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PRODUCT FINAL DIFFERENTIAL STRUCTURES ON THE PLANE

In this paper we define two final differential structures on the Cartesian product of differential spaces (Section 1). The first (second) structure is defined with respect to arbitrary (continuous) real functions on such a product. These structures were introduced by Buchner [1]. If M and N are differential spaces, then by $M \times_1 N$ and $M \times_2 N$ we denote the Cartesian product $M \times N$ regarded as a differential space under the first and the second final differential structure, respectively. In the general case, we shall only give an estimation from below of the dimension of tangent space of $M \times_k N$ at a point, where $k = 1,2$ (Proposition 1.2).

Let $\mathbb{R}$ be the set of reals regarded as a differential space under the natural structure $C^\infty(\mathbb{R})$. This paper is devoted to the study of some properties of the differential spaces $\mathbb{R} \times_1 \mathbb{R}$ and $\mathbb{R} \times_2 \mathbb{R}$ from the point of view of differential geometry. These spaces have many common properties (Sections 3 and 4) and they can frequently be considered simultaneously. However, in general, the investigation of $\mathbb{R} \times_2 \mathbb{R}$ is more complicated than that of $\mathbb{R} \times_1 \mathbb{R}$ (Section 2).

The way of generalizing methods applied to the study of $\mathbb{R} \times_k \mathbb{R}$ to those proper for the study of $M \times_k N$ in the general case is not clear. Therefore, by analogy, we pose several open questions for $M \times_k N$ and related objects (Section 5).

1. Preliminaries

By a *differential space* we shall mean a differential space in the sense of Sikorski [6]. If M is such a space, then $\mathcal{E}(M)$ will denote the family of all *real smooth functions* on M. Every differential space M will be regarded as a *topological space* under the $\mathcal{E}(M)$-*topology* which is defined to be the weakest topology on M such that all functions from $\mathcal{E}(M)$ are continuous.

Let N be a non-empty set. Consider a collection $\mathfrak{C}$ of maps $f: M_f \longrightarrow N$, where M_f is a differential space for each $f \in \mathfrak{C}$. Let $F(N)$ denote the family of all *real functions* on N. The *F-final differential structure* on N induced by $\mathfrak{C}$ is defined to be the strongest differential structure $F\mathcal{E}(N,\mathfrak{C})$ on N for which all maps from $\mathfrak{C}$ are smooth (compare [2]). This means that

$$F\mathcal{E}(N,\mathfrak{C}) = \{\alpha \in F(N) : \alpha \circ f \in \mathcal{E}(M_f) \ \forall_{f \in \mathfrak{C}}\}.$$

The structure $F\mathcal{E}(N,\mathfrak{C})$ is also called the differential structure on N *coinduced* by $\mathfrak{C}$ (compare [8]). It is easy to verify that $F\mathcal{E}(N,\mathfrak{C})$ is a differential structure on N. Suppose further that N is a topological space and every map from $\mathfrak{C}$ is continuous. Let $C(N)$ denote the family of all *real continuous functions* on N. We define the *C-final differential structure* on N induced by $\mathfrak{C}$ to be the family $C\mathcal{E}(N,\mathfrak{C}) = C(N) \cap F\mathcal{E}(N,\mathfrak{C})$. One can see that $C\mathcal{E}(N,\mathfrak{C})$ is also a differential structure on N. Moreover, we have

$$C\mathcal{E}(N,\mathfrak{C}) = \{\alpha \in C(N) : \alpha \circ f \in \mathcal{E}(M_f) \ \forall_{f \in \mathfrak{C}}\}.$$

By a *pointed differential space* (M,x) we shall mean a differential space M together with a *base point* $x \in M$. We say that $f: (M,x) \longrightarrow (N,y)$ is a *smooth map* of pointed differential spaces if $f: M \longrightarrow N$ is a smooth map of differential spaces and $f(x) = y$. The *tangent vector space* of (M,x) is defined to be the tangent vector space $T(M,x)$ of M at x. If $f: (M,x) \longrightarrow (N,y)$ is a smooth map of pointed differential spaces, then we define the linear map $Tf: T(M,x) \longrightarrow T(N,y)$ in a usual manner. Let T be the assignment which sends every pointed differential space (M,x) to the vector space $T(M,x)$

and every smooth map f of pointed differential spaces to the linear map Tf. It is easy to verify

Lemma 1.1. *The assignment* T *is a covariant functor from the category of pointed differential spaces to the category of vector spaces.*

Let M and N be differential spaces. For any $x \in M$ and $y \in N$, define the maps $r^x: N \longrightarrow M \times N$ and $l^y: M \longrightarrow M \times N$ by

$$r^x(s) = (x,s) \text{ and } l^y(t) = (t,y).$$

Let us consider the set $\mathfrak{C} = \mathfrak{C}(M,N) = \{r^x, l^y: x \in M, y \in N\} = \{r^x: x \in M\} \cup \{l^y: y \in N\}$. The *product F-final differential structure* on $M \times N$ is defined to be the structure $\mathcal{S}^1(M \times N) = F\mathcal{C}(M \times N, \mathfrak{C})$. We shall regard $M \times N$ as a topological space under the product topology. One can see that, in general, the family $\mathcal{S}^1(M \times N)$ may contain functions which are discontinuous on $M \times N$ in this topology; however, all maps from $\mathfrak{C}$ are always continuous. Therefore, we define the *product C-final differential structure* on $M \times N$ to be the structure $\mathcal{S}^2(M \times N) = C\mathcal{C}(M \times N, \mathfrak{C})$. Denote by $\mathcal{C}(M \times N)$ the family of all real smooth functions on the product $M \times N$ of differential spaces. It is seen that $\mathcal{C}(M \times N) \subset \subset \mathcal{S}^2(M \times N) \subset C(M \times N)$. This implies that the $\mathcal{S}^2(M \times N)$-topology on $M \times N$ is the product one. We shall denote by $M \times_k N$ the differential space $(M \times N, \mathcal{S}^k(M \times N))$ where $k = 1,2$. By applying Lemma 1.1 it is easy to prove

Proposition 1.2. *If* M *and* N *are differential spaces, then, for any* $x \in M$ *and* $y \in N$, *we have*

$$\dim T(M \times_k N,(x,y)) \geq \dim T(M,x) + \dim T(N,y).$$

We shall regard the set $\mathbb{R}$ of reals as a differential space under the natural structure $C^\infty(\mathbb{R})$ of all real smooth functions on $\mathbb{R}$, i.e. we accept that $\mathcal{C}(\mathbb{R}) = C^\infty(\mathbb{R})$. Let us set $\mathcal{S}^k = \mathcal{S}^k(\mathbb{R} \times \mathbb{R})$ for $k = 1,2$. Throughout this paper, all considerations concerning $\mathbb{R} \times_k \mathbb{R}$ or $\mathcal{S}^k$ will be carried out for an *arbitrary but fixed* k.

Let $\mathfrak{C} = \mathfrak{C}(\mathbb{R},\mathbb{R}) = \{r^x, l^y: x,y \in \mathbb{R}\}$. For any $x,y \in \mathbb{R}$ we set

$$R^x = r^x(\mathbb{R}),\ L^y = l^y(\mathbb{R}) \text{ and } K^x = R^x \cup L^x.$$

Denote by $\mathcal{F}^1$ ($\mathcal{F}^2$) the family of all *arbitrary* (*continuous*) *real functions* on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. It is easy to verify

Proposition 1.3. *If $\alpha \in \mathcal{F}^k$, then the following conditions are equivalent:*

(a) $\alpha \in \mathcal{S}^k$,

(b) $\alpha | R^x \in C^\infty(R^x)$ *and* $\alpha | L^y \in C^\infty(L^y)$ *for any* $x, y \in \mathbb{R}$,

(c) $\alpha | K^x \in C^\infty(K^x)$ *for each* $x \in \mathbb{R}$.

If A is a subset of $\mathbb{R}^2$, then by $\mathcal{S}^k(A)$ we shall denote the differential structure on A induced from $\mathbb{R} \times_k \mathbb{R}$. Moreover, it will be convenient to denote by $\mathcal{F}^1(A)$ ($\mathcal{F}^2(A)$) the family of all *arbitrary (continuous) real functions* on A. Let us set $\mathbb{R}^2_o = \mathbb{R}^2 \setminus \{o\}$ and $K = K^0 = \{(x,y) \in \mathbb{R}^2 \colon xy = 0\}$, where $o = (0,0)$. One can see that Proposition 1.3 implies

Corollary 1.4. *Let $\alpha \in \mathcal{F}^k$. Then $\alpha \in \mathcal{S}^k$ if and only if $\alpha | \mathbb{R}^2_o \in \mathcal{S}^k(\mathbb{R}^2_o)$ and $\alpha | K \in C^\infty(K)$.*

Note that, in the general case, the differential spaces $M \times_1 N$ and $M \times_2 N$ may be identical. In particular, this is satisfied if M or N is discrete, i.e. $\mathcal{C}(M) = F(M)$ or $\mathcal{C}(N) = F(N)$, respectively. The following example shows that the differential spaces $\mathbb{R} \times_1 \mathbb{R}$ and $\mathbb{R} \times_2 \mathbb{R}$ are different. In fact, these spaces are non-diffeomorphic (Corollary 4.5).

Example 1.5. Let $\phi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined as follows:

$$\phi(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{for } (x,y) \in \mathbb{R}^2_o, \\ 0 & \text{for } (x,y) = o. \end{cases}$$

It is seen that $\phi \in \mathcal{S}^1 \setminus \mathcal{S}^2$.

In this paper, by the topology on $\mathbb{R}^2$ we mean the Euclidean one, unless otherwise stated. Moreover, $\mathbb{R}^2$ will be regarded as a real normed (vector) space under the coordinatewise operations and the norm defined by $\|p\| = (x^2+y^2)^{1/2}$ for $p = (x,y)$.

2. Tangent vector space

Observe that if $\phi, \psi \in C^\infty(\mathbb{R})$, then the assignment $(x,y) \mapsto (\phi(x), \psi(y))$ defines a smooth map $\phi \times \psi$ from $\mathbb{R} \times_k \mathbb{R}$ to itself. We have

Lemma 2.1. *If ϕ and ψ are smooth diffeomorphisms of* $\mathbb{R}$, *then $\phi \times \psi$ is a diffeomorphism of* $\mathbb{R} \times_k \mathbb{R}$.

From this lemma it follows that any translation of $\mathbb{R}^2$ is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$. Denote by $\mathbb{R}^2_+$ the set $\mathbb{R}^2$ regarded as a group under the coordinatewise addition. If $v \in \mathbb{R}^2_+$, we denote by t_v the translation of $\mathbb{R}^2$ via v, i.e. $t_v(p) = p + v$. Let $\mathcal{I}$ be the group of all *translations* of $\mathbb{R}^2$ which is isomorphic to $\mathbb{R}^2_+$ via the isomorphism $v \mapsto t_v$.

