DEMONSTRATIO MATHEMATICA
Vol. XXIV No 3-4 1991

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VECTORS AND VECTOR FIELDS OF K-TH ORDER ON DIFFERENTIAL SPACES

The purpose of this paper is to introduce the notions of tangent vectors and tangent vector fields of k-th order to a differential space (M,C). In the paper we prove also the fundamental properties of these notions.

1. Introduction

Let M be a set and C a family of real functions on M. By τ_C we denote the weakest topology on M such that all functions of C are continuous. By scC we denote the set of all functions on M which are of the form $\omega\circ(\alpha_1,\dots,\alpha_n)$, where $\alpha_1,\dots,\alpha_n \in C$ and $\omega \in C^{\infty}(\mathbb{R}^n)$, $n \in \mathbb{N}$. Now let A be a subset of M. By C_A we denote the set of all functions $g: A \longrightarrow \mathbb{R}$ such that, for each point $p \in A$, there exist an open neighbourhood U of p and a function $f \in C$ such that $g|U = f|U$.

If $C = (scC)_M$, the set C is said to be a differential structure on M, and the pair (M,C) is called the differential space (d.s, for short) [1], [2]. If (M,C) is a d.s and A is a subset of M then (A,C_A) is also a d.s and it is called the differential subspace of (M,C).

By a tangent vector to a differential space (M,C) at a point $p \in M$ we mean a linear mapping $v: C \longrightarrow \mathbb{R}$ satisfying the Leibniz condition

$$(1.1) \qquad v(\alpha\cdot\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha),$$

for any $\alpha,\beta \in C$. Next, by T_pM we denote the tangent space to (M,C) at the point p.

One can prove that if $f,g \in C$ and $f|U = g|U$, for some open neighbourhood U of p then, for any $v \in T_pM$, $v(f) = v(g)$. If $f|U = \omega\circ(\alpha_1,\dots,\alpha_n)|U$, for some open neighbourhood U of p, where $\alpha_1,\dots,\alpha_n \in C$ and $\omega \in C^\infty(\mathbb{R}^n)$, then

$$(1.2) \qquad v(f) = \sum_{i=1}^{n} \omega'|_i(\alpha_1(p),\dots,\alpha_n(p))v(\alpha_i),$$

for any $v \in T_pM$. Evidently, the formula (1.2) is a generalization of the formula (1.1).

Next, by a tangent vector field to (M,C) we mean a function $X\colon M \longrightarrow \bigcup_{p\in M} T_pM$ such that $X(p) \in T_pM$, for any $p \in M$. A tangent vector field X to (M,C) is said to be smooth if the function Xf, given by $(Xf)(p) = X(p)(f)$, is smooth for any $f \in C$. Now, by $\mathfrak{X}(M)$ we denote the C module of all smooth tangent vector fields to (M,C).

A differential space (M,C) is said to be of constant differential dimension n if, for any point $p \in M$, there exist an open set U in τ_C and C_U-basis $X_1,\dots,X_n \in \mathfrak{X}(U)$ of C_U- module $\mathfrak{X}(U)$ such that $p \in U$ and $\mathrm{Lin}(X_1(q),\dots,X_n(q)) = T_qM$, for any $q \in U$.

A mapping $F\colon M \longrightarrow N$ is said to be smooth mapping of a differential space (M,C) into a differential space (N,D) if $f\circ F \in C$, for any $f \in D$. If F is a smooth mapping of (M,C) into (N,D) we shall write $F\colon (M,C) \longrightarrow (N,D)$. Now, the function $dF\colon \bigcup_{p\in M} T_pM \longrightarrow \bigcup_{q\in N} T_qN$, defined by the formula

$$(1.3) \qquad dF(v)(\beta) = v(\beta\circ F),$$

for any $v \in T_pM$ and $\beta \in D$, is called the differential of F, or the tangent map of F.

The restriction of dF to the set T_pM, i.e. $dF|T_pM$, is called the differential of F at the point p and it is denoted by d_pF.

Now, let (M,C) be a differential space and let C_0 be a subset of C, then C_0 is said to be a set of generators of the differential structure C on M if $C = (scC_0)_M$.

2. Vectors and vector fields of k-th order on differential spaces

Let (M,C) be a d.s. Now we shall give another, but equivalent, definition of a tangent vector to (M,C) at a point $p \in M$.

Definition 2.1 A linear mapping $v: C \longrightarrow \mathbb{R}$ is said to be a tangent vector to (M,C) at a point $p \in M$, if the following conditions are fulfilled

(2.1) $$v(f) = 0 \quad \text{if } f \text{ is constant}$$

and

(2.2) $$v|a_p^2 = 0,$$

where $a_p^2 := \{(f-f(p))\cdot(g-g(p));\ f,g \in C\}$.

