U$ the sequence $X_1(q),\dots,X_n(q)$ is a vector basis of $T_q(M,C)$ and $X_1,\dots,X_n$ is a C_U-basis of C_U-module $\mathfrak{X}(U)$.

Now let us put [1]

$$T^rM = \left\{(v_1,\dots,v_r) \in TM\times\dots\times TM\colon \pi(v_1)=\dots=\pi(v_r)\right\}$$

as well as

$$T^rC = (TC\times\dots\times TC)_{T^rM} \quad \text{for } r = 1,2,\dots .$$

Let $\pi_i\colon T^rM \longrightarrow TM$, for $i = 1,\dots,r$ be the mapping defined by

$$\pi_i(v_1,\dots,v_r) = v_i \quad \text{for } (v_1,\dots,v_r) \in T^rM.$$

A function $\omega\colon T^rM \longrightarrow \mathbb{R}$ is said to be the r-form on (M,C) if the mapping $\omega_p := \omega|T_pM\times\dots\times T_pM$ is r-linear for any $p \in M$. An r-form ω is called smooth if $\omega \in T^rC$.

For any mapping $F\colon (M,C) \longrightarrow (N,D)$ and a smooth r-form ω on (N,D) $F^*\omega$ is the smooth r-form defined by

$$(F^*\omega)(v_1,\dots,v_r) = \omega(F_*v_1,\dots,F_*v_r) \quad \text{for any } (v_1,\dots,v_r) \in T^rM.$$

Now we recall some properties of the Cartesian product of differential spaces.

Let (M,C) and (N,D) be differential spaces. Let $C \times D$ be the differential structure on $M \times N$ generated by the set of real functions $\{\alpha\circ pr_1\colon \alpha \in C\} \cup \{\beta\circ pr_2\colon \beta \in D\}$, where $pr_1\colon M \times N \longrightarrow M$ and $pr_2\colon M \times N \longrightarrow N$ are the projections.

The differential space $(M\times N\ ,\ C\times D)$ is called the Cartesian product of differential spaces (M,C) and (N,D) [15].

For an arbitrary point $p \in M$ let $j_p: N \longrightarrow M\times N$ be the imbedding given by

(1.1) $$j_p(q) = (p,q) \quad \text{for } q \in N.$$

For an arbitrary point $q \in N$ let $j_q: M \longrightarrow M\times N$ be the imbedding defined by

(1.2) $$j_q(p) = (p,q) \quad \text{for } p \in M.$$

A vector $w \in T_{(p,q)}(M\times N)$ is said to be parallel to (M,C) if $(pr_2)_* w = 0$. A vector $w \in T_{(p,q)}(M\times N)$ is said to be parallel to (N,D) if $(pr_1)_* w = 0$.

It is easy to see that the subspace $(j_q)_{*p}(T_pM)$ is the set of all vectors tangent to $(M\times N\ ,\ C\times D)$ at (p,q) parallel to (M,C) and the subspace $(j_p)_{*q}(T_qN)$ is the set of all vectors tangent to $(M\times N\ ,\ C\times D)$ at (p,q) parallel to (N,D). One can prove [15], that the tangent space $T_{(p,q)}(M\times N)$ is a direct sum of the subspaces $(j_q)_{*p}(T_pM)$ and $(j_p)_{*q}(T_qN)$.

It is easy to prove

Lemma 1.1. Let w_1, w_2 be vectors parallel to (M,C) and z_1, z_2 be vectors parallel to (N,D). Then

(a) $w_1 = w_2$ iff $(pr_1)_* w_1 = (pr_1)_* w_2$,

(b) $z_1 = z_2$ iff $(pr_2)_* w_1 = (pr_2)_* w_2$.

A vector field $Z \in \mathfrak{X}(M\times N)$ is said to be parallel to (M,C) if $Z(p,q)$ is parallel to (M,C) for every $(p,q) \in M\times N$. We denote by $\mathfrak{X}_M(M\times N)$ the set of all smooth vector fields tangent to $(M\times N\ ,\ C\times D)$. $\mathfrak{X}_M(M\times N)$ is a $C\times D$-submodule of the $C\times D$-module $\mathfrak{X}(M\times N)$.

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A vector $Z \in \mathfrak{X}(M\times N)$ is said to be parallel to (N,D) if

$Z(p,q)$ is parallel to (N,D) for every $(p,q) \in M\times N$. We denote by $\mathcal{X}_N(M\times N)$ the set of all smooth vector fields tangent to $(M\times N , C\times D)$ which are parallel to (N,D). It is clear that $\mathcal{X}_N(M\times N)$ is a $C\times D$-submodule of the $C\times D$-module $\mathcal{X}(M\times N)$.

Now let $X \in \mathcal{X}(M)$ be a smooth vector field tangent to (M,C). Let $\bar{X}: M\times N \longrightarrow T(M\times N)$ be defined by

$$\bar{X}(p,q) = (j_q)_{*p}X(p) \quad \text{for } (p,q) \in M\times N. \tag{1.3}$$

It is easy to verify that $\bar{X} \in \mathcal{X}_M(M\times N)$.
Analogously, for any $Y \in \mathcal{X}(N)$ we can define the vector field $\bar{Y} \in \mathcal{X}(M\times N)$ parallel to (N,D) by the formula

$$\bar{Y}(p,q) = (j_p)_{*q}\, Y(q) \quad \text{for } (p,q) \in M\times N. \tag{1.4}$$

Now, let $Z \in \mathcal{X}(M\times N)$ be an arbitrary vector field tangent to $(M\times N , C\times D)$.

Let us define [15]

$$Z_M(p,q) = (j_q\circ pr_1)_{*(p,q)}Z(p,q) \quad \text{for } (p,q) \in M\times N, \tag{1.5}$$

$$Z_N(p,q) = (j_p\circ pr_2)_{*(p,q)}Z(p,q) \quad \text{for } (p,q) \in M\times N. \tag{1.6}$$

It is easy to see that $Z_M \in \mathcal{X}_M(M\times N)$ and $Z_N \in \mathcal{X}_N(M\times N)$. Moreover, $Z = Z_M + Z_N$.

One can prove [15].

Proposition 1.2. The $C\times D$-module $\mathcal{X}(M\times N)$ is a direct sum of $C\times D$-modules $\mathcal{X}_M(M\times N)$ and $\mathcal{X}_N(M\times N)$.

Now let $X \in \mathcal{X}_M(M\times N)$. For any $q \in N$ let $X^q: M \longrightarrow TM$ be defined by

$$X^q(p) = (pr_1)_{*(p.q)}X(p,q) \quad \text{for } p \in M. \tag{1.7}$$

It is easy to see that $X^q \in \mathcal{X}(M)$ for every $q \in N$.
Analogously, for $Y \in \mathcal{X}_N(M\times N)$ and $p \in M$ let $Y^p: N \longrightarrow TN$ be defined by

$$Y^p(q) = (pr_2)_{*(p,q)}X(p,q) \quad \text{for } q \in N. \tag{1.8}$$

One can easily prove that $Y^p \in \mathcal{X}(N)$ for every $p \in M$.

Now we prove

Lemma 1.3. Let (M,C) and (N,D) be differential spaces.

(M,C) is a differential space of differential dimension m if and only if the CxD-module $\mathcal{X}_M(M\times N)$ is an m-dimensional differential module. (N,D) has a differential dimension n if and only if the CxD-module $\mathcal{X}_N(M\times N)$ is an n-dimensional differential module.

Proof. ($\Longrightarrow$) Assume that (M,C) has a differential dimension m. Let (p,q) be an arbitrary point of M×N. Let $V \in \tau_C$ be an open neighbourhood of p such that on V there is a local vector basis $X_1,\dots,X_m \in \mathcal{X}(V)$ of the C-module $\mathcal{X}(M)$. One can verify [15] that the sequence $\bar{X}_1,\dots,\bar{X}_m \in \mathcal{X}(U\times N)$ of vector fields defined by (1.3) is a local vector basis of C×D-module $\mathcal{X}_M(M\times N)$ on $U\times N \in \tau_{C\times D}$.

($\Longleftarrow$) Assume that $\mathcal{X}_M(M\times N)$ is an n-dimensional differential module. $\mathcal{X}_M(M\times N)$ is a CxD-module of Φ-fields, where $\Phi(p,q) = (j_q)_{*p}(T_pM)$ for $(p,q) \in M\times N$.

Since $(j_q)_{*p}\colon T_pM \longrightarrow \Phi(p,q)$ is an isomporphism for every $(p,q) \in M\times N$, $\dim T_pM = \dim \Phi(p,q) = n$ for any $p \in M$. It is enough to show that for an arbitrary vector $u \in TM$ there exist a vector field $X \in \mathcal{X}(M)$ such that $u = X(\pi_M(u))$, where $\pi_M\colon TM \longrightarrow M$ is the projection. Indeed, for the vector $\bar{u} = (j_q)_{*p}u \in \Phi(p,q)$, where $u \in T_pM$, there exists a vector field $Z \in \mathcal{X}_M(M\times N)$ such that $\bar{u} = Z(p,q)$. Hence we have

$$(pr_1)_{*(p,q)}\bar{u} = (pr_1)_{*(p,q)}Z(p,q)$$

or equivalently

$$u = Z^q(p), \quad \text{where } Z^q \in \mathcal{X}(M) \text{ is defined by (1.7).}$$

The second part of Lemma 1.3 can be proved analogously.

Lemma 1.4. Let (M,C) and (N,D) be differential spaces. Then, $\dim T_{(p,q)}(M\times N)$ is constant for any $(p,q) \in M\times N$ if and only if $\dim T_pM$ is constant for any $p \in M$ and $\dim T_qN$ is constant for any $q \in N$.

Proof. This Lemma is a simple consequence of the equality

$$\dim T_{(p,q)}(M\times N) = \dim T_pM + \dim T_qN \quad \text{for any } (p,q) \in M\times N.$$

Now we prove

Proposition 1.5. Let (M,C) and (N,D) be differential spaces. The Cartesian product $(M\times N , C\times D)$ is a differential space of constant differential dimension if and only if (M,C) and (N,D) are differential spaces of constant differential dimension.