For each $p \in \mathbb{R}^2$, denote by $T_p(\mathbb{R} \times_k \mathbb{R})$ the *tangent vector space* of $\mathbb{R} \times_k \mathbb{R}$ at p, where we can assume that

$$T_p(\mathbb{R} \times_k \mathbb{R}) \cap T_q(\mathbb{R} \times_k \mathbb{R}) = \emptyset$$

for distinct points $p,q \in \mathbb{R}^2$. Let us set $T(\mathbb{R} \times_k \mathbb{R}) = \bigcup \{T_p(\mathbb{R} \times_k \mathbb{R}) : p \in \mathbb{R}^2\}$. Denote by $\pi : T(\mathbb{R} \times_k \mathbb{R}) \longrightarrow \mathbb{R} \times_k \mathbb{R}$ the *projection* defined by $\pi^{-1}(p) = T_p(\mathbb{R} \times_k \mathbb{R})$. For each $\alpha \in \mathcal{S}^k$, we define the *differential* $d\alpha : T(\mathbb{R} \times_k \mathbb{R}) \longrightarrow \mathbb{R}$ in a usual manner. Let $\mathcal{T}\mathcal{S}^k$ be the *weakest differential structure* on $T(\mathbb{R} \times_k \mathbb{R})$ such that π and any differential $d\alpha$ ($\alpha \in \mathcal{S}^k$) are smooth (see [3]). We define the *tangent space* of $\mathbb{R} \times_k \mathbb{R}$ to be the differential space $(T(\mathbb{R} \times_k \mathbb{R}), \mathcal{T}\mathcal{S}^k)$ which will be denoted by $T(\mathbb{R} \times_k \mathbb{R})$, as well.

If f is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$, and $p \in \mathbb{R}^2$, then by f_{*p} we denote the *differential of f at* p, that is, the linear isomorphism $f_{*p} : T_p(\mathbb{R} \times_k \mathbb{R}) \longrightarrow T_{f(p)}(\mathbb{R} \times_k \mathbb{R})$ defined by $f_{*p}(v)(\alpha) = v(\alpha \circ f)$, where α ranges over $\mathcal{S}^k$. Thus f defines a diffeomorphism f_* of $T(\mathbb{R} \times_k \mathbb{R})$ such that $f_* | T_p(\mathbb{R} \times_k \mathbb{R}) = f_{*p}$.

By the definition of $\mathcal{S}^k$, any function $\alpha \in \mathcal{S}^k$ has derivatives $\frac{\partial\alpha}{\partial x}, \frac{\partial\alpha}{\partial y}$ which are real functions on $\mathbb{R}^2$, may be not belonging to $\mathcal{S}^k$. Therefore the assignments $\alpha \mapsto \frac{\partial\alpha}{\partial x}(p)$ and $\alpha \mapsto \frac{\partial\alpha}{\partial y}(p)$ define vectors $\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p \in T_p(\mathbb{R} \times_k \mathbb{R})$. A vector $v \in T_p(\mathbb{R} \times_k \mathbb{R})$ is called *standard* if $v = a\frac{\partial}{\partial x}|_p + b\frac{\partial}{\partial y}|_p$ for some $a,b \in \mathbb{R}$. Denote by $T^+_p(\mathbb{R} \times_k \mathbb{R})$ the *vector subspace of* $T_p(\mathbb{R} \times_k \mathbb{R})$ *consisting of all standard vectors at* p. Since the vectors $\frac{\partial}{\partial x}|_p$ and $\frac{\partial}{\partial y}|_p$ are linearly independent and span

$T_p^+(\mathbb{R} \times_k \mathbb{R})$, we get

$$\dim T_p(\mathbb{R} \times_k \mathbb{R}) \geq \dim T_p^+(\mathbb{R} \times_k \mathbb{R}) = 2 \tag{2.1}$$

for each $p \in \mathbb{R}^2$ (compare, Proposition 1.2). Adopt $T^+(\mathbb{R} \times \mathbb{R}) = \bigcup \{T_p^+(\mathbb{R} \times \mathbb{R}) : p \in \mathbb{R}^2\}$. This set will be regarded as a differential subspace of the tangent space $T(\mathbb{R} \times_k \mathbb{R})$. We shall prove that $T^+(\mathbb{R} \times_k \mathbb{R}) = T(\mathbb{R} \times_k \mathbb{R})$ (Theorem 2.10).

We need the well-known

Lemma 2.2. *For each $\alpha \in C^\infty(\mathbb{R})$, there is a unique $\alpha_\bullet \in C^\infty(\mathbb{R})$ such that*

$$\alpha(t) = \alpha(0) + t\alpha_\bullet(t)$$

where $\alpha_\bullet(0) = \frac{d\alpha}{dt}(0)$.

If $\alpha \in \mathcal{S}^k$, then, for any $x, y \in \mathbb{R}$, we define the functions $\alpha^x, \alpha_y \in C^\infty(\mathbb{R})$ by $\alpha^x(y) = \alpha(x,y)$ and $\alpha_y(x) = \alpha(x,y)$. Applying Lemma 2.2, we have defined the functions $(\alpha^x)_\bullet, (\alpha_y)_\bullet \in C^\infty(\mathbb{R})$. Let us set $\alpha_{\bullet 1}(x,y) = (\alpha_y)_\bullet(x)$, $\alpha_{\bullet 2}(x,y) = (\alpha^x)_\bullet(y)$ and note that we have the identities:

$$\begin{aligned} \alpha(x,y) &= \alpha(0,y) + x\alpha_{\bullet 1}(x,y), \\ \alpha(x,y) &= \alpha(x,0) + y\alpha_{\bullet 2}(x,y). \end{aligned} \tag{2.2}$$

Unfortunately, it turns out that there are functions $\alpha \in \mathcal{S}^2$ such that $\alpha_{\bullet 1}$ and $\alpha_{\bullet 2}$ do not belong to $\mathcal{S}^1$. For example, such a function can be of the form:

$$\alpha(x,y) = \begin{cases} \dfrac{x^2y + xy^2}{x^2 + y^2} & \text{for } (x,y) \in \mathbb{R}_o^2, \\ 0 & \text{for } (x,y) = o. \end{cases}$$

We shall consider the families $C^\infty(\mathbb{R}^2)$ and $\mathcal{S}^k$ to be real algebras under the pointwise operations. Let us set

$$m_k = \{\alpha \in \mathcal{S}^k : \alpha(o) = 0\},$$
$$m_k'' = \{\alpha \in \mathcal{S}^k : \alpha(o) = \frac{\partial \alpha}{\partial x}(o) = \frac{\partial \alpha}{\partial y}(o) = 0\}$$

and note that m_k, m_k'' are ideals of $\mathcal{S}^k$ such that

$$m_k^2 \subset m_k'' \subset m_k. \tag{2.3}$$

Obviously, $C^\infty(\mathbb{R}^2)$ is a subalgebra of $\mathcal{S}^k$ and the sets

$$m = m_k \cap C^\infty(\mathbb{R}^2), \ m'' = m_k'' \cap C^\infty(\mathbb{R}^2)$$

are ideals of $C^\infty(\mathbb{R}^2)$ which do not depend on k. Moreover, these ideals can be defined directly with respect to $C^\infty(\mathbb{R}^2)$. Well known is the following

Lemma 2.3. $m'' = m^2$.

We will show that this lemma has analogies for $\mathcal{S}^k$ which will be proved by using different methods for $k = 1$ and $k = 2$.

Proposition 2.4. $m_1'' = m_1^2$.

Proof. Let $\mathbb{R}_o^2 = \mathbb{R}^2 \setminus \{o\}$. Consider the sets

$$E = \{(x,y) \in \mathbb{R}_o^2 : |x| \le \tfrac{1}{2} |y|\},$$

$$F = \{(x,y) \in \mathbb{R}_o^2 : |y| \le \tfrac{1}{2} |x|\}.$$

Since E and F are disjoint closed subsets of the paracompact manifold $\mathbb{R}_o^2$, there is a function $\mu_o \in C^\infty(\mathbb{R}_o^2)$ such that

$$E \subseteq \mu_o^{-1}(0) \text{ and } F \subseteq \mu_o^{-1}(1).$$

We may extend μ_o to a real function μ on $\mathbb{R}^2$ by putting $\mu(o) = 0$. By (2.3), it remains to prove that $m_1'' \subset m_1^2$. Indeed, given $\alpha \in m_1''$, note that we have a decomposition

$$\alpha = \beta + \gamma,$$

where $\beta = \mu\alpha$ and $\gamma = (1-\mu)\alpha$. Observe that $\beta, \gamma \in m_1''$, and that $\beta|E = 0$, $\gamma|F = 0$. Hence and from (2.2) it follows that

$$(2.4) \qquad \beta(x,y) = x\cdot\beta_{\cdot 1}(x,y) \text{ and } \gamma(x,y) = y\cdot\gamma_{\cdot 2}(x,y),$$

where $\beta_{\cdot 1}(0,y) = 0$ and $\gamma_{\cdot 2}(x,0) = 0$. It is easy to see that $\beta_{\cdot 1}, \gamma_{\cdot 2} \in m_1$, which, by (2.4), implies $\beta, \gamma \in m_1^2$. Since $\alpha = \beta + \gamma$, we conclude that $\alpha \in m_1^2$, q.e.d.

The method used in the proof of Proposition 2.4 cannot be applied to prove a version of this proposition for $k = 2$. The following example shows that there is a function $\alpha \in m_2''$ such that, for each decomposition $\alpha = \beta + \gamma$ of the type described in this proof, at least one function $\beta_{\cdot 1}$ or $\gamma_{\cdot 2}$ does not belong to m_2.