The equivalence of both definitions of a tangent vector to (M,C) at $p \in M$ is evident from the identities

$$0 = v[(f-f(p))\cdot(g-g(p))] = v(f\cdot g-f(p)g-g(p)f+f(p)g(p)) =$$
$$= v(f\cdot g)-f(p)v(g)-g(p)v(f).$$

Now, we accept

Definition 2.2 Let $k \in \mathbb{N}$. A linear mapping $v^{(k)}: C \longrightarrow \mathbb{R}$ is said to be a k-th order tangent vector to (M,C) at a point $p \in M$ if the following conditions are fulfilled

(2.3) $$v^{(k)}(f) = 0 \quad \text{if } f \text{ is constant}$$

and

(2.4) $$v^{(k)}|a_p^{k+1} = 0,$$

where $a_p^{k+1} = \{(f_1-f_1(p))\cdot\ldots\cdot(f_{k+1}-f_{k+1}(p));\ f_1,\ldots,f_{k+1} \in C\}$. (We would obtain the same result if we used the linear space $\mathrm{Lin}(a_p^k)$ instead of a_p^k.)

Let us observe that, for any $k \in \mathbb{N}$,

(2.5) $$a_p^{k+1} \subset a_p^k.$$

The set $T_p^{(k)}M$ of all k-th order tangent vectors to (M,C) at a point $p \in M$, for $k \in \mathbb{N}$, has the natural structure of a linear space, and it is called the k-th order tangent space to (M,C) at $p \in M$.

Of course, the vector $v^{(k)} \in T_p^{(k)}M$ is called zero k-th order vector if $v^{(k)}(f) = 0$, for any $f \in C$.

Lemma 2.1 Let (M,C) be a d.s and let $\alpha \in C$ be such that $\alpha|U = 0$, for some open neighbourhood U of $p \in M$. Then $v^{(k)}(\alpha) = 0$, for arbitrary $v^{(k)} \in T_p^{(k)}M$ and $k \in \mathbb{N}$.

Proof. Since (M,C) is a C-regular topological space, there exists a function $\beta \in C$ such that $\beta|A = 1$ for an open neighborhood A of p, $\beta|A_0 = 0$ for an open set A_0 such that $A_0 \cup U = M$ as well as $p \in A \subset U$. Consider now the function $\gamma = 1-\beta$. Evidently, $\gamma(p) = 0$ and

$$\alpha = \alpha\cdot\gamma = \alpha(\gamma-\gamma(p)) = (\alpha-\alpha(p))(\gamma-\gamma(p))^k.$$

Thus, by definition 2.2, we get

$$v^{(k)}(\alpha) = v^{(k)}[(\alpha-\alpha(p))(\gamma-\gamma(p))^k] = 0,$$

for $v^{(k)} \in T_p^{(k)}M$ and $k \in \mathbb{N}$.

Corollary 2.2 Let $f,g \in C$. If $f|U = g|U$, for an open neighbourhood U of $p \in M$, then $v^{(k)}(f) = v^{(k)}(g)$, for any $v^{(k)} \in T_p^{(k)}M$ and $k \in \mathbb{N}$.

Now we shall prove

Proposition 2.3 For an arbitrary $v_1^{(k)},\dots,v_m^{(k)} \in T_p^{(k)}M$, $k \in \mathbb{N}$, the following statements are equivalent:

1. the vectors $v_1^{(k)},\dots,v_m^{(k)}$ are linearly independent,
2. linear mapping $L: C \longrightarrow \mathbb{R}^m$ defined by $L(\alpha) = (v_1^{(k)}(\alpha),\dots,v_m^{(k)}(\alpha))$ for $\alpha \in C$ is onto $\mathbb{R}^m$,
3. there exist functions $\alpha^1,\dots,\alpha^m \in C$ such that $v_i^{(k)}(\alpha^j) = \delta_i^j$, where δ_i^j is the Kronecker delta,
4. there exist functions $\alpha^1,\dots,\alpha^m \in C$ such that the determinant $\det(v_i^{(k)}(\alpha^j))$ of the matrix $(v_i^{(k)}(\alpha^j))$, $i,j = 1,\dots,m$, is different from zero.

Proof. Obviously L(C) is a linear subspace of $\mathbb{R}^m$. Consequently, L(C) is a proper subset of $\mathbb{R}^m$ if and only if there exist real numbers $a^1,\dots,a^m$, $|a^1|+\dots+|a^m| > 0$, and such that, for any $\alpha \in C$, $\sum_{i=1}^{m} a^i v_i^{(k)}(\alpha) = 0$, or equivalently $\sum_{i=1}^{m} a^i v_i^{(k)} = 0$. But the last condition means that the vectors $v_1^{(k)},\dots,v_m^{(k)}$ are linearly dependent. It proves the equivalence of assertions 1 and 2.

Evidently, condition 2 implies condition 3 and condition 3 implies condition 4. So, it suffices to show that assertion 4 implies assertion 2. Indeed, let $[b_1,\dots,b_m]$ be an arbitrary vector in $\mathbb{R}^m$. From condition 4 it follows that there exist real numbers $a^1,\dots,a^m$ such that $\sum_{j=1}^{m} a^j v_i^{(k)}(\alpha^j) = b_i$. In consequence, $L(\sum_{j=1}^{m} a^j\alpha^j)=(b_1,\dots,b_m)$, which proves condition 2.