Proof. ($\Longrightarrow$) Assume that the Cartesian product $(M\times N , C\times D)$ is a differential space of constant differential dimension. Assume that $\dim T_{p_0} M = m$ and $\dim T_{q_0} N = n$ for certain points $p_0 \in M$ and $q_0 \in N$. In view of Lemma 1.4, $\dim T_pM = m$ for any $p \in M$ and $\dim T_qN = n$ for any $q \in N$. It is enough to prove that every vector tangent to (M,C) or (N,D) is extendible to a smooth vector field tangent to (M,C) or (N,D), respectively.

Let $u \in T_pM$ for a point $p \in M$. Then $\bar{u} = (j_q)_{*p}u \in T_{(p,q)}M\times N)$ is a vector parallel to (M,C). Since $(M\times N , C\times D)$ has constant differential dimension there exist a vector field $Z \in \mathcal{X}(M\times N)$ such that $\bar{u} = Z(p,q)$. It is easy to see that Z_M defined by (1.5) is a smooth tangent vector field parallel to (M,C) such that $\bar{u} = Z_M(p,q)$. Hence we have the equality

$$(pr_1)_{*(p,q)}\ \bar{u} = (pr_1)_{*(p,q)} Z_M(p,q)$$

or equivalently

$$u = (Z_M)^q(p).$$

Thus u is extendible to $(Z_M)^q \in \mathcal{X}(M)$.

($\Longleftarrow$) Let (M,C) and (N,D) be differential spaces of differential dimension m and n, respectively. Let (p,q) be an arbitrary point of $M\times N$. Let $X_1,\dots,X_m \in \mathcal{X}(U)$ be a local vector basis of $\mathcal{X}(M)$ on a neighborhood $U \in \tau_C$ of p and $Y_1,\dots,Y_n \in \mathcal{X}(V)$ be a local vector basis of $\mathcal{X}(N)$ on a neighborhood $V \in \tau_D$ of q. It is easy to see [15] that the sequence $\bar{X}_1,\dots,\bar{X}_m,\bar{Y}_1,\dots,\bar{Y}_n$ of vector fields defined by (1.3) - (1.4) is a local vector basis of the $C\times D$-module $\mathcal{X}(M\times N)$ on a neighborhood $U\times V$ of (p,q). This finishes the proof.

Proposition 1.6. Let $X \in \mathcal{X}_M(N\times N)$ and $Y \in \mathcal{X}_N(M\times N)$. Let $c\colon (-\varepsilon,\varepsilon) \longrightarrow M\times N$ be a smooth mapping such that $c(0) = (p,q)$, where $\varepsilon > 0$. Let us put $c_1 = pr_1 \circ c$ and $c_2 = pr_2 \circ c$.
The mapping c is an integral curve of X if and only if c_1 is an integral curve of X^q and $c_2(t) = q$ for any $t \in (-\varepsilon,\varepsilon)$.
The mapping c is an integral curve of Y if and only if c_2 is an integral curve of Y^p and $c_1(t) = p$ for any $t \in (-\varepsilon,\varepsilon)$.

Proof. ($\Longrightarrow$) Let c be an integral curve of X. Then

$$(1.9)\qquad c_{*t}\,\frac{d}{ds}\Big|_t = X(C(t))$$

for any $t \in (-\varepsilon,\varepsilon)$. Hence

$$(pr_1)_{*c(t)}\left(c_{*t}\,\frac{d}{ds}\Big|_t\right) = (pr_1)_{*c(t)}\left(X(c(t))\right)$$

for any $t \in (-\varepsilon,\varepsilon)$ or equivalently

$$(1.10)\qquad (c_1)_{*t}\,\frac{d}{ds}\Big|_t = X^{c_2(t)}\left(c_1(t)\right)$$

for any $t \in (-\varepsilon,\varepsilon)$. Moreover, from (1.9) it follows that

$$(pr_2)_{*c(t)}\left(c_{*t}\,\frac{d}{ds}\Big|_t\right) = (pr_2)_{*c(t)}\left(X(c(t))\right)$$

for any $t \in (-\varepsilon,\varepsilon)$ or equivalently

$$(1.11)\qquad (c_2)_{*t}\,\frac{d}{ds}\Big|_t = 0.$$

Hence $c_2(t) = q$ for every $t \in (-\varepsilon,\varepsilon)$.
($\Longleftarrow$) Now, let c_1 be an integral curve of X^q and $c_2(t) = q$ for any $t \in (-\varepsilon,\varepsilon)$. Thus

$$(c_1)_{*t}\,\frac{d}{ds}\Big|_t = X^{c_2(t)}\left(c_1(t)\right)$$

for any $t \in (-\varepsilon,\varepsilon)$ or equivalently by (1.7)

$$(pr_1)_{*c(t)}\left(c_{*t}\,\frac{d}{ds}\Big|_t\right) = (pr_1)_{*c(t)}\left(X(c(t))\right)$$

for $t \in (-\varepsilon,\varepsilon)$. It is easy to see that the vector $c_{*t}\,\frac{d}{ds}\Big|_t$ is parallel to (M,C). From Lemma 1.1 it follows that

$$c_{*t}\frac{d}{ds}\Big|_t = X\big(c(t)\big) \quad \text{for } t \in (-\varepsilon,\varepsilon).$$

Analogously one can prove the second part of the proposition.

Proposition 1.7. Let (M,C) and (N,D) be differential spaces. The Cartesian product $(M\times N\ ,\ C\times D)$ is a finitely generated differential space if and only if (M,C) and (N,D) are finitely generated differential spaces .

Proof. ($\Longrightarrow$) Let $C\times D$ be generated by a set $\{\varphi_1,\dots,\varphi_k\}$. Then $(C\times D)_{M\times\{q\}}$ is finitely generated by $\left\{\varphi_1|M\times\{q\},\dots,\varphi_k|M\times\{q\}\right\}$ for every $q \in N$. Since $j_q\colon (M,C) \longrightarrow \big(M\times\{q\}\ ,\ (C\times D)_{M\times\{q\}}\big)$ is a diffeomorphism, C is a differential structure generated by the set $\{\varphi_1 \circ j_q,\dots,\varphi_k \circ j_q\}$ for any arbitrary $q \in N$. Analogously one can prove that for any $p \in M$ the set $\{\varphi_1 \circ j_p,\dots,\varphi_k \circ j_p\}$ generates D.

($\Longleftarrow$) It is easy to see that if C is generated by $\{f_1,\dots,f_m\}$ and D is generated by $\{g_1,\dots,g_m\}$, then $C\times D$ is generated by the set $\{f_1 \circ pr_1,\dots,f_m \circ pr_1\} \cup \{g_1 \circ pr_2,\dots,g_n \circ pr_2\}$.

2. Smooth functions and forms of class C^k on a differential space

Let (M,C) be a differential space. A function $f\colon M \longrightarrow \mathbb{R}$ is said to be of class C^k if for any point $p \in M$ there exist an open neighborhood $V \in \tau_C$ of p and functions $f_1,\dots,f_n \in C$, $\sigma\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^k such that $f|V = \sigma \circ (f_1,\dots,f_n)|V$. It is easy to see that the set $\mathcal{F}^k(M)$ of all real functions on (M,C) of class C^k is a linear ring over $\mathbb{R}$.

One can easily prove

Lemma 2.1. Let (M,C) be a differential space with the differential structure C generated by a set C_0. A real function $f\colon M \longrightarrow \mathbb{R}$ is of class C^k (shortly C^k function) on (M,C) if and only if for $p \in M$ there exist a neighbourhood $U \in \tau_C$ of p and functions $f_1,\dots,f_n \in C_0$, a function $\sigma\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^k, $n \in \mathbb{N}$, such that

$$f|U = \sigma \circ (f_1, \ldots, f_n)|U.$$

Lemma 2.2. Let $\sigma: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^k function. If there exists a point $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ such that

(•) $\sigma(ku) = k\sigma(u)$ for any $k \in \mathbb{R}$,

then

$$\sigma(u) = \sum_{i=1}^{n} \frac{\partial\sigma}{\partial x_i}(0) \cdot u_i \,.$$

Proof. Indeed, $\sigma'_{|u}(0) = \lim_{t \to 0} \frac{\sigma(tu) - \sigma(0)}{t} = \lim_{t \to 0} \frac{t\sigma(u)}{t} = \sigma(u)$.

Hence $\sigma(u) = \sigma'_{|u}(0) = \sum_{i=1}^{n} \frac{\partial\sigma}{\partial x_i}(0) \cdot u_i$.

Definition 2.1. An r-form $\omega: T^rM \longrightarrow \mathbb{R}$ is said to be smooth of class C^k on (M,C) (shortly C^k r-form) if ω is a C^k function on the differential space (T^rM, T^rC).

Proposition 2.3. Let (M,C) be a differential space with the differential structure C generated by a set C_0, $p \in M$ an arbitrary point, $\omega: T^rM \longrightarrow \mathbb{R}$ a smooth r-form of class C^k on (M,C) and $r \le k$.

Then there exist a smooth mapping $F: (M,C) \longrightarrow (\mathbb{R}^n, \varepsilon_n)$ with the coordinates $F_1, \ldots, F_n \in C_0$, $n \in \mathbb{N}$, an r-form $\theta: T^r\mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^k on $(\mathbb{R}^n, \varepsilon_n)$ and an open neighbourhood $V \in \tau_C$ of p such that

$$\omega|\pi_0^{-1}(V) = F^*\theta|\pi_0^{-1}(V),$$

where $\pi_0: T^rM \longrightarrow M$ is the projection $(v_1, \ldots, v_r) \longmapsto p = \pi(v_1) = \ldots = \pi(v_r)$.