Example 2.5. Consider the sequence $\{p_n\}$ of points of $\mathbb{R}^2$, where $p_n = (4^{-n}, 4^{-n})$ for $n = 1,2,\ldots$. Let us set $U_n = \{p \in \mathbb{R}^2 : \|p - p_n\| < 4^{-n-1}\}$ and note that, for every natural n, we can choose a function $\alpha_n \in C^\infty(\mathbb{R}^2)$ such that $\alpha_n(p_n) = 2^{-n}$, $0 \le \alpha_n(p) \le 2^{-n}$ and $\operatorname{supp} \alpha_n \subset U_n$. It is seen that $\{U_n\}$ is a discrete family of open subsets of $\mathbb{R}^2_o$, i.e. for each $p \in \mathbb{R}^2_o$, there is an open neighbourhood V of p in $\mathbb{R}^2_o$ such that V has a non-empty intersection with at most one U_n. Thus, the real function α is well-defined on $\mathbb{R}^2$ by $\alpha(p) = \sum_{n=1}^{\infty} \alpha_n(p)$. Observe that $\alpha \in \mathcal{S}^1$ and it is smooth on $\mathbb{R}^2_o$ and continuous at o, which, by Corollary 1.4, implies that $\alpha \in \mathcal{S}^2$. Let $\mu_o, \mu, \beta, \gamma, \beta_{\cdot 1}, \gamma_{\cdot 2}$ be the functions defined with respect to α analogously as in the proof of Proposition 2.4. First, note that if μ_o is unbounded, then the functions β and γ may be discontinuous. But we can additionally assume that

$$0 \le \mu_o(x,y) \le 1,$$

which implies that $\beta, \gamma \in \mathcal{S}^2$. Unfortunately, this additional assumption is not sufficient to avoid a contradiction. Indeed, if we suppose that both the functions $\beta_{\cdot 1}, \gamma_{\cdot 2}$ belong to m_2, then $\beta_{\cdot 1} + \gamma_{\cdot 2} \in m_2$. But, on the other hand, from (2.4) we have

$$(\beta_{\cdot 1} + \gamma_{\cdot 2})(p_n) = 4^n\beta(p_n) + 4^n\gamma(p_n) =$$
$$= 4^n\alpha(p_n)\mu(p_n) + 4^n\alpha(p_n)(1 - \mu(p_n)) = 2^n,$$

which means that $\beta_{\cdot 1} + \gamma_{\cdot 2}$ is not continuous at 0.

Applying Corollary 1.4 it is easy to prove

Lemma 2.6. *If* $\alpha \in \mathcal{S}^k$ *and there are a neighbourhood* U *of* o *in* $\mathbb{R}^2$ *and a constant* $c > 0$ *such that* $|\alpha(p)| \le c\|p\|^2$ *for each* $p \in U$, *then* $\alpha \in m_k^2$.

One can prove that if $\alpha \in m'' = m^2$ (Lemma 2.3), then, for each compact neighbourhood U of o in $\mathbb{R}^2$, i.e. $o \in \operatorname{int} U$, there is a constant $c_U > 0$ such that $|\alpha(p)| \le c_U\|p\|^2$ for $p \in U$. Observe that, for the function α constructed in Example 2.5, there is no such neighbourhood U of o. Indeed, suppose to the contrary that there is a neighbourhood U of o in $\mathbb{R}^2$ such that $|\alpha(p)| \le c\|p\|^2$ for $p \in U$ and some constant $c > 0$. Since the sequence $p_n = (4^{-n}, 4^{-n})$ converges to o, there is $m \in \mathbb{N}$ such that $p_n \in U$ for $n \ge m$. Thus, if $n \ge m$, then $2^{-n} = |\alpha(p_n)| \le c\|p_n\|^2 = 2c \cdot 4^{-2n}$, which is impossible.

Denote by $\mathcal{A}^2$ the family of all real functions α on $\mathbb{R}^2$ such that $\alpha|\mathbb{R}^2_o \in \mathcal{S}^2(\mathbb{R}^2_o)$ and α is bounded in some neighbourhood of o. Note that $\mathcal{A}^2$ is a real algebra under the pointwise operations, such that $\mathcal{S}^2$ is its subalgebra. Let us put $m_k(K) = \{\alpha \in \mathcal{S}^k: \alpha|K = 0\}$ and note that this set is an ideal of the algebra $\mathcal{S}^k$. It is seen that from Corollary 1.4 we get

Lemma 2.7. $m_2(K)$ *is an ideal of* $\mathcal{A}^2$. *In particular, we have* $\mathcal{A}^2 m_2(K) \subset m_2(K)$.

From Proposition 2.4 it follows that $m_1(K) \subset m_1^2$ and a similar result for $k = 2$ gives us

Lemma 2.8. $m_2(K) \subset m_2^2$.

Proof. Let us take $\alpha \in m_2(K)$. Consider the following sets $E = \{p \in \mathbb{R}_o^2: |\alpha(p)| \le \|p\|^2\}$ and $F = \{p \in \mathbb{R}_o^2: |\alpha(p)| \ge 2\|p\|^2\}$. Obviously, $E \ne \emptyset$ but F may be empty. If $F = \emptyset$, then $|\alpha(p)| \le 2\|p\|^2$ for each $p \in \mathbb{R}^2$, and so, $\alpha \in m_2^2$ by Lemma 2.6. Therefore, in the sequel, we can suppose that $F \ne \emptyset$. Since E and F are disjoint non-empty closed subsets of the paracompact manifold $\mathbb{R}_o^2$, there is a function $\mu_o \in C^\infty(\mathbb{R}_o^2)$ such that $E \subset \mu_o^{-1}(0)$, $F \subset \mu_o^{-1}(1)$ and $0 \le \mu_o(x,y) \le 1$. Let μ be the extension of μ_o to a real function μ on $\mathbb{R}^2$ by putting $\mu(o) = 0$. We thus have a decomposition

$$\alpha = \beta + \gamma, \tag{2.5}$$

where $\beta = \mu\alpha$ and $\gamma = (1-\mu)\alpha$. Moreover, from Lemma 2.7 it follows that $\beta, \gamma \in m_2(K)$. Note that $|\gamma(p)| \le 2\|p\|^2$ for each $p \in \mathbb{R}^2$, and so, Lemma 2.6 implies $\gamma \in m_2^2$. Thus, by (2.5), it remains to prove that $\beta \in m_2^2$.

For any $n \in \mathbb{Z}$, let us set

$$U_n = \{p \in \mathbb{R}^2: 8^{n-1} < \|p\| < 8^{n+1}\} \tag{2.6}$$

and note that $\{U_n: n \in \mathbb{Z}\}$ is an open covering of $\mathbb{R}_o^2$. Since $\mathbb{R}_o^2$ is a paracompact manifold, there is a smooth partition $\{\lambda_n\}$ of unity, subordinated to the covering $\{U_n\}_{n \in \mathbb{Z}}$. We may extend every λ_n to a smooth function on $\mathbb{R}^2$, also denoted as λ_n, by putting $\lambda_n(o) = 0$. This means that $\{\lambda_n\}$ is a family of smooth functions on $\mathbb{R}^2$ satisfying the following conditions:

(a) $\operatorname{supp} \lambda_n \subset U_n$,

(b) $0 \le \lambda_n(p) \le 1$,

(c) $\sum_{n \in \mathbb{Z}} \lambda_n = \chi$, where χ is the characteristic function on $\mathbb{R}^2$ of the set $\mathbb{R}^2_o$ and the series is locally finite at each point of $\mathbb{R}^2_o$. Observe that if $|n - m| \geq 2$, then $\lambda_n \cdot \lambda_m = 0$. Hence, by squaring both sides of equality (c), we get

$$\sum_{n\in\mathbb{Z}} \lambda_n^2 + 2 \sum_{n\in\mathbb{Z}} \lambda_n \lambda_{n+1} = \chi. \tag{2.7}$$

It is clear that, for each $n \in \mathbb{Z}$, we can choose a function $\omega_n \in C^\infty(\mathbb{R})$ such that

(d) $\omega_n(t) = t^{1/3}$ for $|t| \geq 8^{n-1}$,

(e) $|\omega_n(t)| \leq |t|^{1/3}$ for $|t| < 8^{n-1}$.

If $i,k \in \mathbb{Z}$, $k \geq 2$ and $0 \leq i < k$, we set

$$\mathbb{Z}^{i:k} = \{n \in \mathbb{Z}\colon n \equiv i \pmod{k}\}.$$

For $i = 0,1$, consider the functions τ_i, υ_i defined on $\mathbb{R}^2$ by

$$\tau_i = \sum_{n\in\mathbb{Z}^{i:2}} \lambda_n \cdot \omega_n(\alpha) \ , \ \upsilon_i = \sum_{n\in\mathbb{Z}^{i:2}} \lambda_n \cdot (\omega_n(\alpha))^2. \tag{2.8}$$

Obviously, these functions are well-defined because the series are locally finite on $\mathbb{R}^2_o$. Moreover, it is seen that

$$\tau_i, \upsilon_i | \mathbb{R}^2_o \in \mathcal{S}^2(\mathbb{R}^2_o) \subset C(\mathbb{R}^2_o). \tag{2.9}$$

Let us take $0 < \varepsilon < 1$. Consider the set

$$W = \{p \in \mathbb{R}^2\colon |\alpha(p)|^{1/3} < \varepsilon\}$$

and note that it is an open neighborhood of o because $\alpha \in \mathcal{S}^2 \subset C(\mathbb{R}^2)$. Observe that (2.8) and conditions (b)-(e) imply that, for each $p \in W$, we have

$$|\tau_i(p)| \leq \sum_{n\in\mathbb{Z}^{i:2}} \lambda_n(p) \cdot |\omega_n(\alpha(p))| \leq |\alpha(p)|^{1/3} < \varepsilon,$$

$$|\upsilon_i(p)| \leq \sum_{n\in\mathbb{Z}^{i:2}} \lambda_n(p) \cdot |\omega_n(\alpha(p))|^2 \leq |\alpha(p)|^{2/3} < \varepsilon^2 < \varepsilon,$$

which means that τ_i and υ_i are continuous at o. Hence and from (2.9) we conclude that $\tau_i, \upsilon_i \in C(\mathbb{R}^2) = \mathcal{F}^2$. Moreover, note that $\tau_i | K = \upsilon_i | K = 0$, and so, from Corollary 1.4 it follows that

$$\tau_i, \upsilon_i \in m_2(K) \subset m_2. \tag{2.10}$$

Next, for $j = 0,1,2$, consider the functions ϕ_j, ψ_j defined on $\mathbb{R}^2$ by

$$\phi_j = \sum_{n\in\mathbb{Z}^{j:3}} \lambda_n \cdot \omega_n(\alpha) \ , \ \psi_j = \sum_{n\in\mathbb{Z}^{j:3}} \lambda_{n+1} \cdot (\omega_n(\alpha))^2 \tag{2.11}$$

and, similarly as for τ_i, υ_i, note that

$$\phi_j, \psi_j \in m_2(K) \subset m_2. \tag{2.12}$$

Observe now that, for each $n \in \mathbb{Z}^{i:2}$ $(i = 0,1)$, we have

$$\lambda_n^2 \cdot (\omega_n(\alpha))^3 \mu = \lambda_n^2 \cdot \alpha\mu. \tag{2.13}$$

Indeed, if $p \in E \cup (\mathbb{R}^2 \setminus U_n)$ then both sides of this equality assume 0 at p. Otherwise, if $p \in U_n \setminus E$, then, from (2.6) and the definition of E, one has $|\alpha(p)| > \|p\| > 8^{n-1}$; thus, by (d), $\omega_n(\alpha)(p) = (\alpha(p))^{1/3}$, which implies (2.13) at p.