Now, let (A,C_A) be a differential subspace of (M,C) and let $p \in A$. Then, for any $v^{(k)} \in T_p^{(k)}A$, $k \in \mathbb{N}$, the formula

$$\bar{v}^{(k)}(\alpha) = v^{(k)}(\alpha|A), \tag{2.6}$$

for any $\alpha \in C$, defines the tangent vector $\bar{v}^{(k)}$ of k-th order to (M,C) at the point $p \in M$. Identifying the vector $v^{(k)} \in T_p^{(k)}A$ with the vector $\bar{v}^{(k)} \in T_p^{(k)}M$, defined by (2.6), one can easily prove

Proposition 2.4 The tangent space $T_p^{(k)}A$ to a differential subspace (A,C_A) of (M,C) at $p \in A$ is a linear subspace of the space $T_p^{(k)}M$.

Moreover, if A is an open subset of M then, for any $p \in A$, $T_p^{(k)}A = T_p^{(k)}M$, $k \in \mathbb{N}$.

From definition 2.2 and (2.5) it follows the inclusion $T_p^{(k)}M \subset T_p^{(k+1)}M$, for any $p \in M$ and $k \in \mathbb{N}$.

Now, we shall prove

Lemma 2.5 Let $F\colon M \longrightarrow N$ be a smooth mapping of (M,C) into (N,D). Let $v^{(k)} \in T_p^{(k)}(M)$, where $k \in \mathbb{N}$. Then the formula

$$w(\alpha) = v^{(k)}(\alpha\circ F), \tag{2.7}$$

for any $\alpha \in D$, defines a tangent vector of k-th order $\bar{v}^{(k)} = w$ to (N,D) at the point $q = F(p) \in N$.

Proof. From (2.7) it follows that $w = \bar{v}^{(k)}\colon C \longrightarrow \mathbb{R}$ is a linear mapping. To show that $w = \bar{v}^{(k)} \in T^{(k)}_{F(p)}N$ it suffices to observe that

$$w[(\alpha_1-\alpha_1(q))\cdot\ldots\cdot(\alpha_{k+1}-\alpha_{k+1}(q))] =$$

$$= v^{(k)}[(\alpha_1-\alpha_1(q))\cdot\ldots\cdot(\alpha_{k+1}-\alpha_{k+1}(q))\circ F] =$$

$$= v^{(k)}[(\tilde{\alpha}_1-\tilde{\alpha}_1(p))\cdot\ldots\cdot(\tilde{\alpha}_{k+1}-\tilde{\alpha}_{k+1}(p))] = 0,$$

for any $\alpha_1,\ldots\alpha_{k+1}\in D$, where $\tilde{\alpha}_i = \alpha_i\circ F$ for $i=1,2,\ldots,k+1$, and $q=F(p)$.

Definition 2.3 A function which assigns to each tangent vector of k-th order $v^{(k)}$ to (M,C) the tangent vector of k-th order $w=\bar{v}^{(k)}$ to (N,D), defined by (2.7), is called the differential of k-order of F and is denoted by $d^{(k)}F$.

Hence by definition we have

$$d^{(k)}F : \bigcup_{p\in M} T^{(k)}_p M \longrightarrow \bigcup_{q\in N} T^{(k)}_q N$$

and

$$d^{(k)}F(v^{(k)})(\alpha) = v^{(k)}(\alpha\circ F), \tag{2.8}$$

for an arbitrary $v^{(k)}\in T^{(k)}_p M$, $\alpha\in D$, $k\in\mathbb{N}$ and $p\in M$.

The restriction of $d^{(k)}F$ to the set $T^{(k)}_p M$, that is $d^{(k)}F|T^{(k)}_p M$, is called the differential of k-th order of F at the point $p\in M$, and it is denoted by $d^{(k)}_p F$.
Thus

$$d^{(k)}_p F(v^{(k)}) = d^{(k)}F(v^{(k)}),$$

for $v^{(k)} \in T^{(k)}_p M$. Evidently $d^{(k)}_p F : T^{(k)}_p M \longrightarrow T^{(k)}_{F(p)}N$.

It is easy to prove

Lemma 2.6 The differential of k-th order of F at a point $p\in M$ is a linear mapping of $T^{(k)}_p M$ into $T^{(k)}_{F(p)}N$.