Proof. There exist a neighbourhood $V \in \tau_C$ of p and functions $F_1, \ldots, F_n \in C_0$, $n \in \mathbb{N}$ and a C^k function $\sigma: \mathbb{R}^{(r+1)n} \longrightarrow \mathbb{R}$ such that

$$\omega|\pi_0^{-1}(V) =$$
$$= \sigma(F_1 \circ \pi_0, \ldots, F_n \circ \pi_0, dF_1 \circ \pi_1, \ldots dF_n \circ \pi_1, \ldots, dF_1 \circ \pi_r, \ldots$$
$$\ldots, dF_n \circ \pi_r)|\pi_0^{-1}(V).$$

Let $\theta: T^r\mathbb{R}^n \longrightarrow \mathbb{R}$ be the r-form of class C^k defined by

$$(2.1)\ \theta = \sum_{i_1,\dots,i_r=1}^{n} \frac{\partial^r \sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\dots\partial x_{rn+i}} \circ \iota_{n,(r+1)n} dx_{i_1} \otimes \dots \otimes dx_{i_r},$$

where $\iota_{n,(r+1)n} : \mathbb{R}^n \longrightarrow \mathbb{R}^{(r+1)n}$ is given by

$$(2.2)\qquad \iota_{n,(r+1)n}(x_1,\dots,x_n) = (x_1,\dots,x_n,0,\dots,0)$$

for $(x_1,\dots,x_n) \in \mathbb{R}^n$.

We will show that $\omega|\pi_0^{-1}(V) = F^*\theta|\pi_0^{-1}(V)$.

Let us consider the C^k function $\alpha: \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$(2.3)\qquad \alpha(x_1,\dots,x_n) = \sigma\big(F_1(p),\dots,F_n(p),x_1,\dots,x_n,v_2(F_1),\dots \\ \dots,v_2(F_n),\dots,v_r(F_1),\dots,v_r(F_n)\big) \quad \text{for } (x_1,\dots,x_n) \in \mathbb{R}^n.$$

It is easy to observe that for the point $u = \big(v_1(F_1),\dots,v_1(F_n)\big)$ the function α satisfies (•). Thus from Lemma 2.2 it follows that

$$\alpha(u) = \sigma\big(F_1(p),\dots,F_n(p),v_1(F_1),\dots,v_1(F_n),\dots,v_r(F_1),\dots \\ \dots,v_r(F_n)\big) = \sum_{i_1=1}^{n} \frac{\partial\sigma}{\partial x_{n+i_1}}(F_1(p),\dots,F_n(p),0,\dots,0, \\ v_2(F_1),\dots,v_2(F_n),\dots,v_r(F_1),\dots,v_r(F_n)\big)\cdot v_1(F_{i_1}).$$

Now using Lemma 1 (r-1) times, in the similar way one checks that

$$\sigma(F_1(p),\dots,F_n(p),v_1(F_1),\dots,v_1(F_n), \ ,\dots,v_r(F_1),\dots,v_r(F_n)\big) = \\ \sum_{i_1,\dots,i_r=1}^{n} \frac{\partial^r\sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\dots\partial x_{rn+i_r}}(F_1(p),\dots,F_n(p),0,\dots,0)\ \cdot \\ \cdot\ v_1(F_{i_1})\cdot\ldots\cdot v_n(F_{i_n}),$$

or equivalently

$$\omega(v_1,\dots,v_r) = F^*(\theta)(v_1,\dots,v_r)$$

for an arbitrary $(v_1,\dots,v_n) \in \pi_0^{-1}(V)$.

Therefore $\omega|\pi_0^{-1}(V) = F^*\theta|\pi_0^{-1}(V)$.

Corollary 2.4. Let (M,C) be a differential space and $p \in M$ an arbitrary point. If there exists a non-degenerate r-form ω

of class C^k $(r \le k)$ on (M,C), then there is an open neighbourhood $U \in \tau_C$ of p such that (U,C_U) can be immersed in the Euclidean space. Moreover, $\dim T_q(M,C) < +\infty$ for any $q \in M$.

Proof. The mapping $F|U$ in Proposition 2.3 is a smooth immersion. Indeed, since ω is non-degenerate and $\omega|\pi_0^{-1}(U) = F^{*}\theta|\pi_0^{-1}(U)$ for some open set U containing p, F_{*q} is injective for every $q \in U$. Thus
$F_{*q}\colon T_q(M,C) \longrightarrow T_{F(q)}(\mathbb{R}^n,\varepsilon_n)$ is an isomorphism onto the image. Hence

$$\dim T_q(M,C) = \dim F_{*q}\big(T_q(M,C)\big) \le \dim T_{F(q)}(\mathbb{R}^n,\varepsilon_n) = n.$$

Definition 2.2. A smooth C^k 2-form $g\colon T^2M \longrightarrow \mathbb{R}$ on a differential space (M,C) is said to be a C^k Lorentz metric on (M,C) if for any $p \in M$ the 2-form $g_p := g|T_pM \times T_pM$ is symmetric, non-degenerate and g_p has the signature $(\dim T_pM - 1,\ 1)$.

Now we prove

Proposition 2.5. Let (M,C) be a connected differential space of constant differential dimension n and g a symmetric, non-degenerate, smooth 2-form of class C^k on (M,C). If g_p has the signature (k,l) at a certain point $p \in M$, then g has the signature (k,l).

Proof. Assume that the signature of g at a certain point $p \in M$ is equal to (k,l). Let $v_1,\dots,v_n$ be a basis of T_pM such that the matrix $\big(g(v_i,v_j)\big)_{\substack{1\le i\le n\\ 1\le j\le n}}$ has the diagonal form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_k & & & \\ & 0 & & \lambda_{k+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_{k+l} \end{pmatrix},$$

where $\lambda_1,\dots,\lambda_k > 0$ and $\lambda_{k+1},\dots,\lambda_{k+l} < 0$.

Since (M,C) is a differential space of differential dimension n, there exists an open neighborhood $U \in \tau_C$ of p and smooth vector fields $W_1,\dots,W_n \in \mathcal{X}(U)$ such that $W_i(p) = v_i$ for $i=1,\dots,n$, and $W_1(q),\dots,W_n(q)$ is a basis of T_qM for every $q \in U$.

Let $\Delta_i\colon U \longrightarrow \mathbb{R}$ be the smooth function defined by

$$(2.4)\qquad \Delta_i(q) = \begin{vmatrix} g_q(W_1(q),W_1(q))\dots g_q(W_1(q),W_i(q)) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ g_q(W_i(q),W_1(q))\dots g_q(W_i(q),W_i(q)) \end{vmatrix}$$

for $q \in U$, $i=1,\dots,n$.

It is easy to see that $\Delta_i(p) > 0$, for $i = 1,\dots,k$, and $\operatorname{sgn}\Delta_i(q) = (-1)^{i-k}$ for $i = k+1,\dots,n$. In view of Proposition 6.1 in [4], for any $q \in U$ there exists a basis $e_1,\dots,e_n$ of the tangent space T_qM such that

$$(2.5)\qquad g(v,v) = \frac{\Delta_0(q)}{\Delta_1(q)}\,\xi_1^2 + \frac{\Delta_1(q)}{\Delta_2(q)}\,\xi_2^2 + \dots + \frac{\Delta_{n-1}(q)}{\Delta_n(q)}\,\xi_n^2 ,$$

for any $v \in T_qM$, where $v = \sum_{i=1}^{n} \xi_i e_i$.

Let $V \in \tau_C$ be open connected neighborhood of p such that $V \subset U$ and $\Delta_i(q) > 0$ for $q \in V$, $i = 1,\dots,k$ and $\operatorname{sgn}\Delta_i(q) = (-1)^{i-k}$ for $i = k+1,\dots,n$, $q \in V$. Hence from (2.5) it follows that the signature of g is constant on V. Thus the signature of g is locally constant on M. Since (M,τ_C) is a connected topological space, the signature of g is constant on (M,C).

Proposition 2.6. Let (M,C) be a differential space with the differential structure C generated by a set $C_0 = \{f_1,\dots,f_n\}$ and let $p \in M$ be a point such that $\dim T_pM = n$. If $g\colon T^2M \longrightarrow \mathbb{R}$ is a symmetric, non-degenerate C^k 2-form ($k\geq 2$) of signature (k,l) at p, then there exist an open neighbourhood $U \in \tau_C$ of p and a pseudo-Riemannian C^k metric η of signature (k,l) on some open subspace of $(\mathbb{R}^n,\varepsilon_n)$ such that $g|\pi_0^{-1}(U) = F^*\eta|\pi_0^{-1}(U)$,where $F = (f_1,\dots,f_n)$.

Proof. There exist a neighbourhood $V \in \tau_C$ of p and a C^k function $\sigma\colon \mathbb{R}^{3n} \longrightarrow \mathbb{R}$ such that

$g|\pi_0^{-1}(V) = \sigma \circ (f_1 \circ \pi_0, \dots, f_n \circ \pi_0, df_1 \circ \pi_1, \dots, df_n \circ \pi_1, df_1 \circ \pi_2, \dots,$
$df_n \circ \pi_2)|\pi_0^{-1}(V)$.

Let $\eta\colon T^2\mathbb{R}^n \longrightarrow \mathbb{R}$ be the 2-form defined by

$$\eta = \sum_{i,j=1}^{n} \frac{\partial^2\sigma}{\partial x_{n+i}\partial x_{2n+j}} \circ \iota_{n,3n} dx_i \otimes dx_j, \tag{2.6}$$

where $\iota_{n,3n}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$ is the mapping
$(x_1,\dots,x_n) \longmapsto (x_1,\dots,x_n,0,\dots,0)$.
Analogously as in the proof of Proposition 2.3, one can check that $g|\pi_0^{-1}(V) = F^*\eta|\pi_0^{-1}(V)$. It is easy to see that η is a symmetric, non-degenerate 2-form (on U).

There exists an open connected neighbourhood A of F(p) such that $F^{-1}(A) \subset V$ and $\det\left(\frac{\partial^2\sigma}{\partial x_{n+i}\partial x_{n+j}}(\iota_{n,3n}(q))\right) \neq 0$, for $q \in A$.

Since $F_{*p}\colon T_pM \longrightarrow T_{F(p)}\mathbb{R}^n$ is an isomorphism, η has the signature (k,l) at F(p). From Proposition 2.5 it follows that η has the signature (k,l) on (A,ε_{nA}). Now, if we put $U = F^{-1}(A)$, we have $g|\pi_0^{-1}(U) = F^*\eta|\pi_0^{-1}(U)$. This finishes the proof.