Observe that from (2.8) we get

$$\tau_i \upsilon_i = \sum_{n \in \mathbb{Z}^{i:2}} \lambda_n^2 \cdot (\omega_n(\alpha))^3 \tag{2.14}$$

because $\lambda_n \lambda_m = 0$ for distinct $n, m \in \mathbb{Z}^{i:2}$. Thus, from (2.13), (2.14) and the definition of β it follows that

$$\tau_i \upsilon_i \mu = \sum_{n \in \mathbb{Z}^{i:2}} \lambda_n^2 \cdot \beta \quad (i = 0,1) \tag{2.15}$$

Analogously, one can observe that, for each $n \in \mathbb{Z}^{j:3}$ $(j = 0,1,2)$, we have

$$\lambda_n \lambda_{n+1} \cdot (\omega_n(\alpha))^3 \mu = \lambda_n \lambda_{n+1} \alpha\mu$$

Moreover, (2.11) implies

$$\phi_j \psi_j = \sum_{n \in \mathbb{Z}^{j:3}} \lambda_n \lambda_{n+1} \cdot (\omega_n(\alpha))^3,$$

and so, in a way similar as for τ_i, υ_i, we get

$$\phi_j \psi_j \mu = \sum_{n \in \mathbb{Z}^{j:3}} \lambda_n \lambda_{n+1} \cdot \beta \quad (j = 0,1,2). \tag{2.16}$$

Now, from (2.15),(2.16) and (2.7) it follows that

$$\sum_{i=0}^{1} \tau_i \upsilon_i \mu + 2 \sum_{j=0}^{2} \phi_j \psi_j \mu = \Big(\sum_{n \in \mathbb{Z}} \lambda_n^2 + 2 \sum_{n \in \mathbb{Z}} \lambda_n \lambda_{n+1} \Big) \beta = \chi\beta = \beta. \tag{2.17}$$

Finally, note that Lemma 2.7 implies that all $\upsilon_i \mu$ and $\psi_j \mu$ belong to $m_2(K)$ because $\upsilon_i, \psi_j \in m_2(K)$ and $\mu \in \mathcal{A}^2$. Thus, from (2.10),(2.12) and (2.17) we conclude that $\beta \in m_2^2$, q.e.d.

For any $\alpha \in \mathcal{S}^k$ we set $\alpha_+(x,y) = \alpha(x,0) + \alpha(0,y) - \alpha(0,0)$ and $\alpha_- = \alpha - \alpha_+$, that is, we have a decomposition

$$\alpha = \alpha_+ + \alpha_-,$$

where $\alpha_+ \in C^\infty(\mathbb{R}^2)$, $\alpha_+|K = \alpha|K$ and $\alpha_- \in m_k(K)$.

From (2.3), Lemmas 2.3 and 2.8 and this decomposition it follows

Proposition 2.9. $m_2'' = m_2^2$.

Theorem 2.10. $T(\mathbb{R} \times_k \mathbb{R}) = T^+(\mathbb{R} \times_k \mathbb{R})$.

Proof. Clearly, it suffices to show that the following condition holds:

(+) $\dim T_p(\mathbb{R} \times_k \mathbb{R}) = 2$ *for any* $p \in \mathbb{R}$.

Indeed, for any $\alpha \in \mathcal{S}^k$ we can define the functions $\alpha'(x,y) = \alpha(o) + x\cdot\frac{\partial\alpha}{\partial x}(o) + y\cdot\frac{\partial\alpha}{\partial y}(o)$ and $\alpha'' = \alpha - \alpha' \in m_k''$. Obviously, we have a decomposition

$$\alpha = \alpha' + \alpha'',$$

which proves that $\dim(m_k/m_k'') = 2$, i.e. the congruent classes of x and y form a base of m_k/m_k''. Hence $\dim(m_k/m_k^2) = 2$ because $m_k^2 = m_k''$ by Propositions 2.4 and 2.9. Since the vector space $T_o(\mathbb{R} \times_k \mathbb{R})$ is isomorphic to the dual space of m_k/m_k^2, it follows that $\dim T_o(\mathbb{R} \times_k \mathbb{R}) = 2$. This implies that condition (+) is fulfilled because, by Lemma 2.1, any translation of $\mathbb{R}^2$ is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$, q.e.d.

3. Vector fields

In the sequel, the families $\mathcal{F} = \mathcal{F}^1$ and $\mathcal{S}^k$ $(k = 1,2)$ will be regarded as rings (real algebras) under the pointwise operations. Denote by $\mathfrak{X}(\mathbb{R} \times_k \mathbb{R})$ the module over $\mathcal{F}$ of all vector fields on $\mathbb{R} \times_k \mathbb{R}$, that is, $X \in \mathfrak{X}(\mathbb{R} \times_k \mathbb{R})$ if $X: \mathbb{R} \times_k \mathbb{R} \longrightarrow T(\mathbb{R} \times_k \mathbb{R})$ is a map such that $X_p = X(p) \in T_p(\mathbb{R} \times_k \mathbb{R})$ for each $p \in \mathbb{R}^2$ or, equivalently, if $X: \mathcal{S}^k \longrightarrow \mathcal{F}$ is a linear map such that $X(\alpha\beta) = X(\alpha)\beta + \alpha X(\beta)$.

Denote by $\mathcal{D}^2(k)$ the group of all diffeomorphisms of $\mathbb{R} \times_k \mathbb{R}$. If $f \in \mathcal{D}^2(k)$ and $X \in \mathfrak{X}(\mathbb{R} \times_k \mathbb{R})$, we define the vector field $f_\#(X)$ on $\mathbb{R} \times_k \mathbb{R}$ by $(f_\#(X))_p = f_{*p}(X_p)$. Obviously, the assignment $f \mapsto f_\#$ defines an isomorphism from $\mathcal{D}^2(k)$ into the group of all automorphisms of the module $\mathfrak{X}(\mathbb{R} \times_k \mathbb{R})$. We say that $X \in \mathfrak{X}(\mathbb{R} \times_k \mathbb{R})$ is invariant if $t_\#(X) = X$ for each $t \in \mathfrak{T}$. If $v \in T_o(\mathbb{R} \times_k \mathbb{R})$, we define the vector field X^v on $\mathbb{R} \times_k \mathbb{R}$ by $X_p^v = t_{op\#}(v)$, where t_{op} is a unique translation of $\mathbb{R}^2$ mapping o to p. It is seen that X is invariant iff $X = X^v$ for $v = X_o$.

Denote by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ the vector fields on $\mathbb{R} \times_k \mathbb{R}$ defined by the assignments $p \mapsto \frac{\partial}{\partial x}|_p$ and $p \mapsto \frac{\partial}{\partial y}|_p$. It is clear that they are invariant. Moreover, note that from Theorem 2.10 it follows that every invariant vector field on $\mathbb{R} \times_k \mathbb{R}$ is of the form $a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ for some constants $a,b \in \mathbb{R}$. A vector field $X \in \mathcal{X}(\mathbb{R} \times_k \mathbb{R})$ is called standard if

$$X = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \text{ for some } \alpha,\beta \in \mathcal{F}.$$

From Theorem 2.10 we get

Corollary 3.1. *Every vector field on* $\mathbb{R} \times_k \mathbb{R}$ *is standard.*

Call a vector field $X \in \mathcal{X}(\mathbb{R} \times_k \mathbb{R})$ *smooth* if $X(\mathcal{S}^k) \subset \mathcal{S}^k$. Obviously, the zero vector field on $\mathbb{R} \times_k \mathbb{R}$ is smooth.

Theorem 3.2. *There is no non-zero smooth vector field on* $\mathbb{R} \times_k \mathbb{R}$.

Proof. Suppose to the contrary that there is a non-zero smooth vector field X on $\mathbb{R} \times_k \mathbb{R}$. From Corollary 3.1 we get

$$X = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}, \tag{3.1}$$

where $\alpha,\beta \in \mathcal{F}$. Moreover, observe that $\alpha,\beta \in \mathcal{S}^k$ because X is smooth and $\alpha = X(\pi_1)$, $\beta = X(\pi_2)$, where $\pi_i \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the projection onto the i-th axis $(i = 1,2)$. Since the vector field X is non-zero, we may assume without loss of generality that there is $p \in \mathbb{R}^2$ such that $\alpha(p) > 0$. Let us take $\alpha(p) > \varepsilon > 0$ and consider the $\mathcal{S}^k$-open neighbourhood U of p defined by

$$U = \{(x,y) \in \mathbb{R}^2 \colon \alpha(x,y) > \varepsilon\}.$$

Let λ be a real smooth function on $\mathbb{R}$ such that $\lambda(t) > 0$ for each $t \in \mathbb{R}$, and $\lambda(t) = t$ for $t \geq \varepsilon$. Define the function

$$\alpha_*(x,y) = \frac{1}{\lambda(\alpha(x,y))}$$

and note that $\alpha_* \in \mathcal{S}^k$. Thus, for each $\phi \in \mathcal{S}^2 \subset \mathcal{S}^k$, from (3.1) we get

$$\alpha_* X(\phi) = \alpha_* \alpha \frac{\partial \phi}{\partial x} + \alpha_* \beta \frac{\partial \phi}{\partial y} .$$

Hence

$$\frac{\partial \phi}{\partial x}|U = (\alpha_* X(\phi) - \alpha_* \beta \frac{\partial \phi}{\partial y})|U$$

because $\alpha_* \alpha|U = 1|U$, which implies that $\frac{\partial \phi}{\partial x}|U \in \mathcal{S}^1(U)$ provided that $\frac{\partial\phi}{\partial y}|U \in \mathcal{S}^1(U)$. But there is a function $\phi \in \mathcal{S}^2$, namely,

$$\phi(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{for } (x,y) \in \mathbb{R}^2_o, \\ 0 & \text{for } (x,y) = o, \end{cases}$$

such that, for any neighbourhood V of o, we have $\frac{\partial\phi}{\partial y}|V \in \mathcal{S}^1(V)$ but $\frac{\partial\phi}{\partial x}|V \notin \mathcal{S}^1(V)$, which gives a contradiction for p = o. This leads to a contradiction in the general case for any p since, by Lemma 2.1, the translation of $\mathbb{R}^2$ via any vector is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$, q.e.d.