Similarly as in the theory of manifolds we prove

Proposition 2.7 Let (M,C), (N,D) and (P,Q) be differential spaces and let $F: M \longrightarrow N$ as well as $G: N \longrightarrow P$ be smooth mappings. Then, for any $k \in \mathbb{N}$,

$$d^{(k)}(G\circ F) = d^{(k)}G\circ d^{(k)}F.$$

Thus, for any $p \in M$, we have

(2.9) $$d_p^{(k)}(G\circ F) = d_{F(p)}^{(k)}G\circ d_p^{(k)}F.$$

Now we shall generalize formula (1.2). Let $v^{(k)}\in T_p^{(k)}M$ and let f be an arbitrary function from C. By definition of a differential structure, there exist an open neighborhood U of p and functions $\alpha_1,\dots\alpha_n\in$ C and $\omega \in C^\infty(\mathbb{R}^n)$ such that

$$f|U = \omega\circ(\alpha_1,\dots,\alpha_n)|U$$

Hence

(2.10) $$v^{(k)}(f) = v^{(k)}[\omega\circ(\alpha_1,\dots,\alpha_n)] = v^{(k)}(\omega\circ\Phi),$$

where $\Phi := (\alpha_1,\dots,\alpha_n)\colon M \longrightarrow \mathbb{R}^n$. From (2.10) by (2.8) and (2.9) we get

$$v^{(k)}(f) = v^{(k)}(\omega\circ\Phi) = d_p^{(k)}(\omega\circ\Phi)(v^{(k)}) = $$
$$= [d_p^{(k)}\Phi(v^{(k)})](\omega) .$$

Evidently, $d_p^{(k)}\Phi(v^{(k)}) \in T_{\Phi(p)}^{(k)}\mathbb{R}^n$ for any $v^{(k)} \in T_p^{(k)}M$ and $k \in \mathbb{N}$.

Let us denote by $\partial^{(m)}_{i_1\dots i_m}$, where $m = 1,2,\dots k$ and $1\le i_1,\dots,i_m\le n$, the vectors of the partial derivations at the point $q=\Phi(p)$ of $\mathbb{R}^n$. Obviously these vectors form a basis of $T_q^{(k)}\mathbb{R}^n$. Now, any vector $d_p^{(k)}\Phi(v^{(k)}) \in T_q^{(k)}\mathbb{R}^n$ has the decomposition with respect to this basis of the form

(2.11) $$d_p^{(k)}\Phi(v^{(k)}) = \sum_{m=1}^{k}\ \sum_{1\le i_1<\dots<i_m\le n} \lambda^{i_1\dots i_m}\,\partial^{(m)}_{i_1\dots i_m}|q,$$

where

(2.12) $$\lambda^{i_1\dots i_m} = d_p^{(k)}\Phi(v^{(k)})(\tilde\pi_{i_1},\dots,\tilde\pi_{i_m}) ,$$

and $\tilde\pi_i = \pi_i-\pi_i(q)$ whereas π_i, $i=1,2,\dots,n$, is the canonical projection in $\mathbb{R}^n$ onto its i-th coordinate. On the other hand, we have

(2.13) $$d_p^{(k)}\Phi(v^{(k)})(\tilde\pi_{i_1},\dots,\tilde\pi_{i_m}) = v^{(k)}(\Phi\circ(\tilde\pi_{i_1},\dots,\tilde\pi_{i_m})) =$$
$$= v^{(k)}(\tilde\alpha_{i_1},\dots,\tilde\alpha_{i_m}).$$

where $\tilde\alpha_i = \alpha_i-\alpha_i(p)$, for $i=i_1,\dots,i_m$. From (2.11), (2.12) and

(2.13) we get finally the formula :

$$v^{(k)}(f) = v^{(k)}(\omega\circ(\alpha_1,\dots\alpha_n)) =$$

$$(*)\qquad = \sum_{m=1}^{k}\Big(\sum_{1\le i_1<\dots<i_m\le n}(\omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots\alpha_n(p))\cdot$$

$$\cdot v^{(k)}[(\alpha_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(\alpha_{i_m}-\alpha_{i_m}(p)]\Big),$$

It is easy to observe that by putting in (*) k=1 we get the formula (1.2). Next, by putting k=2 in (*) we get

$$v^{(2)}(f) = v^{(2)}(\omega\circ(\alpha_1,\dots,\alpha_n)) =$$

$$= \sum_{i=1}^{n}\omega'_{|i}(\alpha_1(p),\dots,\alpha_n(p))v^{(2)}(\alpha_i-\alpha_i(p)) +$$

$$+ \sum_{1\le i<j\le n}\omega''_{|ij}(\alpha_1(p),\dots,\alpha_n(p))v^{(2)}[(\alpha_i-\alpha_i(p))\cdot(\alpha_j-\alpha_j(p))]$$

From (*) it follows:

Corollary 2.8 For any $k\in\mathbb{N}$, a tangent vector of k-th order $v^{(k)}\in T^{(k)}_pM$ is uniquely determined by its values on all products of the form: $(\alpha_1-\alpha_1(p))\cdot\ldots\cdot(\alpha_m-\alpha_m(p))$, where $m=1,2,\dots,k$ and $\alpha_1,\dots\alpha_m\in C$.