Proposition 2.7. Let (M,C) be a differential space of class D_0. If $g\colon T^2M \longrightarrow \mathbb{R}$ is a symmetric, non-degenerate C^k 2-form of the signature (k,l) at a point p, then there exist an open neighbourhood $U \in \tau_C$ of p and a pseudo-Riemannian manifold $(\tilde{M},\tilde{g})$ of dimension $n = \dim T_pM$ such that $\tilde{g}$ is a C^k 2-form of the signature (k,l), $C_U = C^\infty(\tilde{M})_U$ and $g|\pi_0^{-1}(U) = \iota_U^*\tilde{g}$, where $\iota_U\colon U \longrightarrow \tilde{M}$ is the inclusion mapping.

Proof. There exist an open neighbourhood $V \in \tau_C$ of p and a manifold $\bar{M}$ containing V such that $C_V = C^\infty(\bar{M})_V$ [17]. Let $x = (x^1,\dots,x^n)$ be a chart on M defined on V_1 such that $U = V_1 \cap M \subset V$. It is clear that (U,C_U) is a differential space finitely generated by the set $\{x^1|U,\dots,x^n|U\}$. From Proposition 2.6 it follows that there exists a pseudo-Riemannian C^k metric η of the signature (k,l) on some open connected set $W \ni x(p)$ such that

$g|\pi_0^{-1}(x^{-1}(W)) = (X|x^{-1}(W))^*\eta$. Let us put $\tilde{M} = x^{-1}(W)$ and $\tilde{g} = x^*\eta$. Of course, $\tilde{g}$ is a C^k pseudo-Riemannian metric on $\tilde{M}$ of the signature (k,1) and $\tilde{g}|\pi_0^{-1}(U) = \iota_U^*\tilde{g}$, where $U = \tilde{M}$.

Now we prove

Lemma 2.8. Let (M,C) be a differential space and a subset $A \subset M$. A real function $f\colon A \longrightarrow \mathbb{R}$ is smooth of class C^k on (A,C_A) if and only if, for any point $p \in A$, there is a neighbourhood $U \in \tau_C|A$ and a function $g\colon M \longrightarrow \mathbb{R}$ smooth of class C^k on (M,C) such that $f|U = g|U$.

Proof. ($\Longrightarrow$) Let $f\colon A \longrightarrow \mathbb{R}$ be a smooth function of class C^k on (A,C_A) and $p \in M$ an arbitrary point. There exist a neighbourhood $W \in \tau_C|A$ and functions $f_1,\dots,f_n \in C_A$, $\sigma\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^k such that $f|W = \sigma\circ(f_1,\dots,f_n)|W$. There is a neighbourhood $W_1 \in \tau_C|A$ of p and functions $g_1,\dots,g_n \in C$ such that $f_i|W_1 = g_i|W_1$ for $i=1,\dots,n$. Of course, the composition $g = \sigma\circ(g_1,\dots,g_n)$ is of class C^k on (M,C) and $f|U = g|U$, where $U = W_1 \cap W$.
($\Longleftarrow$) Now, let $f\colon A \longrightarrow \mathbb{R}$ be a real function such that, for any $p \in A$, there exist an open neighbourhood $U \in \tau_C|A$ of p and a function $g\colon M \longrightarrow \mathbb{R}$ smooth of class C^k on (M,C) and $f|U = g|U$. We will show that f is smooth of class C^k on (A,C_A). For any point $q \in A \subset M$, there exist a neighbourhood $W \in \tau_C$ of q and functions $g_1,\dots,g_n \in C$, $\sigma\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^k such that $g|W = \sigma\circ(g_1,\dots,g_n)|W$. It is easy to see that

$$f|W \cap U = \sigma \circ (g_1|A,\dots,g_n|A)|W \cap U.$$

This proves that f is smooth of class C^k on (A,C_A).

Lemma 2.9. Let $F\colon (M,C) \longrightarrow (N,D)$ be a smooth mapping between differential spaces. If $f\colon N \longrightarrow \mathbb{R}$ is a smooth function of class C^k on (N,D), then the function $F\circ f$ is smooth of class C^k on (M,C).

Proof. Let f be a smooth real function of class C^k on (N,D) and $p \in M$ be an arbitrary point. There exist a neighborhood $V \ni f(p)$ open in τ_D and functions $f_1,\dots,f_n \in D$, $\sigma: \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^k such that

$$f|U = \sigma \circ (f_1,\dots,f_n)|U.$$

Then $f\circ F|F^{-1}(V) = \sigma\circ(f_1\circ F,\dots,f_n\circ F)|F^{-1}(V)$. Since $F^{-1}(V) \ni p$ is open in τ_C and $f_i\circ F \in C$, for $i = 1,\dots,n$, $f\circ F$ is smooth of class C^k on (M,C).

Lemma 2.10. Let (M,C) be a differential space of constant differential dimension n. Then a 2-form $g: T^2M \longrightarrow \mathbb{R}$ is smooth of class C^k on (M,C) if and only if for any local vector basis $W_1,\dots,W_n \in \mathfrak{X}(U)$ on $U \in \tau_C$ the coordinates $g_{ij} = g \circ (W_i,W_j)$, $i,j=1,\dots,n$, are smooth functions of class C^k on (U,C_U).

Proof. If $g: T^2M \longrightarrow \mathbb{R}$ is smooth of class C^k on (M,C) then in view of Lemma 2.9 the composition $g\circ(W_i,W_j)$ is smooth of class C^k on (U,C_U). Conversely, if the coordinates g_{ij}, $i,j=1,\dots,n$, of g with respect to a local vector basis $W_1,\dots,W_n$ on $U \in \tau_C$, are smooth functions of class C^k on (U,C_U), then evidently $g|\pi_0^{-1}(U) = \sum_{i,j=1}^{n} g_{ij}\circ \pi\cdot W_i^* \otimes W_j^*$ is smooth of class C^k on (U,C_U). There exists an open covering $\mathcal{U}$ of (M,τ_C) such that $g|\pi_0^{-1}(V)$ is smooth of class C^k for any $V \in \mathcal{U}$. This proves that g is smooth of class C^k on (M,C).

Definition 2.3. Let $F: M \longrightarrow N$ be a mapping from a differential space (M,C) into a differential space (N,D). F is said to be a smooth mapping of class C^k from (M,C) into (N,D) if $F^*(\mathcal{F}^k(N)) \subset \mathcal{F}^k(M)$.

It is easy to see that $f \in \mathcal{F}^k(M)$ iff a mapping $f: M \longrightarrow \mathbb{R}$ is smooth mapping of class C^k from (M,C) into $(\mathbb{R},\varepsilon)$.

It is easy to prove

Lemma 2.11. Let $F: (M,C) \longrightarrow (N,D)$ be a smooth mapping between differential spaces. If $f: N \longrightarrow \mathbb{R}$ is smooth of class C^k on (N,D), then $f\circ F$ is smooth of class C^k on (M,C). Moreover

F is a smooth mapping of class C^k from (M,C) into (N,D) for $k = 1,2,\ldots$.

Definition 2.4. A vector field X tangent to (M,C) is said to be smooth of class C^k if $X: M \longrightarrow TM$ is a smooth mapping of class C^k from (M,C) into (TM, TC).

Let $\mathfrak{X}^r(M)$ be the $\mathcal{F}^r(M)$-module of all smooth vector fields of class C^r tangent to (M,C).

One can easily prove

Lemma 2.12. Let (M,C) be a differential space of constant differential dimension n. Then a vector field X tangent to (M,C) is smooth of class C^k on (M,C) if and only if for any local vector basis $W_1,\ldots,W_n \in \mathfrak{X}(U)$ on $U \in \tau_C$, the coordinates $\varphi_i = W_i^* \circ (X|U)$, $i=1,\ldots,n$, of X are smooth functions of class C^k on (U,C_U).

Now let $N \subset \mathbb{R}^n$ be a subset. Consider the differential space (N,D), where $D := (\varepsilon_n)_N$. Denote by $\mathcal{F}^r(N)$ the linear ring of all smooth real functions of class C^r on (N,D).

Let us put $0^r(N) = \{f \in \mathcal{F}^r(\mathbb{R}^n) : f|N \equiv 0\}$.

Let $p \in N$ be an arbitrary point. Let us consider the following linear subspaces of $\mathbb{R}^n$:

$$N_p^r := \{h \in \mathbb{R}^n\colon f'_{|h}(p) = 0 \quad \text{for any} \quad f \in 0^r(N)\},$$
$$G_p^r := \{(\mathrm{grad}\, f)(p)\colon \quad f \in 0^r(N)\}.$$

Proposition 2.13. $G_p^r \oplus N_p^r = \mathbb{R}^n$ and G_p^r is orthogonal to N_p^r with respect to the standard metric on $\mathbb{R}^n$.

Proof. It is easy to see that

$$N_p^r = \{h \in \mathbb{R}^n\colon (\mathrm{grad} f)(p)\cdot h = 0 \quad \text{for any } f \in 0^r(N)\} = G_p^{\perp}.$$

Since the standard metric is non-degenerate, $G_p^r \oplus N_p^r = \mathbb{R}^n$.

Corollary 2.14. The following conditions are equivalent:

(i) $\dim N_p^r = n$,

(ii) $f'_{|h}(p) = 0$ for any $f \in 0^r(N)$ and $h \in \mathbb{R}^n$.

Proof. From Proposition 2.13 it follows that $\dim N_p^r = n$

iff $\dim G_p^r = 0$. It is clear that $G_p = 0$ iff $(\mathrm{grad} f)(p) = 0$ for any $f \in \mathcal{O}^r(N)$. This is equivalent to (ii).

Example 2.1. Let $N = \{(t, t^{\frac{5}{3}}) : t \in \mathbb{R}\}$. Of course N is the graph of a C^1 function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $x \longmapsto x^{\frac{5}{3}}$. From Proposition 2.13 it follows that $\dim N_p^1 = 1$, for any $p \in N$. $N \subset \mathbb{R}^2$ is a subspace such that $\dim N_p^1 = 1 < 2$. Let $\alpha: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the C^1 function defined by

$$\alpha(x,y) = x^{\frac{5}{3}} - y \quad \text{for } (x,y) \in \mathbb{R}^2.$$

It is clear that $\alpha \in \mathcal{O}^1(N)$ but $\alpha'_{|2}(p) = -1$, for any $p \in N$.