It is known that if G is a differential group (see [4]), then each vector $v \in T_e(G)$ determines a unique left invariant vector field X_v on G such that $X_v(e) = v$, where e denotes the identity of G, and any such field is smooth (see [5]). Thus, if $\dim T_e(G) > 0$, then there are non-zero smooth vector fields on G. Of course, $\mathbb{R} \times_k \mathbb{R}$ is a group under the coordinatewise addition and $\dim T_o(\mathbb{R} \times_k \mathbb{R}) = 2$ by Theorem 2.10. Hence and from Theorem 3.2 we obtain

Corollary 3.3. *The group $\mathbb{R} \times_k \mathbb{R}$ is non-differential.*

In particular, $\mathbb{R} \times_k \mathbb{R}$ is not a differential vector space under the coordinatewise operations because these operations, namely coordinatewise subtraction and multiplication by scalars, are non-smooth. Moreover, the multiplication of $v \in \mathbb{R} \times_k \mathbb{R}$ by scalars, i.e. the function $t \mapsto tv$, is smooth if and only if v is horizontal or vertical (see Theorem 4.1).

A triple $\xi = (E,\pi,M)$ is said to be a *differential bundle* if $\pi: E \longrightarrow M$ is a smooth surjection of differential spaces and, for each $p \in M$, the fibre $E_p = \pi^{-1}(p)$ has a given vector structure such that E_p is a differential vector space, where the differential structure on E_p is induced from E. We shall denote by $\Gamma(\xi)$ the $\mathcal{E}(M)$-module of all smooth cross-sections of ξ. It is known that with every differential space M we can associate the differential tangent bundle $TB(M) = (T(M),\pi_M,M)$ as follows. T(M) is the differential tangent space of M (see [3]) and $\pi_M: T(M) \longrightarrow M$ is the canonical surjection. Moreover, every fibre $T_p(M) = \pi_M^{-1}(p)$ is a differential vector space under the canonical vector structure and the differential structure induced from T(M). The $\mathcal{E}(M)$-module of all smooth vector fields on M is defined to be the module $\mathfrak{X}(M) = \Gamma(TB(M))$.

Consider the differential tangent bundle $TB(\mathbb{R} \times_k \mathbb{R}) = (T(\mathbb{R} \times_k \mathbb{R}), \pi_k, \mathbb{R} \times_k \mathbb{R})$, where $\pi_k = \pi_{\mathbb{R} \times_k \mathbb{R}}$. By Theorem 2.10, for each $p \in \mathbb{R} \times_k \mathbb{R}$, we have $\pi_k^{-1}(p) = T_p(\mathbb{R} \times_k \mathbb{R}) = T_p^+(\mathbb{R} \times_k \mathbb{R})$. Moreover, the differential vector space $T_p(\mathbb{R} \times_k \mathbb{R})$ is isomorphic to the differential space $\mathbb{R}^2$ with structure $\mathcal{E}(\mathbb{R}^2) = C^\infty(\mathbb{R}^2)$, via the isomorphism $\tau_{k,p}: T_p(\mathbb{R} \times_k \mathbb{R}) \longrightarrow \mathbb{R}^2$ defined by $\tau_{k,p}(a \frac{\partial}{\partial x}|_p + b \frac{\partial}{\partial y}|_p) = (a,b)$.

Corollary 3.4. *The differential space* $\mathbb{R} \times_k \mathbb{R}$ *is not diffeomorphic to the differential space* $\mathbb{R}^2$.

Proof. The assertion follows from the fact that $\mathfrak{X}(\mathbb{R} \times_k \mathbb{R})$ is the zero module by Theorem 3.2, but $\mathfrak{X}(\mathbb{R}^2)$ is not, q.e.d.

In the next section, we shall prove that the differential spaces $\mathbb{R} \times_1 \mathbb{R}$ and $\mathbb{R} \times_2 \mathbb{R}$ are non-diffeomorphic.

Following Trafny [7], we can define the *category of differential bundles* with morphisms given by *smooth bundle maps*. Recall that a differential bundle $\xi = (E,\pi,M)$ is *locally trivial* at a *point* $p \in M$ if there is a neighbourhood U of p such that the differential bundle $\xi|U$ is isomorphic to a trivial differential bundle. We say that ξ is *locally non-trivial* at p if it is not locally trivial at p. A differential bundle $\xi = (E,\pi,M)$ is said to be *locally non-trivial* if it is locally non-trivial at each point $p \in M$. It is easy to prove

Lemma 3.5. *If a differential bundle* $\xi = (E,\pi,M)$ *is locally trivial at* $p \in M$, *then, for every* $v \in E_p$, *there is* $\sigma \in \Gamma(\xi)$ *such that* $\sigma(p) = v$.

From Theorem 2.10, Theorem 3.2 and Lemma 3.5 we get

Corollary 3.6. *The differential tangent bundle* $TB(\mathbb{R} \times_k \mathbb{R})$ *is locally non-trivial.*

Let $\iota k: \mathbb{R} \times_k \mathbb{R} \longrightarrow \mathbb{R}^2$ be the identity map regarded as a smooth map of differential spaces. For any $p \in \mathbb{R} \times_k \mathbb{R}$ and $v \in T_p(\mathbb{R} \times_k \mathbb{R})$, we define $T(\iota k)_p: T_p(\mathbb{R} \times_k \mathbb{R}) \longrightarrow T_p(\mathbb{R}^2)$ by $(T(\iota k)_p(v)(\alpha) = v(\alpha \circ \iota k)$, where α ranges over $C^\infty(\mathbb{R}^2)$. Next, the map $T(\iota k)\colon T(\mathbb{R}\times_k\mathbb{R})\longrightarrow T(\mathbb{R}^2)$ is defined by $T(\iota k)(v) = T(\iota k)_p(v)$ for $v\in T_p(\mathbb{R}\times_k\mathbb{R})$ and called the differential of ιk. Note that $T(\iota k)$ is a smooth map of differential spaces. Moreover, it is seen that the pair $(\iota k, T(\iota k))$ defines a smooth bundle map from $(T(\mathbb{R}\times_k\mathbb{R}), \pi k, \mathbb{R}\times_k\mathbb{R})$ onto $(T(\mathbb{R}^2), \pi_{\mathbb{R}^2}, \mathbb{R}^2)$, which means that the following diagram of smooth maps of differential spaces is commutative:

$$\begin{array}{ccc} T(\mathbb{R}\times_k\mathbb{R}) & \xrightarrow{T(\iota k)} & T(\mathbb{R}^2) \\ \downarrow{\scriptstyle \pi k} & & \downarrow{\scriptstyle \pi_{\mathbb{R}^2}} \\ \mathbb{R}\times_k\mathbb{R} & \xrightarrow{\iota k} & \mathbb{R}^2 \end{array}$$

where $T(\iota k)_p\colon T_p(\mathbb{R}\times_k\mathbb{R})\longrightarrow T_p(\mathbb{R}^2)$ is an isomorphism of differential vector spaces for each $p\in\mathbb{R}^2$.

Let M and N be differential spaces. We say that $g\colon T(M)\longrightarrow T(N)$ is a *fibre diffeomorphism* if g is a diffeomorphism of differential spaces and, for each $p\in M$, there is a unique $q\in N$ such that $g(T_p(M)) = T_q(N)$. In addition, if g is an isomorphism of differential vector spaces from $T_p(M)$ onto $T_q(N)$ for p and q as above, then g is called a *bundle diffeomorphism*. The tangent differential spaces T(M) and T(N) are called *fibre (bundle) diffeomorphic* if there is a fibre (bundle) diffeomorphism $g\colon T(M)\longrightarrow T(N)$. Note that if $g\colon T(M)\longrightarrow T(N)$ is a fibre (bundle) diffeomorphism, then so is the inverse map $g^{-1}\colon T(N)\longrightarrow T(M)$. It is easy to verify

Lemma 3.7. *Let* M *and* N *be differential spaces. If* $g\colon T(M)\longrightarrow T(N)$ *is a fibre diffeomorphism, then there is a unique diffeomorphism* $f\colon M\longrightarrow N$ *such that the following diagram of smooth maps of differential spaces is commutative:*

$$\begin{array}{ccc} T(M) & \xrightarrow{g} & T(N) \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N \end{array}$$

This lemma immediately implies

Proposition 3.8. *Let* M *and* N *be differential spaces. Then the following conditions are equivalent:*

(a) M *and* N *are diffeomorphic,*

(b) T(M) *and* T(N) *are bundle diffeomorphic,*

(c) T(M) *and* T(N) *are fibre diffeomorphic.*

From this proposition and Corollary 3.4 we get

Corollary 3.9. *The tangent differential spaces* $T(\mathbb{R}\times_k\mathbb{R})$ *and* $T(\mathbb{R}^2)$ *are not fibre diffeomorphic. In particular, the smooth map* $T(\iota k)\colon T(\mathbb{R}\times_k\mathbb{R})\longrightarrow T(\mathbb{R}^2)$ *is not a diffeomorphism.*

Note that this corollary gives no answer to the question: Are the differential spaces $T(\mathbb{R}\times_k\mathbb{R})$ and $T(\mathbb{R}^2)$ diffeomorphic (see Question 5.4)?

4. Regular curves and diffeomorphisms

Let $c\colon I\longrightarrow\mathbb{R}\times_k\mathbb{R}$ be a smooth curve, that is, c is a smooth map of differential spaces, where I is an open (non-empty) interval of $\mathbb{R}$, regarded as a differential space under the structure $C^\infty(I)$ of all real smooth functions on I. Note that the map c determines the differential $c_* = T(c)\colon T(I)\longrightarrow T(\mathbb{R}\times_k\mathbb{R})$. Let $\partial_s = \frac{\partial}{\partial t}|_s$ be the standard tangent vector of I at s. We say that c is *regular* at $s\in I$ provided that $c_*\partial_s$ is a non-zero vector of $T_{c(s)}(\mathbb{R}\times_k\mathbb{R})$. By a regular curve in $\mathbb{R}\times_k\mathbb{R}$ we mean a smooth curve $c\colon I\longrightarrow\mathbb{R}\times_k\mathbb{R}$ which is regular at each $s\in I$.