Let now C_o be a set of generators of differential structure C on M. For any $l\in\mathbb{N}$ and $p\in M$, let us put:

$$\mathring{a}^i_p := \{(\alpha_1-\alpha_1(p))\cdot\ldots\cdot(\alpha_i-\alpha_i(p)),\alpha_1,\dots,\alpha_i\in C_o\}$$

One can easily prove

Lemma 2.9 Let $k\in\mathbb{N}$ and let $\underset{o}{w} : \bigcup_{i=1}^{k}\mathring{a}^i_p \longrightarrow \mathbb{R}$ be a mapping such that, for any $\alpha_1,\dots,\alpha_n\in C_o$ and $\omega\in C^\infty(\mathbb{R}^n)$, the equality

$$\omega\circ(\alpha_1,\dots,\alpha_n) = 0$$

implies the equality

$$\sum_{m=1}^{k}\Big(\sum_{1\le i_1<\dots<i_m\le n}\omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p))\cdot$$

$$\cdot\underset{o}{w}[(\alpha_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(\alpha_{i_m}-\alpha_{i_m}(p))]\Big) = 0.$$

Then there exists exactly one tangent vector of k-th order

$v^{(k)} \in T_p^{(k)}M$ such that $v^{(k)}|\bigcup_{i=1}^{k} \mathring{a}_p^i = \gamma$.

Now for any $p\in M$ we shall denote by A_p^l the set of all smooth functions $f\in C$ for which there exist an open neighborhood $U\in\tau_C$ of p and functions $f_1,\dots,f_n\in C$, $\omega \in C^\infty(\mathbb{R}^n)$, for some $n\in \mathbb{N}$, such that

1^o $$f|U = \omega\circ(f_1,\dots,f_n)|U$$

2^o $$\omega^{(k)}_{|i_1\dots i_k}(f_1(p),\dots,f_n(p)) = 0,$$

for any k=1,2,...,l and $k \in \mathbb{N}$, $p\in M$. Moreover, it can easily be seen that A_p^l is a differential substructure of the differential structure C. Consequently, A_p^l is a *linear subspace of C* and $A_p^{k+1} \subset A_p^k$.

Definition 2.4 Let $k \in \mathbb{N}$. A subset $\mathcal{F}^{(k)}$ of C is said to be a local basis of k-th order of differential structure C on M at a point $p\in M$ if, for any $f \in C$, there is exactly one decomposition of the form:

$$f = \lambda^1 f_1+\dots+\lambda^m f_m+ g ,$$

where $f_1,\dots,f_m \in \mathcal{F}^{(k)}$, $\lambda^1,\dots\lambda^m \in \mathbb{R}$ and $g \in A_p^k$.

One can prove

Proposition 2.10 Let (M,C) be a d.s. For any point p of M there exists a local basis of k-th order $\mathcal{F}^{(k)}$ of differential structure C at the point p such that $F^{(k)}\subset \bigcup_{i=1}^{k} \mathring{a}_p^i$, $k\in\mathbb{N}$.

Lemma 2.11. Let $\mathcal{F}^{(k)}$, $k\in\mathbb{N}$, be a local basis of k-th order of differential structure C on M at a point $p\in M$. Then for every function $\gamma : \mathcal{F}^{(k)} \longrightarrow \mathbb{R}$ there exists exactly one tangent vector of k-th order $v^{(k)}\in T_p^{(k)}M$ such that $v^{(k)}|\mathcal{F}^{(k)} = \gamma$.

Let us consider the quotient linear space C/A_p^l and let $[f]_p^l$ denotes the equivalence class of $f\in C$.

Lemma 2.12 Let (M,C) be a d.s, and let $p\in M$ be an arbitrary point. Then

1° $[\omega\circ(\alpha_1,\dots,\alpha_n)]^1_p =$

$$= \sum_{m=1}^{1} \Big(\sum_{1\le i_1<\dots<i_m\le n} \omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p))\cdot$$

$$\cdot[(\alpha_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(\alpha_{i_n}-\alpha_{i_m}(p))]^1_p\Big).$$

2° If $g,f\in C$ and $f|U = g|U$, for a neighborhood U of p, then $[f]^1_p = [g]^1_p$.

Proof. ad 1°. It is enough to show that

$$\omega\circ(\alpha_1,\dots,\alpha_n) - \sum_{m=1}^{1} \Big(\sum_{1\le i_1<\dots<i_m\le n} \omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p))\cdot$$

$$\cdot(\alpha_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(\alpha_{i_m}-\alpha_{i_m}(p))$$

belongs to A^1_p.

Let $\theta\in C^{\infty}(\mathbb{R})$ be a function given by the formula

$$\theta(x_1,\dots,x_n) = \omega(x_1,\dots,x_n) -$$

$$-\sum_{m=1}^{1} \Big(\sum_{1\le i_1<\dots<i_m\le n} \omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p))\cdot$$

$$\cdot(x_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(x_{i_m}-\alpha_{i_m}(p)),$$

for any $(x_1,\dots x_n)\in\mathbb{R}^n$. Hence

$$\theta(\alpha_1,\dots,\alpha_n) = \omega\circ(\alpha_1,\dots,\alpha_n) -$$

$$-\sum_{m=1}^{1} \Big(\sum_{1\le i_1<\dots<i_m\le n} \omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p))\cdot$$

$$\cdot(\alpha_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(\alpha_{i_m}-\alpha_{i_m}(p))$$

and

$$\theta^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p)) = 0$$

for any $m = 1,2,\dots,1$ and $1\le i_1,\dots,i_m\le n$. Consequently

$$\omega\circ(\alpha_1,\dots,\alpha_n) - \sum_{m=1}^{1} \Big(\sum_{1\le i_1<\dots<i_m\le n} \omega^{(m)}_{|i_1\dots i_m}(\alpha_1(p),\dots,\alpha_n(p))\cdot$$

$$\cdot(\alpha_{i_1}-\alpha_{i_1}(p))\cdot\ldots\cdot(\alpha_{i_m}-\alpha_{i_m}(p))$$

belongs to A_p^1.