One can easily prove

Lemma 2.15. Let $p \in N$ be an arbitrary point of a subset $N \subset \mathbb{R}^n$. If $r_1 < r_2$ then $\mathcal{O}_p^{r_1}(N) \supset \mathcal{O}_p^{r_2}(N)$ and $N_p^{r_1}$ is a linear subspace of $N_p^{r_2}$. Moreover, if $\dim N_p^{r_1} = n$ then $\dim N_p^{r_2} = n$.

Example 2.2. Let N be the graph of C^1 function $f: \mathbb{R} \longrightarrow \mathbb{R}$ which is not C^2 at any point. Then $N_p^1 \subset N_p^\infty$, $\dim N_p^1 = 1$ and $\dim N_p^\infty = 2$, for every $p \in N$.

Now we prove

Proposition 2.16. If $\dim N_p^r = k \geq 1$, then there exist an open neighbourhood $U \in \tau_D$ of the point p and a k-dimensional C^r surface $S \subset \mathbb{R}^n$ including U and $\mathcal{F}^r(N)_U = C^r(S)_U$, where $C^r(S) := \mathcal{F}^r(\mathbb{R}^n)_S$. Moreover, the integer $k = \dim N_p^r$ is the smallest dimension of such a C^r surface S.

Proof. Clearly, $\dim G_p^r = n-k$. Let $h_1, \ldots, h_{n-k} \in \mathbb{R}^n$ be a vector basis of G_p^r. There exist functions $f_1, \ldots, f_n \in \mathcal{O}^r(N)$ such that $h_i = (\mathrm{grad} f_i)(p)$, for $i = 1, \ldots, n-k$. Since $\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{\substack{1 \leq i \leq n-k \\ 1 \leq j \leq n}} = n-k$, the mapping $(f_1, \ldots, f_{n-k}): \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$ is regular at p. There exists a neighbourhood V of p open in top $\mathbb{R}^n$ such that rank $\left(\frac{\partial f_i}{\partial x_j}(q)\right) = n - k$ for $q \in V$. From the implicit function theorem [16] it follows that the set $S := \{q \in V: f_1(q) = \dots = f_{n-k}(q) = 0\}$ is a k-dimensional C^r surface in $\mathbb{R}^n$. Of course, the set $U = N \cap V$ is open in τ_D and $U \subset S$. It is easy to observe that $\mathcal{F}^r(N)_U = C^r(S)_U$. Since $U \subset S$, $0^r(U) \supset 0^r(S)$. Thus $U^r_p \subset S^r_p$. It is easy to observe that $\dim U^r_p = \dim N^r_p$ and $\dim S^r_p = \dim S$. Therefore $\dim S \geq \dim N^r_p = k$. This finishes the proof.

Now let $N \subset \mathbb{R}^n$ be a subset such that $\dim N^r_q = n$ for any $q \in N$. From Lemma 2.1 it follows that for any function $\alpha \in \mathcal{F}^r(N)$ and a point $q \in N$ there exist a neighbourhood $V \in$ top $\mathbb{R}^n$ of q and a smooth function $\beta: \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^r such that $\alpha | V \cap N = \beta | V \cap N$. The function β is said to be a *local extension* of α at the point q.

From Corollary 2.14 it follows the correctness of the following definition:

Definition 2.5. Let $N \subset \mathbb{R}^n$ be a set such that $\dim N^r_p = n$ for $q \in N$. i-th partial derivative of a smooth function $\alpha \in \mathcal{F}^r(N)$ of class C^r at q, ($r \geq 1$), is defined to be the i-th partial derivative of its local extension β at this point

(2.7) $\quad \alpha'_{|i}(q) := \beta'_{|i}(q) \quad$ for $q \in N$, $i = 1, \dots, n$.

Now let M be an n-dimensional differential C^∞ manifold and M_0 be a subset of M. If $x = (x^1, \dots, x^n)$ is a chart on $U \in$ top M, then the restriction $x_0 = x | U \cap M$ is said to be a chart on $U_0 = U \cap M \subset M_0$.

Let $C_0 = C^\infty(M)_{M_0}$ be a differential structure on M_0. Let us assume that for any point $p \in M_0$ there exists a chart $x_0 = x|U_0$, $U_0 \ni p$, such that $\dim x_0(U_0)^r_q = n$, for any $q \in U_0$. Then (M_0, C_0) is said to be of constant C^r dimension n. The chart x_0 allows us to define tangent vectors at $p \in U_0$:

(2.8) $\frac{\partial}{\partial x_0^i}|_p\,(\psi) := (\psi\circ x_0^{-1})'_{|i}(x_0(p))$ for $\psi \in C_0$, $i=1,\dots,n$.

The correctness of (2.8) follows from (2.7).

It is easy to see that the vector fields $\frac{\partial}{\partial x_0^1},\dots,\frac{\partial}{\partial x_0^n}$ are local vector basis of the $\mathcal{F}^r(M_0)$-module $\mathfrak{X}^r(M_0)$ of all smooth vector fields of class C^r tangent to (M_0,C_0).

Lemma 2.17. Let (M_0,C_0) be a subspace of an n-dimensional C^∞ manifold M. If (M_0,C_0) is of a constant C^r dimension n then (M_0,C_0) has constant differential dimension n.

Proof. Since (M_0,C_0) is a differential subspace of $(M,C^\infty(M))$, $\dim T_pM_0 \le \dim T_pM = n$ for any $p \in M$. Let $x = (x^1,\dots,x^n)$ be a chart on an open neighbourhood $U \in \text{top } M$ of a point $p \in M_0$ and let $x_0 = x|U\cap M_0$. It is easy to see that vectors $\frac{\partial}{\partial x_0^1}|_q,\dots,\frac{\partial}{\partial x_0^n}|_q$ defined by (2.8) are linearly independent for every $q \in U_0 = U\cap M_0$. Thus $\frac{\partial}{\partial x_0^1}|_q,\dots,\frac{\partial}{\partial x_0^n}|_q$ is a basis of T_qM_0 for $q \in U_0$. Therefore $\frac{\partial}{\partial x_0^1},\dots,\frac{\partial}{\partial x_0^n}$ is a local vector basis of C_0-module $\mathfrak{X}(M_0)$ on a neighbourhood U_0 of p. This finishes the proof.

Proposition 2.18. Let (M_0,C_0) be a differential subspace of M and let (M_0,C_0) be of constant C^r dimension $n = \dim M$. If g is a C^r $(r\ge 2)$ Lorentz metric on M, then $\mathring{g} = \iota_{M_0}^* g$ is a C^r Lorentz metric on M_0, where $\iota_{M_0} : M_0 \longrightarrow M$ is the inclusion map.

Proof. It is enough to prove that for every $p \in M_0$ the signature of $\mathring{g}$ at p is equal to $(n-1,1)$. Let $x = (x^1,\dots,x^n)$ be a chart on an open neighbourhood $U \in \text{top } M$ of a point $p \in M_0$ and let $x_0 = x|U\cap M_0$.

It is easy to check the equality

(2.9) $$(\iota_{M_0})_{*p}\left(\frac{\partial}{\partial x_0^i}|_p\right) = \frac{\partial}{\partial x^i}|_p \quad \text{for } i = 1,\dots,n.$$

From Lemma 2.15 it follows that $\dim T_pM_0 = \dim T_pM = n$, for any $p \in M_0$. Thus $(\iota_{M_0})_{*p}\colon T_pM_0 \longrightarrow T_pM$ is an isomorphism for $p \in M_0$. Now it is clear that $\mathring{g}_p = (\iota_{M_0}^{*} g)_p$ has the signature $(n-1,n)$ for any $p \in M_0$.

Now for any $r \in \mathbb{N}$ and $p \in M$, let $\tilde{T}_p^rM$ be the set of linear mappings $v\colon \mathcal{F}^r(M) \longrightarrow \mathbb{R}$ satisfying the following condition

$$(2.10) \qquad v\big(\sigma\circ(f_1,\dots,f_n)\big) = \sum_{i=1}^{n} \sigma'_{|i}\big(f_1(p),\dots,f_n(p)\big)\cdot v(f_i)$$

for $\sigma \in \varepsilon_n^r$, $f_1,\dots,f_n \in C$.

Clearly $\tilde{T}_p^rM$, is a linear space over $\mathbb{R}$. It is easy to see that $\tilde{T}_p^rM$ is a linear subspace of the tangent space T_p^rM to $\big(M,\mathcal{F}^r(M)\big)$ at the point p. Since $C \subset \mathcal{F}^r(M)$, the mapping id: $\big(M,\mathcal{F}^r(M)\big) \longrightarrow (M,C)$ is smooth. Let us put $L_p^r = \mathrm{id}_{*p}|\tilde{T}_p^rM$.

Lemma 2.19. For any $p \in M$ and $r \in \mathbb{N}$, the mapping $L_p^r\colon \tilde{T}_p^rM \longrightarrow T_pM$ is a monomorphism. If $\dim \tilde{T}_p^rM = \dim T_pM$, then L_p^r is an isomorphism.

Proof. It is easy to see that

$$(2.11) \qquad L_p^r(v) = v|C \quad \text{for any} \quad v \in \tilde{T}_p^rM.$$

We will show that L_p^r is a monomorphism. Let $L_p^r(v) = 0$ for a vector $v \in \tilde{T}_p^rM$. By (2.11) $v|C = 0$.

We will prove that $v = 0$. Let $f \in \mathcal{F}^r(M)$. There exist a neighbourhood $U \in \tau_C$ of p, $n \in \mathbb{N}$, functions $f_1,\dots,f_n \in C$ and $\sigma \in \varepsilon_n^r$ such that

$$f|U = \sigma \circ (f_1,\dots,f_n)|U.$$

Hence and from (2.10) we have

$$v(f) = v\big(\sigma \circ (f_1,\dots,f_n)\big) = \sum_{i=1}^{n} \sigma'_{|i}\big(f_1(p),\dots,f_n(p)\big)\cdot v(f_i) =$$

$$= \sum_{i=1}^{n} \sigma'_{|i}\big(f_1(p),\dots,f_n(p)\big)\cdot 0 = 0 .$$

Therefore $v(f) = 0$ for any $f \in \mathcal{F}^r(M)$. Thus $v = 0$.