A line in $\mathbb{R}^2$ is called *vertical* (*horizontal*) if it is of the form $\{a\}\times\mathbb{R}$ ($\mathbb{R}\times\{b\}$). By a *principal line* we mean a line in $\mathbb{R}^2$ which is vertical or horizontal. We have

Theorem 4.1. *Every regular curve in* $\mathbb{R}\times_k\mathbb{R}$ *is contained in a principal line.*

Proof. Let us take an arbitrary regular curve $c\colon I\longrightarrow\mathbb{R}\times_k\mathbb{R}$. Obviously, c is a regular smooth curve in the classical sense because $C^\infty(\mathbb{R}^2)\subset\mathcal{S}^k$. Let $\phi=\pi_1\circ c$ and $\psi = \pi_2\circ c$, i.e. $c(t)=(\phi(t),\psi(t))$ for $t\in I$. First, we prove that the following condition is satisfied:

(A) *for each* $s\in I$, *the non-zero vector* $c_*\partial_s = \alpha(s)\frac{\partial}{\partial x}|_{c(s)} + \beta(s)\frac{\partial}{\partial y}|_{c(s)}$ *is principal, i.e.* $\alpha(s)\beta(s) = 0$, *where* $\alpha = \frac{\partial\phi}{\partial t}$, $\beta = \frac{\partial\psi}{\partial t}$ *and* $\frac{\partial}{\partial x}|_{c(s)}$, $\frac{\partial}{\partial y}|_{c(s)}$ *are the standard tangent vectors of* $\mathbb{R}\times_k\mathbb{R}$ *at* $c(s)$.

Indeed, suppose to the contrary that there is $s\in I$ such that the vector $c_*\partial_s = \alpha(s)\frac{\partial}{\partial x}|_{c(s)} + \beta(s)\frac{\partial}{\partial y}|_{c(s)}$ is not principal or, equivalently, $\alpha(s)\beta(s) \neq 0$. Without loss of generality we may assume that $s = 0$. Since, by Lemma 2.1, the translation of $\mathbb{R}^2$ via the vector $\overrightarrow{po}$ is a diffeomorphism of $\mathbb{R}\times_k\mathbb{R}$, we may further assume that $p = o$.

Observe that the map c is a local C^∞ diffeomorphism at 0 because $c_*\partial_0 \neq 0$. Therefore there is $\varepsilon > 0$ such that $c|[-\varepsilon,\varepsilon]$ is a C^∞ diffeomorphism onto its image in $\mathbb{R}^2$. Moreover, since $\alpha(0)\beta(0) \neq 0$ and $\alpha,\beta\in C^\infty(\mathbb{R})$, we can choose ε so small that $C\cap K = \{o\}$ where $C = c([-\varepsilon,\varepsilon])$ and $K = \{(x,y)\in\mathbb{R}^2: xy = 0\}$.

Consider the function $m: t\mapsto |t|$ for $-\varepsilon\le t\le\varepsilon$ and note that there is a unique real function μ on C such that $\mu\circ c = m$. Accept the following notations: $C_o = C\setminus\{o\}$, $K_o = K\setminus\{o\}$ and $F = C_o\cup K_o$. Define the real function μ_o on F as follows:

$$\mu_o(p) = \begin{cases} \mu(p) & \text{for } p\in C_o, \\ 0 & \text{for } p\in K_o \end{cases}$$

and note that $\mu_o\in C^\infty(F)$.

Since F is a closed subset of $\mathbb{R}^2_o$, $\sup\{|\mu_o(p)|: p\in F\} = \varepsilon$, and $\mathbb{R}^2_o$ is a paracompact manifold, therefore we conclude that there is an extension $\bar{\mu}\in C^\infty(\mathbb{R}^2_o)$ of μ_o such that $\sup\{|\bar{\mu}(p)|: p\in\mathbb{R}^2_o\}\le 2\varepsilon$. It is seen that the function $\lambda: \mathbb{R}^2\longrightarrow\mathbb{R}$ defined by

$$\lambda(x,y) = \begin{cases} x\bar{\mu}(x,y) & \text{for } (x,y)\in\mathbb{R}^2_o, \\ 0 & \text{for } (x,y) = o \end{cases}$$

is continuous on $\mathbb{R}^2$ and $\lambda\in\mathcal{S}^1$ by Corollary 1.4, which implies that $\lambda\in\mathcal{S}^2$. Since $c: I\longrightarrow\mathbb{R}\times_k\mathbb{R}$ is a smooth map and $\mathcal{S}^2\subset\mathcal{S}^k$, it follows that $\lambda\circ c\in C^\infty(I)$. Hence we get

$$\lambda\circ c|[-\varepsilon,\varepsilon] = \phi\cdot m|[-\varepsilon,\varepsilon]\in C^\infty([-\varepsilon,\varepsilon]). \tag{4.1}$$

On the other hand, $(\lambda\circ c)(t) = \phi(t)|t|$ for $-\varepsilon\le t\le\varepsilon$, and so, the second derivative of this function satisfies the equalities:

$$(\lambda\circ c)''(t) = \alpha'(t)t + 2\alpha(t) \text{ for } 0 < t \le \varepsilon,$$
$$(\lambda\circ c)''(t) = -\alpha'(t)t - 2\alpha(t) \text{ for } -\varepsilon \le t < 0.$$

Hence we get

$$\lim_{t\to 0+} (\lambda\circ c)''(t) = 2\alpha(0) \ne \lim_{t\to 0-} (\lambda\circ c)''(t) = -2\alpha(0)$$

because $\alpha(0) \ne 0$, which contradicts (4.1). This completes the proof of condition (A).

Finally, note that condition (A) is satisfied for all regular smooth parametrizations of c. In particular, we can consider the natural parametrization of c with respect to the arc parameter s. Then $\alpha \equiv 1$, $\beta \equiv 0$ or $\alpha \equiv 0$, $\beta \equiv 1$, which implies that $c_*\partial_s = \frac{\partial}{\partial x}|_{c(s)}$ for each $s \in I$ or $c_*\partial_s = \frac{\partial}{\partial y}|_{c(s)}$ for each $s \in I$, q.e.d.

This theorem immediately implies

Corollary 4.2. *If f is a diffeomorphism of* $\mathbb{R} \times_k \mathbb{R}$, *then f maps every principal line onto a principal line.*

The following example shows that a non-regular smooth curve in $\mathbb{R} \times_k \mathbb{R}$ need not be contained in a principal line.

Example 4.3. Let $c\colon \mathbb{R} \longrightarrow \mathbb{R} \times_k \mathbb{R}$ be a curve defined by

$$c(t) = \begin{cases} (\exp(1/t),0) & \text{for } t < 0, \\ (0,0) & \text{for } t = 0, \\ (0,\exp(-1/t)) & \text{for } t > 0. \end{cases}$$

Note that c is smooth and regular except for $t = 0$. Moreover, observe that the image of c is contained in the set $(\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$, but it is not contained in $\mathbb{R} \times \{0\}$ or $\{0\} \times \mathbb{R}$ separately.

A map $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is said to be *principal* if it maps every principal line into a principal line. We say that f is *0-principal* (*1-principal*) provided that f maps every vertical line into a vertical (horizontal) line and every horizontal line into a horizontal (vertical) line. It is seen that f may be 0-principal and 1-principal simultaneously. We say that f is *exactly 0-principal* (*1-principal*) if it is 0-principal (1-principal) but not 1-principal (0-principal). In this case, we write $\tau(f) = 0$ ($\tau(f) = 1$) and say that f is of *P-type*

$\tau = \tau(f)$. If f is exactly τ-principal for some $\tau \in \{0,1\}$, then f is said to be of *definite P-type*. For example, if s is a bijection of $\mathbb{R}^2$ defined by $(x,y) \mapsto (y,x)$, then s is of P-type 1. Furthermore, if ϕ and ψ are bijections of $\mathbb{R}$, then the map $\phi \times \psi$ defined by $(x,y) \mapsto (\phi(x),\psi(y))$ is a bijection of $\mathbb{R}^2$ and it is of P-type 0. Note that s is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$, and so is the map $\phi \times \psi$ provided that ϕ and ψ are smooth diffeomorphisms of $\mathbb{R}$.

A map $f\colon M \longrightarrow N$ of differential spaces is said to be a *smooth embedding* if f regarded as a map from M onto the differential subspace $f(M)$ of N is a diffeomorphism. We have

Theorem 4.4. *Let f be a smooth bijective map from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$, where $k,\ell \in \{1,2\}$, such that f restricted to every prncipal line is a smooth embedding. Then $k = \ell$ and f is of the form $\phi \times \psi$ or of the form $s\circ(\phi \times \psi)$, where ϕ and ψ are smooth diffeomorphisms of $\mathbb{R}$. In particular, f is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$.*

Proof. For any $x,y \in \mathbb{R}$, let us set $R^x = \{x\} \times \mathbb{R}$, $L^y = \mathbb{R} \times \{y\}$ and define the maps $r^x\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $l^y\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ by $r^x(s) = (x,s)$ and $l^y(t) = (t,y)$. We can regard the sets R^x, L^y as differential subspaces of $\mathbb{R} \times_k \mathbb{R}$ or $\mathbb{R} \times_\ell \mathbb{R}$ and the maps r^x, l^y as smooth curves in $\mathbb{R} \times_k \mathbb{R}$. Let us set $c^x = f\circ r^x$ and $d^y = f\circ l^y$. It is seen that c^x, d^y are smooth curves in $\mathbb{R} \times_\ell \mathbb{R}$.

Step 1. Consider now the curve $d = d^0\colon \mathbb{R} \longrightarrow \mathbb{R} \times_\ell \mathbb{R}$. Since f maps the line L^0 diffeomorphically onto its image in $\mathbb{R} \times_\ell \mathbb{R}$, there is a function $\alpha \in \mathcal{S}^\ell(f(L^0))$ such that $\alpha\circ f = \pi_1|L^0$. Then, for any $t \in \mathbb{R}$, there are an $\mathcal{S}^\ell$-open neighbourhood U of $f(t,0)$ in $\mathbb{R} \times_\ell \mathbb{R}$ and a function $\beta \in \mathcal{S}^\ell$, such that $\beta|U \cap f(L^0) = \alpha|U \cap f(L^0)$. Obviously, $\pi_1\circ l^0 = \mathrm{id}_{\mathbb{R}}$ and note that we have the following equalities: $(d_*\partial_t)(\beta) = \partial_t(\beta\circ d) = \partial_t(\alpha\circ d) = \partial_t(\alpha\circ f\circ l^0) = \partial_t(\pi_1\circ l^0) = \partial_t(\mathrm{id}_{\mathbb{R}}) = 1$. Hence it follows that $d_*\partial_t \neq 0$ for any $t \in \mathbb{R}$, so the curve d is regular.

Analogously as for d, one can prove that all curves c^x and d^y are regular. Hence and from Theorem 4.1 it follows that each image $c^x(\mathbb{R}) = f(R^x)$ is contained in a principal line and so is each $d^y(\mathbb{R}) = f(L^y)$. Suppose first that the image of d is contained in a horizontal line, that is, $d(\mathbb{R}) = f(L^0) \subset L^e$ for some $e \in \mathbb{R}$. We shall prove that, in this case, f is 0-principal, i.e. f maps every vertical line into a vertical line and every horizontal line into a horizontal line.