2^o is obvious.

Let $v^{(k)} \in T_p^{(k)}M$ be a tangent vector of k-th order. Evidently, $v^{(k)}{}_{|A_p^k} = 0$ for any $v^{(k)} \in T_p^{(k)}M$. Hence $v^{(k)}$ induces a linear function $l_{v^{(k)}} \in (C/A_p^k)^*$ defined by

$$(2.14) \qquad l_{v^{(k)}}([f]_p^1) = v^{(k)}(f),$$

for any $f \in C$.

Lemma 2.13 The mapping $I: T_p^{(k)}M \longrightarrow (C/A_p^k)^*$ defined by $I(v^{(k)}) = l_{v^{(k)}}$, for any $v^{(k)} \in T_p^{(k)}M$, is an isomporphism of linear spaces.

Proof. The linearity of the mapping I is clear. Obviously if $l_{v^{(k)}} = 0$ for some $v^{(k)} \in {}_pT^{(k)}M$, then $v^{(k)} = 0$. Hence I is a monomorphism. Now we shall show that I is an epimorphism. For any $l \in (C/A_p^k)^*$, let $v_l: C \longrightarrow \mathbb{R}$ be a mapping defined by

$$v_l(f) := l([f]_p^k),$$

for $f \in C$. It is easy to see that $v_l \in T_p^{(k)}M$.

Corollary 2.14 Let (M,C) be a d.s and $p \in M$. Then for any $k \in \mathbb{N}$

$$\dim T_p^{(k)}M = 0 \text{ if and only if } C = A_p^k .$$

Proposition 2.15 Let (M,C) be a d.s and $p \in M$. If $\dim T_pM = 0$ then for any $k \in \mathbb{N}$ $\dim T_p^{(k)}M = 0$.

Proof. Let $\dim T_pM = 0$. Now we shall show that $\dim T_p^{(2)}M = 0$. It is enough to verify that for any $v^{(2)} \in T_p^{(2)}M$ $v^{(2)}{}_{|a_p^2} = 0$.

Really, since $\dim T_pM = 0$ therefore $C = A_p^1$. For any $\alpha_1, \alpha_2 \in C$ there exist an open neighborhood U of p and functions $f_1, \ldots f_n \in C$, $\theta_1, \theta_2 \in C^\infty(\mathbb{R}^n)$ such that

1^o $\alpha_1 - \alpha_1(p)|U = \theta_1 \circ (f_1, \ldots, f_n)|U,$

2^o $\alpha_1 - \alpha_2(p)|U = \theta_2 \circ (f_1, \ldots, f_n)|U,$

3° $\theta'_{1|i}(f_1(p),\dots,f_n(p)) = 0$, $\theta'_{2|i}(f_1(p),\dots,f_n(p))=0$.

Now it is easy to check that

$$v^{(2)}(\theta_1\circ(f_1,\dots f_n)\cdot\theta_2\circ(f_1,\dots f_n)) =$$
$$=\sum_{i=1}^{n}\theta'_{|i}(f_1(p),\dots,f_n(p))v^{(2)}(f_i-f_i(p)) +$$
$$+\sum_{1\le i<j\le n}\theta''_{|ij}(f_1(p),\dots,f_n(p))v^2[(f_i-f_i(p))\cdot(f_j-f_j(p))],$$

where $\theta:=\theta_1\cdot\theta_2$. Evidently,

$$\theta'_{|i}(f_1(p),\dots f_n(p)) =$$
$$= \theta'_{1|i}(f_1(p),\dots f_n(p))\cdot\theta_2(f_1(p),\dots f_n(p)) +$$
$$+ \theta_1(f_1(p),\dots,f_n(p))\cdot\theta'_{2|i}(f_1(p),\dots,f_n(p)) = 0,$$

for $i=1,2,\dots,n$. Hence one can easily prove that

$$\theta''_{|ij}(f_1(p),\dots f_n(p)) = 0,$$

for $i,j=1,2,\dots,n$. Thus for any $\alpha_1,\alpha_2\in C$

$$v^{(2)}[(\alpha_1-\alpha_1(p))\cdot(\alpha_2-\alpha_2(p)) = 0$$

which gives $\dim T_p^{(2)}M = 0$.

Assume now that $\dim T_p^{(k)}M = 0$ for some $k\in\mathbb{N}$. We will show that $\dim T_p^{(k+1)}M = 0$. If $\dim T_p^{(k)}M = 0$ then from Corollary 2.14 it follows that $C = A_p^k$.