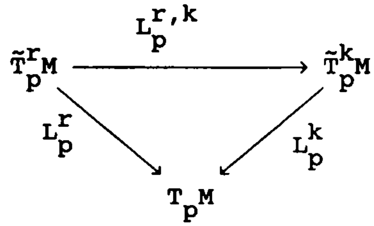
Lemma 2.20. If $\dim \tilde{T}_p^rM = \dim T_pM = n$, then $\dim \tilde{T}_p^kM = n$ for every $k > r$. Moreover, for $k > r$ the mapping L_p^k is an isomorphism.

Proof. For $k > r$ we have the smooth mapping $\mathrm{id}\colon (M,\mathcal{F}^r(M)) \longrightarrow (M,\mathcal{F}^k(M))$. Let us notice that $\mathrm{id}_{*p}(\tilde{T}^r_pM) \subset \tilde{T}^k_pM$. In fact, for any $v \in \tilde{T}^r_pM$ vector $\mathrm{id}_{*p}v$ satisfies (2.10).

Let $L^{r,k}_p\colon \tilde{T}^r_pM \longrightarrow \tilde{T}^k_pM$ be the mapping defined by

$$L^{r,k}_p = \mathrm{id}_{*p}|\tilde{T}^r_pM. \tag{2.12}$$

It is evident that $L^{r,k}_p$ is a monomorphism. The following diagram



is commutative. Since $L^{r,k}_p$ and L^k_p are monomorphisms, $\dim \tilde{T}^r_pM \le \dim \tilde{T}^k_pM \le \dim T_pM$. Thus $n \le \dim \tilde{T}^k_nM \le n$. Hence $\dim \tilde{T}^k_pM = n$.

Now we prove

Lemma 2.21. Let (M,C) be a differential space with the differential structure C generated by C_0.

Then for any mapping $v_0\colon C_0 \longrightarrow \mathbb{R}$ satisfying the condition

(•) for any $\sigma \in \varepsilon^r_n, f_1,\dots,f_n \in C_0$, $n \in \mathbb{N}$

if $\sigma \circ (f_1,\dots,f_n) = 0$, then

$$\sum_{i=1}^{n} \sigma'_{|i}\big(f_1(p),\dots,f_n(p)\big)\cdot v_0(f_i) = 0,$$

there exists a unique vector $v \in \tilde{T}^r_pM$ such that $v|C_0 = v_0$.

Proof. Let $v\colon \mathcal{F}^r(M) \longrightarrow \mathbb{R}$ be the mapping given by

$$v(f) = \sum_{i=1}^{n} \sigma'_{|i}\big(f_1(p),\dots,f_n(p)\big)\cdot v_0(f_i) \tag{2.13}$$

for $f \in \mathcal{F}^r(M)$, where $f_1,\dots,f_n \in C_0$ and $\sigma \in \varepsilon^r_n$ are such functions that there is an open neighbourhood $U \in \tau_C$ of p and

$f|U = \sigma \circ (f_1,\dots,f_f)|U$.

From (•) it follows the correctness of definition (2.13) and the uniquess of the vector v satisfying the condition $v|C_0 = v_0$.

Proposition 2.22. Let $N \subset \mathbb{R}^n$ be a subset with the differential structure $D = (\varepsilon_n)_N$. Then for any $p \in N$ the mapping $I_p^r\colon \tilde{T}_p^r N \longrightarrow N_p^r$ defined by

(2.14) $$I_p^r(v) = \big(v(\pi_1|N),\dots,v(\pi_n|N)\big) \quad \text{for } v \in \tilde{T}_p^r N,$$

is an isomorphism of linear spaces.

Proof. First, we prove that I_p^r is a monomorphism. If $I_p^r(v) = 0$ for a vector $v \in \tilde{T}_p^r N$, then $v(\pi_1|N) = \dots = v(\pi_n|N) = 0$. By condition (2.10), for any $\alpha \in \mathcal{F}^r(N)$,

$$v(\alpha) = \sum_{i=1}^{n} \sigma'_{|i}(p)\cdot v(\pi_i|N) = 0,$$

where $\sigma \in \varepsilon_n^r$ is a function such that there exist a neighbourhood $U \in \tau_D$ of p and $\alpha|U = \sigma \circ (\pi_1|N,\dots,\pi_n|N)|U$. Therefore $v = 0$.

Now we verify that I_p^r is an epimorphism. Let $h \in N_p^r$. It means that $f'_{|h}(p) = 0$ for any $f \in \mathcal{F}^r(\mathbb{R}^n)$ such that $f|N = 0$.

Let $v_{0h}\colon \{\pi_1|N,\dots,\pi_n|N\} \longrightarrow \mathbb{R}$ be the mapping defined by

(2.15) $$v_{0h}(\pi_i|N) = h_i \quad \text{for } i = 1,\dots,n.$$

It is easy to see that v_{0h} satisfies the condition (•) from Lemma 2.21. Thus, in view of Lemma 2.21, there exists a unique vector $v_h \in \tilde{T}_p^r N$ such that $v_h(\pi_i|N) = h_i$, for $i = 1,\dots,n$, or equivalently $I_p^r(v_h) = h$. This finishes the proof.

Proposition 2.23. Let (M,C) be a differential space of constant differential dimension n. Then for any $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\mathcal{F}^k(M)$-module $\mathcal{X}^k(M)$ is an n-dimensional differential module.

Proof. One can prove [15] that for any point $p \in M$ there exist a neighbourhood $U \in \tau_C$ of p, a local vector basis $W_1,\dots,W_n \in \mathcal{X}(U)$ of the $\mathcal{F}(M)$-module $\mathcal{X}(M)$ and smooth functions $\alpha_1,\dots,\alpha_n \in C_U$ such that $W_i(\alpha_j) = \delta_{ij}$, for $i,j = 1,\dots,n$.

Let $X \in \mathcal{X}^k(M)$ be an arbitrary vector field. Then for any point $p \in U$, $X(p) = \sum_{i=1}^{n} \varphi^i(p) W_i(p)$, where $\varphi^i: U \longrightarrow \mathbb{R}$, $i = 1,\dots,n$, are unique real functions. Moreover, since $\varphi^i = X \circ d\alpha_i$, for $i = 1,\dots,n$, $\varphi^i \in \mathcal{F}^k(M)$, for $i = 1,\dots,n$. Thus $W_1,\dots,W_n$ is a local vector basis of the $\mathcal{F}^k(M)$-module $\mathfrak{X}^k(M)$.

Now we prove

Lemma 2.24. Let (M,C) be a differential space satisfying the condition: there is an $r \in \mathbb{N}$ such that $\dim T_pM = \dim \tilde{T}^r_pM$ for any $p \in M$. Then for any $X \in \mathfrak{X}^k(M)$, $f \in \mathcal{F}^l(M)$, $k, l \in \mathbb{N}$, $k, l \geq r$, the function $Xf : M \longrightarrow \mathbb{R}$ defined by

$$(2.16)\qquad (Xf)(p) = (L^l_p)^{-1}\big(X(p)\big)(f) \quad \text{for } p \in M,$$

is a smooth function of class C^s on (M,C), where $s = \min(k,l-1)$.

Proof. For any point $p \in M$ there exist a neighbourhood $U \in \tau_C$ of p and functions $\sigma \in \varepsilon^l_n$, $f_1,\dots,f_n \in C$ such that $f|U = \sigma \circ (f_1,\dots,f_n)|U$. Then

$$(Xf)(p) = (L^l_p)^{-1}\big(X(p)\big)(f) = \sum_{i=1}^{n} \sigma'_{|i} \circ (f_1,\dots,f_n)(p) \cdot (Xf_i)(p)$$

for $p \in U$.

Thus $Xf|U = \sum_{i=1}^{n} \sigma'_{|i} \circ (f_1,\dots,f_n)|U \cdot (Xf_i)|U$. Clearly,

$$\sigma'_{|i} \circ (f_1,\dots,f_n) \in \mathcal{F}^{l-1}(U)$$

and $Xf_i \in \mathcal{F}^k(U)$ for $i = 1,\dots,n$. Hence $Xf|U \in \mathcal{F}^{\min(k,l-1)}(U)$. Therefore $Xf \in \mathcal{F}^s(M)$.

Definition 2.6. A linear mapping $\tilde{X}: \mathcal{F}(M) \longrightarrow \mathcal{F}^k(M)$ satisfying

$$(2.17)\qquad \tilde{X}(\alpha\beta) = \tilde{X}\alpha\cdot\beta + \alpha\cdot\tilde{X}\beta \quad \text{for any } \alpha,\beta \in \mathcal{F}(M)$$

is said to be a C^k derivation of $\mathcal{F}(M)$.

Let us denote by $\mathrm{Der}^k\big(\mathcal{F}(M)\big)$ the $\mathcal{F}^k(M)$-module of all C^k-derivations of $\mathcal{F}(M)$. For any $X \in \mathfrak{X}^k(M)$, the mapping $\partial_X: \mathcal{F}(M) \longrightarrow \mathcal{F}^k(M)$ given by

$$(2.18)\qquad (\partial_X\alpha)(p) = (X\alpha)(p) \quad \text{for } p \in M,\ \alpha \in \mathcal{F}(M)$$

is a C^k-derivation of the linear ring $\mathcal{F}(M)$.

Now one can prove

Proposition 2.25. The mapping Θ^k: $\mathfrak{X}^k(M) \longrightarrow \mathrm{Der}^k(\mathcal{F}(M))$ given by

$$(2.19) \qquad \Theta^k(X) = \partial_X \quad \text{for } X \in \mathfrak{X}^k(M)$$

is an isomorphism of $\mathcal{F}^k(M)$-modules.

Proof. It is clear that Θ^k is a monomorphism. To prove that Θ^k is an epimorphism it is enough to notice that for any $\tilde{X} \in \mathrm{Der}^k(\mathcal{F}(M))$ the vector field $X: M \longrightarrow TM$ defined by

$$(2.20) \qquad X(p)(\alpha) = (\tilde{X}\alpha)(p) \quad \text{for } \alpha \in \mathcal{F}(M) \text{ and } p \in M,$$

is a vector field from $\mathfrak{X}^k(M)$ such that $\Theta^k(X) = \tilde{X}$.