Indeed, consider first the image $f(R^x)$ of a vertical line R^x. Since f is bijective and $R^x \cap L^0 = \{(x,0)\}$, we get

$$(4.2) \qquad f(R^x) \cap f(L^0) = \{f(x,0)\}.$$

Thus $f(x,0) \in f(R^x) \cap L^e$, and if $f(R^x)$ were contained in a horizontal line, then $f(R^x) \subset L^e$. In this case, $f(R^x)$ and $f(L^0)$ are connected open subsets of L^e. Note that the functions $\rho = \pi_1 \circ f \circ r^x$ and $\lambda = \pi_1 \circ f \circ l^0$ are smooth embeddings from $\mathbb{R}$ to $\mathbb{R}$, which implies that $\rho(\mathbb{R})$ and $\lambda(\mathbb{R})$ are open intervals of $\mathbb{R}$. Since the set

$$\rho(\mathbb{R}) \cap \lambda(\mathbb{R}) = \pi_1(f(R^x)) \cap \pi_1(f(L^0)) = \pi_1(f(R^x) \cap f(L^0))$$

is non-empty by (4.2), we conclude that $\rho(\mathbb{R}) \cap \lambda(\mathbb{R})$ is an open interval of $\mathbb{R}$, too. On the other hand, π_1 defines a bijection from $f(R^x) \cap f(L^0)$ onto $\rho(\mathbb{R}) \cap \lambda(\mathbb{R})$, so the set $f(R^x) \cap f(L^0)$ would be infinite, which contradicts (4.2). Consequently, we have proved that, for each $x \in \mathbb{R}$, there is a unique $\phi(x)$ such that $f(R^x) \subset R^{\phi(x)}$. Since $(x,0) \in R^x \cap L^0$, it follows that

$$f(x,0) \in f(R^x) \cap f(L^0) \subset R^{\phi(x)} \cap L^e,$$

whence $f(x,0) = (\phi(x),e)$. This implies that $\phi = \pi_1 \circ f \circ l^0$ is a smooth embedding from $\mathbb{R}$ to $\mathbb{R}$. More precisely, observe that ϕ is a smooth diffeomorphism of $\mathbb{R}$. Indeed, it remains to show that ϕ is a surjection on $\mathbb{R}$. Since $\bigcup \{R^x : x \in \mathbb{R}\} = \mathbb{R}^2$ and f is a bijection of $\mathbb{R}^2$, we get

$$\mathbb{R}^2 = \bigcup \{f(R^x) : x \in \mathbb{R}\} \subset \bigcup \{R^{\phi(x)} : x \in \mathbb{R}\},$$

which implies that $\phi(\mathbb{R}) = \mathbb{R}$. Note that $f(R^x) = R^{\phi(x)}$ for each $x \in \mathbb{R}$, or else f could not be a bijection of $\mathbb{R}^2$.

Step 2. Consider now the map $g = s \circ f \circ s$. It is seen that g put instead of f also satisfies the assumptions of our theorem. Moreover, note that $g(L^0) = (s \circ f)(R^0) = L^{\phi(0)}$, and so, we can apply to g the results of Step 1. Therefore we conclude that there is a smooth diffeomorphism ψ of $\mathbb{R}$ such that $g(R^y) = R^{\psi(y)}$ for each $y \in \mathbb{R}$. This and the fact that

$f = \mathfrak{s}\circ\mathfrak{g}\circ\mathfrak{s}$ imply $f(L^y) = (\mathfrak{s}\circ\mathfrak{g})(R^y) = L^{\psi(y)}$. To sum up, we have proved that there are smooth diffeomorphisms ϕ and ψ of $\mathbb{R}$ such that $f(R^x) = R^{\phi(x)}$ and $f(L^y) = L^{\psi(y)}$ for any $x,y \in \mathbb{R}$. This yields $f(x,y) = (\phi(x),\psi(y))$ for $(x,y) \in \mathbb{R}^2$ because $\{f(x,y)\} = f(R^x \cap L^y) = f(R^x) \cap f(L^y) = R^{\phi(x)} \cap L^{\psi(y)} = \{(\phi(x),\psi(y))\}$, so $f = \phi \times \psi$.

Step 3. Suppose next that $f(L^0) \subset R^e$ for some $e \in \mathbb{R}$. It is seen that the map $\mathfrak{h} = \mathfrak{s}\circ f$ put instead of f also satisfies the assumptions of our theorem. Moreover, we have $\mathfrak{h}(L^0) \subset L^e$, and so, the results of Steps 1 and 2 can be applied to $\mathfrak{h}$. Thus there are smooth diffeomorphisms ϕ and ψ of $\mathbb{R}$ such that $\mathfrak{h} = \phi \times \psi$, whence $f = \mathfrak{s}\circ\mathfrak{h} = \mathfrak{s}\circ(\phi \times \psi)$.

Finally, to show that $k = \ell$, note first that from the form of f it follows that the inverse map f^{-1} can be regarded as a diffeomorphism of $\mathbb{R} \times_\ell \mathbb{R}$. Therefore the map $id_{\mathbb{R}^2} = f^{-1}\circ f$ defines a diffeomorphism from $\mathbb{R} \times_k \mathbb{R}$ onto $\mathbb{R} \times_\ell \mathbb{R}$. But this is possible only in case $k = \ell$ (Example 1.5). Clearly, f is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$ and $\mathbb{R} \times_\ell \mathbb{R}$ simultaneously, q.e.d.

This theorem immediately implies

Corollary 4.5. *The differential spaces* $\mathbb{R} \times_1 \mathbb{R}$ *and* $\mathbb{R} \times_2 \mathbb{R}$ *are non-diffeomorphic.*

Note that from Theorem 4.4 it follows that every diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$ is of definite P-type. Of course, if f and $\mathfrak{g}$ are diffeomorphisms of $\mathbb{R} \times_k \mathbb{R}$, then

$$\tau(f\mathfrak{g}) = \tau(f) + \tau(\mathfrak{g}) \pmod 2.$$

Let $\iota_{12}\colon \mathbb{R} \times_1 \mathbb{R} \longrightarrow \mathbb{R} \times_2 \mathbb{R}$ be the identity map regarded as a smooth map of differential spaces. Then the following diagram of smooth maps of differential spaces is commutative:

$$\begin{array}{ccc} T(\mathbb{R} \times_1 \mathbb{R}) & \xrightarrow{T(\iota_{12})} & T(\mathbb{R} \times_2 \mathbb{R}) \\ \Big\downarrow \pi_1 & & \Big\downarrow \pi_2 \\ \mathbb{R} \times_1 \mathbb{R} & \xrightarrow{\iota_{12}} & \mathbb{R} \times_2 \mathbb{R} \end{array}$$

where $T(\iota_{12})$ denotes the differential of ι_{12} and π_1,π_2 denote the corresponding canonical projections. Of course, the map $T(\iota_{12})_p$ defined to be the restriction of $T(\iota_{12})$ to $T_p(\mathbb{R} \times_1 \mathbb{R})$

is an isomorphism of differential vector spaces from $T_p(\mathbb{R} \times_1 \mathbb{R})$ onto $T_p(\mathbb{R} \times_2 \mathbb{R})$. From Corollary 4.5 and Proposition 3.8 we get

Corollary 4.6. *The tangent differential spaces* $T(\mathbb{R} \times_1 \mathbb{R})$ *and* $T(\mathbb{R} \times_2 \mathbb{R})$ *are not fibre diffeomorphic. In particular, the smooth map* $T(\iota_{12})$ *is not a diffeomorphism.*

Applying a method similar to that used in the proof of Theorem 4.4, one has

Proposition 4.7. *Every principal smooth diffeomorphism of* $\mathbb{R}^2$ *is of the form* $\phi \times \psi$ *or of the form* $s \circ (\phi \times \psi)$, *where* ϕ *and* ψ *are smooth diffeomorphisms of* $\mathbb{R}$.

From Theorem 4.4 and Proposition 4.7 we get

Corollary 4.8. *Every diffeomorphism of* $\mathbb{R} \times_k \mathbb{R}$ *is a principal smooth diffeomorphism of* $\mathbb{R}^2$. *Conversely, every principal smooth diffeomorphism of* $\mathbb{R}^2$ *is a diffeomorphism of* $\mathbb{R} \times_k \mathbb{R}$. *This correspondence defines an isomorphism between the corresponding groups of diffeomorphisms.*

If $r \in \mathbb{Z}_2$, we denote by $\mathcal{D}^2\{r\}$ the collection of all r-principal smooth diffeomorphisms of $\mathbb{R}^2$. Let us set $\mathcal{D}^2\{\#\} = \mathcal{D}^2\{0\} \cup \mathcal{D}^2\{1\}$. Obviously, $\mathcal{D}^2\{\#\}$ can be regarded as the group of all principal smooth diffeomorphisms of $\mathbb{R}^2$. Denote by a_s the inner automorphism of $\mathcal{D}^2\{\#\}$ determined by s, i.e. $a_s(f) = s \circ f \circ s^{-1}$. Let $\tau\colon \mathcal{D}^2\{\#\} \longrightarrow \mathbb{Z}_2$ be the map defined by the assignment $f \mapsto \tau(f)$. It is easy to verify

Proposition 4.9. *The map* $\tau\colon \mathcal{D}^2\{\#\} \longrightarrow \mathbb{Z}_2$ *is an epimorphism of groups such that* $\mathcal{D}^2\{r\} = \tau^{-1}(r)$. *In particular,* $\mathcal{D}^2\{0\}$ *is a normal divisor of* $\mathcal{D}^2\{\#\}$ *and the factor group* $\mathcal{D}^2\{\#\}/\mathcal{D}^2\{0\}$ *is isomorphic to* $\mathbb{Z}_2$. *Moreover,* $s \circ \mathcal{D}^2\{0\} = \mathcal{D}^2\{0\} \circ s = \mathcal{D}^2\{1\}$ *and the cosets* $\mathcal{D}^2\{0\}$, $\mathcal{D}^2\{1\}$ *are invariant under* a_s, *i.e.* $s \circ \mathcal{D}^2\{r\} \circ s^{-1} = \mathcal{D}^2\{r\}$.

Since the group $\mathcal{D}^2\{\#\}$ is isomorphic to the group $\mathcal{D}^2(k)$ by Corollary 4.8, it follows that this proposition can be reformulated for $\mathcal{D}^2(k)$.