Let $v^{(k+1)}\in T_p^{(k+1)}M$ be an arbitrary tangent vector of (k+1)-th order. We should show that $v^{(k+1)}|\alpha_p^{k+1}= 0$.

Indeed, for any $\alpha_1,\dots\alpha_{k+1}\in C$, there exist an open neighborhood U of p and functions $f_1,\dots f_n\in C$, for some $n\in\mathbb{N}$, as well as functions $\theta_1,\dots,\theta_{k+1}\in C^\infty(\mathbb{R}^n)$ such that

1° $\alpha_i-\alpha_i(p)|U = \theta_i\circ(f_1,\dots,f_n)|U$,

for $i=1,2,\dots,k+1$.

2° $\theta^{(l)}_{j\,|i_1\dots i_l}(f_1(p),\dots,f_n(p)) = 0$, for $l=1,2,\dots,k$ and $1\le i_1,\dots,i_l\le n$. Of course,

$$\theta_j(f_1(p),\dots,f_n(p)) = 0,$$

for $j=1,2,\dots,k+1$. It is evident that

$$(\alpha_1 - \alpha_1(p))\cdot\ldots\cdot(\alpha_{k+1} - \alpha_{k+1}(p))\,|U = \theta\circ(f_1,\dots,f_n)\,|U ,$$

where $\theta := \theta_1\cdot\ldots\cdot\theta_{k+1}$.

One can see that $\theta^{(l)}{}_{|i_1\ldots i_l}(f_1(p),\dots,f_n(p))=0$ for $l=1,2,\dots k+1$ and $1\le i_1,\dots,i_l\le n$.

Thus

$$v^{(k+1)}[(\alpha_1 - \alpha_1(p))\cdot\ldots\cdot(\alpha_{k+1} - \alpha_{k+1}(p))] =$$

$$= v^{(k+1)}(\theta\circ(f_1,\dots,f_n)) =$$

$$= \sum_{m=1}^{k+1} \Big(\sum_{1\le i_1<\ldots<i_m\le n} \theta^{(m)}{}_{|i_1\ldots i_m}(f_1(p),\dots,f_n(p))\cdot$$

$$\cdot v^{(k+1)}[(f_{i_1} - f_{i_1}(p))\cdot\ldots\cdot(f_{i_m} - f_{i_m}(p))] = 0.$$

Hence $T_p^{(k+1)}M = \{0\}$. So, we have proved that if $\dim T_p^{(k)}M = 0$ then $\dim T_p^{(k+1)}M = 0$. Consequently, we have proved, by induction, that $\dim T_p^{(k)}M = 0$ for any $k\in\mathbb{N}$.

Definition 2.5 A function $X^{(k)}: M \longrightarrow \bigcup_{p\in M} T_p^{(k)}M$, where $k\in\mathbb{N}$, is said to be a tangent vector field of k-th order to (M,C) if $X^{(k)}(p) \in T_p^{(k)}M$ for any $p\in M$.

A tangent vector field of k-th order $X^{(k)}$ to (M,C) is said to be a *smooth* if a function $X^{(k)}f$, given by $(X^{(k)}f)(p) = X^{(k)}(p)(f)$, is a smooth function on M for any $f\in C$.

Evidently the set $\mathfrak{X}^{(k)}(M)$ of all smooth tangent vector fields of k-th order to (M,C) is a C-module, for any $k\in\mathbb{N}$.

Now let (M,C) be a d.s and let $k\in\mathbb{N}$. A C-module $\mathfrak{X}^{(k)}(M)$ is said to be a *differential module* of dimension r if for any $p\in M$, $\dim T_p^{(k)}M = r$ as well as for any point $p\in M$ there exist an open neighborhood U of p and C_U-basis $X_1^{(k)},\dots,X_r^{(k)}$ such that $\mathrm{Lin}\,(X_1^{(k)}(q),\dots,X_r^{(k)}(q)) = T_p^{(k)}M$ for $q \in U$.

In the case k=1, we exchangeably say that (M,C) is a d.s of constant differential dimension.

Let X be a smooth tangent vector field on (M,C). Now, by a smooth tangent vector field X on (M,C) we mean the linear mapping $X : C \longrightarrow C$ satisfying the condition

$$X(f\cdot g) = f(X)g + gX(f),$$

for any $f,g \in C$.

One can easily prove

Lemma 2.16 Let $X_1,\ldots,X_k \in \mathfrak{X}(M)$, where $k\in\mathbb{N}$. Then $X_1\circ\ldots\circ X_k$ is a smooth tangent vector field of k-th order to (M,C).

Lemma 2.17 Let $X\in\mathfrak{X}(M)$ and let $v^{(k)} \in T_p^{(k)}M$, where $k\in\mathbb{N}$. Then the mapping $W:C \longrightarrow \mathbb{R}$ defined by $W(f) = v^{(k)}(X(f))$, for any $f\in C$, is a tangent vector of (k+1)-th order to (M,C) at the point $p\in M$.