Definition 2.7. Assume that (M,C) is a differential space satisfying the following condition: there exists $r \in \mathbb{N}$ such that $\dim T_pM = \dim \tilde{T}^r_pM$ for any $p \in M$. For any $X,Y \in \mathfrak{X}^k(M)$, $k \geq r$, denote by $[X,Y]: \mathcal{F}(M) \longrightarrow \mathcal{F}^{k-1}(M)$ the mapping defined by

$$(2.21) \qquad [X,Y](f) = X(Yf) - Y(Xf) \quad \text{for } f \in \mathcal{F}(M).$$

One can verify that $[X,Y]$ is a C^{k-1}-derivation of $\mathcal{F}(M)$.

From Proposition 2.25 it follows that there exists a unique vector field $[X,Y] \in \mathfrak{X}^{k-1}(M)$ such that $\partial_{[X,Y]} = [X,Y]$. The vector field $[X,Y]$ is said to be the Lie bracket of $X,Y \in \mathfrak{X}^k(M)$. One can check that Θ^k defined by (2.19) is an isomorphism of the Lie algebras $(\mathfrak{X}^k(M), [\cdot,\cdot])$ and $(\mathrm{Der}^k(\mathcal{F}(M)), [\cdot,\cdot])$.

Now for any n-form $\omega: T^nM \longrightarrow \mathbb{R}$ of class C^k on (M,C) and for $l = 0,1,2,\ldots,$ let $\tilde{\omega}: \mathfrak{X}^l(M)\times\ldots\times\mathfrak{X}^l(M) \longrightarrow \mathcal{F}^s(M)$ be the $\mathcal{F}^l(M)$-module-linear mapping given by

$$(2.22) \qquad \tilde{\omega}(X_1,\ldots,X_n) = \omega \circ (X_1,\ldots,X_n)$$

for $X_1,\ldots,X_n \in \mathfrak{X}^l(M)$, where $s = \min(k,l)$.

It can be proved

Lemma 2.26. Let (M,C) be a differential space of constant differential dimension and g a semi-Riemannian metric on (M,C) of class C^r, $r = 0,1,2,\dots$. Then for any $\mathcal{F}^k(M)$-linear mapping $\varphi\colon \mathfrak{X}^1(M) \longrightarrow \mathcal{F}^k(M)$, $k = 0,1,2,\dots$, there exists a unique vector field $A \in \mathfrak{X}^s(M)$, $s = \min(k,r)$, such that

$$\varphi(Z) = \tilde{g}(A,Z) \quad \text{for any } Z \in \mathfrak{X}^l(M),\ l = 1,2,\dots . \tag{2.23}$$

Proposition 2.27. Let (M,C) be a differential space satisfying the condition: $\dim T_pM = \dim \tilde{T}_p^1M$ for any $p \in M$.

Then for any semi-Riemannian metric $g\colon T^2M \longrightarrow \mathbb{R}$ of class C^r there exists a unique covariant derivative of class C^r [15] such that

$$Z\tilde{g}(X,Y) = \tilde{g}(\nabla_Z X,Y) + \tilde{g}(X,\nabla_Z Y), \tag{2.24}$$

$$\nabla_X Y = \nabla_Y X + [X,Y], \tag{2.25}$$

for any $X,Y,Z \in \mathfrak{X}^1(M)$.

Proof. For any $X,Y \in \mathfrak{X}^1(M)$ let $\varphi_{X,Y}\colon \mathfrak{X}^1(M) \longrightarrow \mathcal{F}^0M$ be $\mathcal{F}^0(M)$-linear mapping given by

$$\varphi_{X,Y}(Z) = \frac{1}{2}\big[\partial_X\tilde{g}(Y,Z) + \partial_Y\tilde{g}(Z,X) - \partial_Z\tilde{g}(X,Y) + \\ + \tilde{g}([X,Y],Z) + \tilde{g}([Z,X],Y) - \tilde{g}([Y,Z],X)\big], \tag{2.26}$$

for $Z \in \mathfrak{X}^1(M)$.

From Lemma 2.26 it follows that for any $\varphi_{X,Y}$, $X,Y \in \mathfrak{X}^1(M)$, there exists a unique vector field $\nabla_X Y \in \mathfrak{X}^0(M)$ such that

$$\varphi_{X,Y}(Z) = \tilde{g}(\nabla_X Y,Z) \quad \text{for any } Z \in \mathfrak{X}^1(M).$$

Let $\nabla\colon \mathfrak{X}^1(M)\times\mathfrak{X}^1(M) \longrightarrow \mathfrak{X}^0(M)$ be the mapping defined by

$$\nabla(X,Y) = \nabla_X Y \quad \text{for any } X,Y \in \mathfrak{X}^1(M). \tag{2.27}$$

It can be proved, in the standard way, that ∇ is a covariant derivative of class C^r [7] i.e. for any $X,Y \in \mathfrak{X}^{r+1}(M)$,

$\nabla_X Y \in \mathfrak{X}^r(M)$.

3. Singularities of the fundamental differential space

Definition 3.1. A pair $((\bar{M},\bar{C}) \, , \, (M,\bar{C}_M))$ is said to be the fundamental differential space (shortly F-d-space), if $(\bar{M},\bar{C})$ is a differential space and M a subset of $\bar{M}$ dense in $(\bar{M},\tau_{\bar{C}})$ such that $(M,\bar{C}_M)$ is an n-dimensional C^∞ manifold.

The set $\partial M = \bar{M} - M$ is called the boundary of the F-d-space $((\bar{M},\bar{C}) \, , \, (M,\bar{C}_M))$.

If $((\bar{M},\bar{C}) \, , \, (M,\bar{C}_M))$ and $((\bar{N},\bar{D}) \, , \, (N,\bar{D}_N))$ are fundamental differential spaces, then

$((\bar{M}\times\bar{N} \, , \, \bar{C}\times\bar{D}) \, , \, (M\times N,(\bar{C}\times\bar{D})_{M\times N}))$ is a fundamental differential space with the boundary $\partial(M\times N) = \partial M\times\bar{N} \cup \bar{M}\times\partial N$.

Definition 3.2. A boundary point $p \in \partial M$ is called regular if there exists a neighbourhood $U \in \tau_{\bar{C}}$ of p such that the differential subspace $(U,\bar{C}_U)$ has constant differential dimension n. A boundary point $p \in \partial M$ is called singular if p is not regular. A boundary point $p \in \partial M$ is said to be of class D_0 (shortly D_0-point) if there exists a neighborhood $U \in \tau_{\bar{C}}$ of p such that $(U,\bar{C}_U)$ is a differential space of class D_0. A boundary point $p \in \partial M$ is called a non-D_0-point if p is not of class D_0.

Now we can present the following diagram:

$p \in \partial M$ boundary point			
p regular point		p singular point	
p D_0-regular point	p non-D_0-regular point	p D_0-singular point	p non-D_0-singular point

Example 3.1. Let $\bar{M} = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \wedge y \geq 0\}$ and $\bar{C} = (\varepsilon_2)_{\bar{M}}$. The boundary points $\partial M = \{(x,y) \in \bar{M} : x=0 \vee y=0\}$ are D_0-regular.

Example 3.2. Let $\bar{C}$ be the differential structure on $\bar{M} = \mathbb{R}^2$ generated by the set $\{\pi_1, \pi_2, f\}$, where $\pi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\pi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are the natural projections and $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the function defined by

$$f(x,y) = \sqrt{x^2+y^2} \quad \text{for} \quad (x,y) \in \mathbb{R}^2.$$

Let $M := \mathbb{R}^2 \setminus \{(0,0)\}$. It is easy to observe that $((\bar{M},\bar{C}),(M,\bar{C}_M))$ is a fundamental differential space and $\partial M = \{(0,0)\}$. The point $(0,0)$ is a D_0-singular point. It is clear that $\dim T_{(0,0)}(\bar{M},\bar{C}) = 3$ and $\dim T_p(\bar{M},\bar{C}) = 2$ for $p \neq (0,0)$.

Example 3.3. Let $\bar{C}$ be the differential structure on $\bar{M} = \mathbb{R}^2$ generated by the set $\{\pi_1, \pi_2\} \cup \{f_n : n \in \mathbb{N}\}$, where $f_n : \mathbb{R}^2 \longrightarrow \mathbb{R}$ for $n \in \mathbb{N}$ is the function given by

$$f_n(x,y) = \sqrt[3^n]{x^2+y^2} \quad \text{for} \quad (x,y) \in \mathbb{R}^2.$$

One can see that $((\bar{M},\bar{C}),(M,\bar{C}_M))$, where $M = \mathbb{R}^2 \setminus \{(0,0)\}$ is a fundamental differential space with the boundary $\partial M = \{(0,0)\}$. The point $(0,0)$ is non-D_0-singular.

Example 3.4. Let $N = \{ \frac{1}{n} \in \mathbb{R}: n \in \mathbb{N} \} \cup \{0\}$. Let D be the differential structure on N generated by the set $\{id_N\} \cup \{f_n : n \in \mathbb{N}\}$, where $f_n : N \longrightarrow \mathbb{R}$ for $n \in \mathbb{N}$, is defined by

$$f_n(x,y) = \sqrt[3^n]{x} \quad \text{for} \quad x \in N.$$

It is easy to see that $\dim T_x(N,D) = 0$ for $x \in N$. Let us take the Cartesian product $(\bar{M},\bar{C}) = (N \times \mathbb{R}^2, D \times \varepsilon_2)$ and let $M = \left\{ \frac{1}{n} \in \mathbb{R}: n \in \mathbb{N}\right\} \times \mathbb{R}^2$. Evidently $((\bar{M},\bar{C}),(M,\bar{C}_M))$ is a fundamental differential space. $(\bar{M},\bar{C})$ is a differential space of constant differential dimension 2. The boundary points are non-D_0-regular.

Definition 3.3. The pair $((\bar{M},\bar{C}),(M,g))$ is said to be the C^k differential space-time if $((\bar{M},\bar{C}),(M,\bar{C}_M))$ is a fundamental differential space and (M,g) is a dense n-dimensional C^k Lorentz submanifold. The set $\partial M = \bar{M} - M$ is called the boundary of the C^k differential space-time. The C^k Lorentz metric g is said to be extendible on the boundary ∂M if there exists a C^k Lorentz metric $\bar{g}$ on $(\bar{M},\bar{C})$ such that $g = \iota_M^*\bar{g}$, where $\iota_M : M \longrightarrow \bar{M}$ is the inclusion mapping.