Give attention to the fact that there is another epimorphism $\delta\colon \mathcal{D}^2\{\#\} \longrightarrow \mathbb{Z}_2$ of groups defined as follows: $\delta(f) = 0$ ($\delta(f) = 1$) if f preserves (changes) an orientation of $\mathbb{R}^2$. Obviously, $\mathcal{D}^2_+\{\#\} = \delta^{-1}(0) \neq \mathcal{D}^2\{0\}$. Observe that the collection $\mathcal{D}^2_+\{0\} = \mathcal{D}^2_+\{\#\} \cap \mathcal{D}^2\{0\}$ is a subgroup of $\mathcal{D}^2\{\#\}$. Moreover, note that $\mathcal{D}^2_+\{0\}$ is a normal divisor of $\mathcal{D}^2_+\{\#\}$, $\mathcal{D}^2\{0\}$ and $\mathcal{D}^2\{\#\}$. This follows from the fact that the assignment $f \mapsto (\delta(f),\tau(f))$ defines an epimorphism of groups from $\mathcal{D}^2\{\#\}$ onto $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is known that every isometry f of $\mathbb{R}^2$ is a linear isomorphism and we have a unique decomposition

$$(*) \qquad f = o(\delta)\circ r(\vartheta)\circ t(v),$$

where $t(v)$, $r(\vartheta)$ and $o(\delta)$ are special isometries which are the translation via the vector v, the rotation via the angle ϑ and the orientation map via the parameter $\delta \in \mathbb{Z}_2$ ($o(0) = \mathrm{id}_{\mathbb{R}^2}$, $o(1) = s$), respectively. By Theorem 4.4, $o(\delta)$ and $t(v)$ are always diffeomorphisms of $\mathbb{R} \times_k \mathbb{R}$ but $r(\vartheta)$ is such a diffeomorphism only in the case when ϑ is a multiple of $\frac{1}{2}\pi$.

Let us set $e = \mathrm{id}_{\mathbb{R}^2}$ and $r = r(\frac{1}{2}\pi)$. We have

Corollary 4.10. *There are eight diffeomorphisms of* $\mathbb{R} \times_k \mathbb{R}$ *which are isometries of* $\mathbb{R}^2$ *preserving* o, *namely,* $e, r, r^2, r^3, s, sr, sr^2, sr^3$.

Proof. The assertion follows from (*) and the observation that, for such a diffeomorphism f of $\mathbb{R} \times_k \mathbb{R}$, we have $\delta \in \{0,1\}$, $v = [0,0]$ and $\vartheta = l\,\frac{1}{2}\pi$, where $l \in \mathbb{Z}$. Therefore $o(\delta) = s^{\delta}$, $t(v) = e$ and $r(\vartheta) = r(l'\frac{1}{2}\pi) = r^{l'}$, where l' is the remainder of l mod 4. Thus the list of all such diffeomorphisms is full, q.e.d.

5. Final remarks and open questions

Consider the set $\mathbb{R}^n$ for a finite $n \geq 2$. Let us set $[n] = \{1,\ldots,n\}$ and suppose that $i \in [n]$. Let σ_i be the permutation of $\mathbb{R}^n$ defined as follows:

$$\sigma_i(x_1,\ldots,x_{n-1},t) = (x_1,\ldots,x_{i-1},t,x_i,\ldots,x_{n-1}).$$

If $x \in \mathbb{R}^{n-1}$, we define the linear injection $1^x_i\colon \mathbb{R} \longrightarrow \mathbb{R}^n$ by

$$1^x_i(t) = \sigma_i(x,t).$$

Let us set $\mathfrak{C}_n = \{1^x_i\colon x \in \mathbb{R}^{n-1},\ i \in [n]\}$. Consider the final differential structures $\mathcal{S}^k(\mathbb{R}^n)$ for $k = 1,2$, defined on $\mathbb{R}^n$ as follows:

$$\mathscr{S}^{k}(\mathbb{R}^{n}) = \begin{cases} F\mathscr{C}(\mathbb{R}^{n},\mathfrak{E}_{n}) & \text{for } k = 1, \\ C\mathscr{C}(\mathbb{R}^{n},\mathfrak{E}_{n}) & \text{for } k = 2. \end{cases}$$

Denote by $\mathbb{R}^{n}(k)$ the differential space $(\mathbb{R}^{n},\mathscr{S}^{k}(\mathbb{R}^{n}))$. In particular, for $n = 2$, we have $\mathbb{R}^{2}(k) = \mathbb{R} \times_{k} \mathbb{R}$.

One may expect that to all the properties which are proved in this paper for $\mathbb{R} \times_{k} \mathbb{R}$ there correspond generalized ones for all $\mathbb{R}^{n}(k)$ where $n \geq 2$ is finite. It seems that these generalized properties can be obtained by means of techniques being a combinatorial n-variant of those used here for $n = 2$. Obviously, for the differential spaces $\mathbb{R}^{n}(k)$, we shall have new properties. For example, one can prove that, for any finite $n,m \geq 2$, the differential space $\mathbb{R}^{n}(k) \times_{k} \mathbb{R}^{m}(k)$ is diffeomorphic to the differential space $\mathbb{R}^{n+m}(k)$.

Let (M,x) and (N,y) be pointed differential spaces. Consider the smooth maps $r^{x}\colon (N,y) \longrightarrow (M \times_{k} N, (x,y))$ and $l^{y}\colon (M,x) \longrightarrow (M \times_{k} N, (x,y))$ defined in Section 1. Applying the functor T (Lemma 1.1) to these maps we get the linear maps

$$T(r^{x})_{y}\colon T(N,y) \longrightarrow T(M \times_{k} N, (x,y)),$$
$$T(l^{y})_{x}\colon T(M,x) \longrightarrow T(M \times_{k} N, (x,y)),$$

which are monomorphisms. Moreover, one can prove that $\operatorname{Im} T(r^{x})_{y} \cap \operatorname{Im} T(l^{y})_{x} = 0$. It is seen that from Theorem 2.10 it follows that if $M = N = \mathbb{R}$, then

$$\operatorname{Im} T(r^{x})_{y} + \operatorname{Im} T(l^{y})_{x} = T(\mathbb{R} \times_{k} \mathbb{R}, (x,y)).$$

So, we may pose

Question 5.1. *Let* (M,x) *and* (N,y) *be pointed differential spaces. Do the following conditions hold*:

(a) $\operatorname{Im} T(r^{x})_{y} + \operatorname{Im} T(l^{y})_{x} = T(M \times_{k} N, (x,y))$,

(b) $\dim T(M,x) + \dim T(N,y) = \dim T(M \times_{k} N, (x,y))$.

Obviously, they are equivalent provided that $\dim T(M,x)$ *and* $\dim T(N,y)$ *are finite.*

Let M and N be differential spaces. Denote by $\mathfrak{X}(M)$ $(\mathfrak{X}^{\infty}(M))$ and $\mathfrak{X}(M \times_{k} N)$ $(\mathfrak{X}^{\infty}(M \times_{k} N))$ the corresponding modules of arbitrary (smooth) vector fields which can be regarded as real vector spaces, as well. If $X \in \mathcal{X}(M)$, then there is a unique $X_{\#} \in \mathcal{X}(M \times_k N)$ such that, for any $p = (x,y) \in M \times N$, we have $X_{\#p} = (T1^{Y})(X_x)$. Obviously, the assignment $X \mapsto X_{\#}$ defines a linear monomorphism $1^N_{\#}: \mathcal{X}(M) \longrightarrow \mathcal{X}(M \times_k N)$ of vector spaces. We say that $1^N_{\#}$ is $\mathcal{X}^{\infty}$-*invariant* if $1^N_{\#}(\mathcal{X}^{\infty}(M)) \subset \mathcal{X}^{\infty}(M \times_k N)$. Note that, in the general case, the monomorphism $1^N_{\#}$ need not be $\mathcal{X}^{\infty}$-invariant. For example, if $M = N = \mathbb{R}$, then from Theorem 3.2 and since $1^{\mathbb{R}}_{\#}$ is a monomorphism, it follows that $1^{\mathbb{R}}_{\#}$ cannot be $\mathcal{X}^{\infty}$-invariant. On the other hand, if N is discrete, i.e. $\mathcal{E}(N) = F(N)$, then $M \times_1 N = M \times_2 N = M \times N$ where $M \times N$ denote the usual product of differential spaces. In this case, one can see that $1^N_{\#}$ is $\mathcal{X}^{\infty}$-invariant. These considerations and Theorem 3.2 lead to the following questions:

Question 5.2. *Let* M *be a differential space. For what differential space* N *is the monomorphism* $1^N_{\#}$ $\mathcal{X}^{\infty}$-*invariant?*

Question 5.3. *Let* M *be a differential space. For what differential space* N *is* $\mathcal{X}^{\infty}(M \times_k N) = 0$?

Note that if, for a differential space N, we have that $\mathcal{X}^{\infty}(M \times_k N) = 0$ and $1^N_{\#}$ is $\mathcal{X}^{\infty}$-invariant, then $\mathcal{X}^{\infty}(M) = 0$. Conversely, if $\mathcal{X}^{\infty}(M) = 0$, then $1^N_{\#}$ is always $\mathcal{X}^{\infty}$-invariant.

If differential spaces M and N are diffeomorphic, then, clearly (Proposition 3.8), the tangent differential spaces T(M) and T(N) are diffeomorphic, too. It is of interest to know an answer to the following

Question 5.4. *Let* M *and* N *be differential spaces. Is it true that if* T(M) *and* T(N) *are diffeomorphic, then the differential spaces* M *and* N *are diffeomorphic, too?*

Note that if we get an affirmative answer to this question, then from Corollaries 3.4 and 4.5 it follows that the tangent differential spaces $T(\mathbb{R} \times_1 \mathbb{R})$, $T(\mathbb{R} \times_2 \mathbb{R})$ and $T(\mathbb{R}^2)$ are non-diffeomorphic.

Now, let us come back to the differential spaces $\mathbb{R} \times_k \mathbb{R}$ ($k = 1,2$). In Section 1 we observed that the $\mathcal{S}^2$-topology on $\mathbb{R} \times \mathbb{R}$ is the original one but the $\mathcal{S}^1$-topology is essentially stronger (see Example 1.5). It would be interesting to know an answer to the following

Question 5.5. *What is the kind of the $\mathcal{S}^1$-topology? In particular, is the $\mathcal{S}^1$-topology locally non-compact?*

Finally, consider the family $\mathfrak{C} = \mathfrak{C}(\mathbb{R},\mathbb{R}) = \{r^x, l^y\colon x,y \in \mathbb{R}\}$. Let τ^1 denote the strongest topology on $\mathbb{R}\times\mathbb{R}$ for which all maps $r^x, l^y\colon \mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$ are continuous. It is seen that the $\mathcal{S}^1$-topology on $\mathbb{R}\times\mathbb{R}$ is weaker than τ^1.

Question 5.6. *Is the $\mathcal{S}^1$-topology on $\mathbb{R}\times\mathbb{R}$ essentially weaker than τ^1?*

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