Proof. Evidently, W is a linear mapping. So it suffices to show that $W[(\alpha_1- \alpha_1(p))\cdot\ldots\cdot(\alpha_{k+2}- \alpha_{k+2}(p))] = 0$ for any $\alpha_1,\ldots,\alpha_{k+2} \in C$. We have

$$W[(\alpha_1- \alpha_1(p))\cdot\ldots\cdot(\alpha_{k+2}- \alpha_{k+2}(p))] =$$
$$v^{(k)}(X[(\alpha_1- \alpha_1(p))\cdot\ldots\cdot(\alpha_{k+2}- \alpha_{k+2}(p))]) =$$
$$v^{(k)}[\sum_{i=1}^{k+2} (\alpha_1- \alpha_1(p))\cdot\ldots\cdot(\alpha_i\hat{-} \alpha_i(p))\cdot\ldots$$
$$\ldots\cdot(\alpha_{(k+2)}- \alpha_{(k+2)}(p))X(\alpha_i- \alpha_i(p)).$$

Similarly we can prove

Lemma 2.18 Let $X_1,\ldots X_k\in \mathfrak{X}(M)$ and $v^{(l)}\in T_p^{(l)}M$, where $k,l\in\mathbb{N}$. Then $v^{(l)}\circ X_1\circ\ldots\circ X_k \in T_p^{(k+l)}M$.

Lemma 2.19 Let $X^{(k)} \in \mathfrak{X}^{(k)}(M)$ and $X^{(k)}(p)\neq 0$ for some $p\in M$. Let $\underset{1}{v},\ldots,\underset{m}{v} \in T_pM$. If the vectors $\underset{1}{v},\ldots,\underset{m}{v}$ are linearly independent then the vectors $\underset{1}{v}\circ X^{(k)},\ldots,\underset{m}{v}\circ X^{(k)}$ are also linearly independent.

Proof. Let k=1. Assume that $\underset{1}{v},\ldots,\underset{m}{v} \in T_pM$ are linearly independent. Suppose that the vectors of 2-th order

$$\underset{1}{Y}^{(2)} = \underset{1}{Y} \circ X, \dots, \underset{m}{Y}^{(2)} = \underset{m}{Y} \circ X$$

are linearly dependent. Hence there exist $\lambda^1, \dots, \lambda^m \in \mathbb{R}$ such that $|\lambda^1| + \dots + |\lambda^m| > 0$ and $\lambda^1 \underset{1}{Y}^{(2)} + \dots + \lambda^m \underset{m}{Y}^{(2)} = 0$ or equivalently

$$\lambda^1 \underset{1}{Y}^{(2)} + \dots + \lambda^m \underset{m}{Y}^{(2)} = (\lambda^1 \underset{1}{Y} + \dots + \lambda^m \underset{m}{Y}) \circ X = 0$$

Let us put $W = \lambda^1 \underset{1}{Y} + \dots + \lambda^m \underset{m}{Y}$. Then $W \neq 0$ and $W \circ X = 0$. Hence $W(X(\lambda)) = 0$ for any $\lambda \in C$. Let $\lambda = f^2$, where $f \in C$. Then we get $W(X(f^2)) = W(2fX(f)) = 2W(f) \cdot X(p)(f) = 0$. Obviously, there exists $f \in C$ such that $W(f) \neq 0$ and $X(p)(f) \neq 0$. So $W(f) \cdot X(p)(f) \neq 0$. Thus we get the contradiction. Therefore, the Lemma is true for $k=1$. In the case $k>1$, the proof runs in a similar way.

Proposition 2.20 Let $\mathfrak{X}(M)$ be a differential module of dimension n. Assume that, for any point $p \in M$, there exists a vector basis $X_1, \dots, X_n \in \mathfrak{X}(U)$ in an open neighborhood U of p satisfying the condition

$$[X_i, X_j] = 0 \text{ for } i,j = 1, \dots, n.$$

Then $\mathfrak{X}^{(k)}(M)$ is a differential module of dimension $m = \sum_{i=1}^{k} \binom{n+i-1}{i}$, for any $k \in \mathbb{N}$.

Proof. Let $\mathfrak{X}(M)$ be a differential module of dimension n satisfying the assumption of Proposition 2.20. Consider the case $k=2$. It is easy to see that the vector fields

$$X_1, \dots, X_n,\ X_1 \circ X_1,\ X_1 \circ X_2, \dots, X_1 \circ X_n,\ X_2 \circ X_2, \dots, X_2 \circ X_n, \dots,\ X_n \circ X_n$$

form $C(U)$-basis of $\mathfrak{X}^{(2)}(U)$. From Lemma 2.19 it follows that, for any $q \in U$, the vectors

$X_1(q), \dots, X_n(q),\ X_1 \circ X_1(q),\ X_1 \circ X_2(q), \dots, X_1 \circ X_n(q),\ X_2 \circ X_2(q), \dots, X_2 \circ X_n(q), \dots,\ X_n \circ X_n(q)$ form a basis of $T_q^{(2)}M$.

In the case $k>2$, one can prove this proposition in a similar way.

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Received May 9, 1991.