Example 3.5. Let us consider the set $\bar{M} = \{(x,y,z) \in \mathbb{R}^3 : x = 0 \vee y = 0\}$. Let $\bar{C}$ be the differential structure on $\bar{M}$ induced from the Euclidean differential space $(\mathbb{R}^3, \varepsilon_3)$. Then $\bar{g} = d\pi_1^2 + d\pi_2^2 - d\pi_3^2$ is an extension of the Lorentz metric from the space-time (M,g), which is the disjoint union of 2-dimensional Minkovski space-times $\left(\{(x,y,z) \in \bar{M} : y \neq 0\}, d\pi_2^2 - d\pi_3^2\right)$ and $\left(\{(x,y,z) \in \bar{M} : x \neq 0\}, d\pi_1^2 - d\pi_3^2\right)$. The set of all points of the axis OZ is the boundary of the C^∞ differential space-time $\left((\bar{M},\bar{C}), (M,g)\right)$.

Definition 3.4. Let $((\bar{M},\bar{C}),(M,g))$ be a C^k- differential space-time. A boundary point $p \in \partial M$ is said to be C^k-metric if there exist a neighbourhood $U \in \tau_{\bar{C}}$ of p and C^k Lorentz metric $\bar{g}$ on $(U,\bar{C}_U)$ such that $\iota_{U\cap M}^*\bar{g} = g_{U\cap M}$, where $\iota_{U\cap M} : U\cap M \longrightarrow \bar{M}$ is the inclusion mapping. A boundary point $p \in \partial M$ is said to be C^k-metric D_0-regular if p is D_0-regular and C^k-metric.

Now we can present the following classification of C^k-metric boundary points.

p C^k-metric point			
p C^k-metric regular point		p C^k-metric singular point	
p C^k-metric D_0-regular point	p C^k-metric non-D_0-regular point	p C^k-metric singular point	p C^k-metric non-D_0 singular point

Proposition 3.1. Let $((\bar{M},\bar{C}),(M,g))$ be a C^k differential space-time. If a point $p \in \partial M$ is C^k-metric and $k \geq 2$, then there exist a neighbourhood $V \in \tau_{\bar{C}}$ of p and the integer $m \in \mathbb{N}$ such that

$$\dim T_q(\bar{M},\bar{C}) \leq m \quad \text{for } q \in V.$$

Proof. From Corollary 2.4 it follows that there exist an open neighbourhood $V \in \tau_{\bar{C}}$ of p and a mapping $F: (V, \bar{C}_V) \longrightarrow (\mathbb{R}^m, \varepsilon_m)$ such that F_{*q} is injective for $q \in V$. Now it is evident that

$$\dim T_q(\bar{M}, \bar{C}) = \dim T_q(V, \bar{C}_V) \le m \quad \text{for any } q \in V.$$

Example 3.6. Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences. Denote by π_i, for $i \in \mathbb{N}$, the projection of $\mathbb{R}^{\mathbb{N}}$ onto the i-th coordinate given by

$$\pi_i(x) = x_i \quad \text{for} \quad x = (x_i) \in \mathbb{R}^{\mathbb{N}}.$$

Let $\varepsilon_{\mathbb{N}}$ be the differential structure on $\mathbb{R}^{\mathbb{N}}$ generated by the set $\{\pi_i : i \in \mathbb{N}\}$ [15]. Let us put

$$M_i = \{ x \in \mathbb{R}^{\mathbb{N}} : x_j = 0 \ \text{ for } j \ne i \}, \ i \in \mathbb{N}.$$

Let $\bar{M} := \bigcup_{i \in \mathbb{N}} M_i$ and $\bar{C} := (\varepsilon_{\mathbb{N}})_{\bar{M}}$, $M = \bar{M} \setminus \{0\}$. It is easy to see that $\big((\bar{M}, \bar{C}), (M, \bar{C}_M)\big)$ is a fundamental differential space such that $\dim T_0 \bar{M} = \infty$ and $\dim T_x \bar{M} = 1$, for $x \ne 0$. There is no non-degenerate 2-form of class C^k $(k \ge 2)$ in a neighbourhood of the singular point $0 = (0) \in \mathbb{R}^{\mathbb{N}}$, because $\dim T_0(\bar{M}, \bar{C}) = \infty$.

Proposition 3.2. Let (M,C) and (N,D) be differential spaces. If $g: T^2M \longrightarrow \mathbb{R}$ a C^k Lorentz metric on (M,C), $h: T^2N \longrightarrow \mathbb{R}$ is a C^k Riemiannian metric on (N,D) and $f: M \longrightarrow (0,+\infty)$ is a smooth function of class C^k on (N,D), then the 2-form $\bar{g}: T^2(M \times N) \longrightarrow \mathbb{R}$ defined by

$$(3.1) \qquad \bar{g}(w_1, w_2) = (pr_1^* g)(w_1, w_2) + f\big(pr_1(\pi(w_1))\big) \cdot \\ \cdot (pr_2^* h)(w_1, w_2) \quad \text{for} \quad (w_1, w_2) \in T^2(M \times N),$$

is a C^k Lorentz metric on $(M \times N, C \times D)$, where $\pi: T(M \times N) \longrightarrow M \times N$ is the natural projection.

Proof. Let $(p,q) \in M \times N$ be an arbitrary point. Let $v_1, \dots, v_m \in T_pM$ be a vector basis of T_pM such that

$\left(g(v_i,v_j)\right) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$ and let $u_1,\dots,u_n \in T_qN$ be a vector basis of T_qN such that $\left(h(u_i,u_j)\right) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$. It is easy to see that

$$w_1 = (j_q)_{*p}v_1, \dots, \quad w_m = (j_q)_{*p}v_m,$$

$$w_{m+1} = (j_p)_{*q}u_1, \dots, \quad w_{m+n} = (j_p)_{*q}u_n$$

is a vector basis of $T_{(p,q)}(M\times N)$ such that

$$\left(\bar{g}(w_i,w_j)\right) = \begin{pmatrix} \left(g(v_i,v_j)\right) & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & f(p)\left(h(u_i,u_j)\right) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & f(p) & & \\ & & & & \ddots & \\ 0 & & & & & f(p) \end{pmatrix}.$$

Now it is obvious that $\bar{g}$ is the C^k Lorentz metric on $(M\times N\ ,\ C\times D)$. Analogously one can prove

Proposition 3.3. Let (M,C) and (N,D) be differential spaces. If $g\colon T^2M \longrightarrow \mathbb{R}$ is a C^k Riemannian metric on (M,C), $h\colon T^2N \longrightarrow \mathbb{R}$ is a C^k Lorentz metric on (N,D) and $f\colon M \longrightarrow (0,+\infty)$ is a smooth function of class C^k on (M,C) then the 2-form $\bar{g}\colon T^2(M\times N) \longrightarrow \mathbb{R}$ defined by

$$(3.2) \qquad \bar{g}(w_1,w_2) = (pr_1^*g)(w_1,w_2) + f\left(pr_1(\pi(w_1))\right)(pr_2^*h)(w_1,w_2)$$

for $(w_1,w_2) \in T^2(M\times N)$, is a C^k Lorentz metric on $(M\times N\ ,\ C\times D)$.

Example 3.7. Let $g = \iota_M^*\eta$ be the metric on the submanifold M from Example 3.1, where η is the Minkowski metric on $(\mathbb{R}^2,\varepsilon_2)$. The metric $\bar{g} = \iota_{\bar{M}}^*\eta$ is an extension of g onto $\bar{M}$. All points of the boundary ∂M are D_0-C^k-metric regular.

Example 3.8. Let $(\bar{M},\bar{C}) = (N\times\mathbb{R}^2\ ,\ D\times\varepsilon_2)$ be the differential space from Example 3.4. Let η be the Minkowski metric on $(\mathbb{R}^2,\varepsilon_2)$. It is easy to observe that $\bar{g} = pr_2^*\eta$ is a Lorentz metric on $(\bar{M},\bar{C})$, where $pr_2\colon N\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the projection. Thus the boundary point (0,0,0) is non-D_0 C^k-metric regular.

Example 3.9. The points of the axis OZ from Example 3.5 are D_0-C^k-metric singular.

Example 3.10. Let (N,D) be the differential space from Example 3.4 and $(\bar{M},\bar{C})$ be the differential space from Example 3.5. The then 2-form $pr_1^*\bar{g} = pr_1^*(d\pi_1^2 + d\pi_2^2 - d\pi_3^2)$ is a Lorentz metric on $(\bar{M}\times N\ ,\ \bar{C}\times D)$. The pair $\big((\bar{M}\times N\ ,\ \bar{C}\times D)\ ,\ (\bar{M}\times N_0, pr_1^* g)\big)$ is a differential space-time, where $N_0 := \{\frac{1}{n} \in \mathbb{R}\colon n \in \mathbb{N}\}$. All points of the boundary are non-D_0-C^k-metric singular.

From Proposition 1.5 it follows

Corollary 3.4. Let $\big((\bar{M},\bar{C})\ ,\ (M,\bar{C}_M)\big)$ and $\big((\bar{N},\bar{D})\ ,\ (N,\bar{D}_N)\big)$ be fundamental differential spaces. Then a boundary point $(p,q) \in \partial(M\times N)$ of the Cartesian product $\big((\bar{M}\times\bar{N}\ ,\ \bar{C}\times\bar{D})\ ,\ (M\times N\ ,\ \bar{C}\times\bar{D}_{M\times N})\big)$ is regular if and only if p and q are regular. A boundary point $(p,q) \in \partial(M\times N)$ is singular iff $p \in \partial M$ is singular or $q \in \partial N$ is singular.

From Proposition 1.7 it follows

Corollary 3.5. Let $\big((\bar{M},\bar{C})\ ,\ (M,\bar{C}_M)\big)$ and $\big((\bar{N},\bar{D})\ ,\ (N,\bar{D}_N)\big)$ be fundamental differential spaces. Then a boundary point $(p,q) \in \partial(M\times N)$ is a D_0 point if and only if p and q are D_0 points.

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