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ON GENERAL HAMILTONIAN DYNAMICAL SYSTEMS IN DIFFERENTIAL SPACES

The purpose of this paper is to show that a major part of the general formalism of the Poisson brackets, well known from the classical mechanics, can be formulated basing on a more general object than differentiable manifold.

1. Introduction

Let M be any set and C a family of real functions on M. By τ_C we denote the weakest topology on M in which all functions of C are continuous. By scC we denote the set of all functions on M of the form $\omega\circ(\alpha_1,\dots,\alpha_n)$, where $\omega \in C^{\infty}(\mathbb{R}^n)$ and $n \in \mathbb{N}$. Now, let A be a subset of M. By C_A we denote the set of all functions $g: A \longrightarrow \mathbb{R}$ such that for each point $p \in A$ there exist an open neighbourhood U of p and a function $f \in C$ such that $g|U = f|U$.

If $C = (\mathrm{sc}C)_M$ then the set C of real functions on M is said to be a differential structure on M, and the pair (M,C) is called a differential space [1], [2]. If (M,C) is a differential space and A is a subset of M then (A,C_A) is also a differential space and it is called a differential subspace of (M,C).

A mapping $F: M \longrightarrow N$ is said to be a smooth mapping of a differential space (M,C) into a differential space (N,D) if, for any $f \in D$, $f\circ F \in C$. If F is a smooth mapping of (M,C) into (N,D), we shall write $F: (M,C) \longrightarrow (N,D)$. Moreover, for an arbitrary mapping $F: M \longrightarrow N$, by $F^*: D \longrightarrow C$ we denote the mapping given by $F^*\alpha = F^*(\alpha) := \alpha\circ F$, for any $\alpha \in D$.

We define the notion of a tangent vector to a differential space (M,C) at a point $p \in M$ as a linear mapping $v: C \longrightarrow \mathbb{R}$ satisfying the condition $v(\alpha\cdot\beta) = \alpha(p)\cdot v(\beta) + \beta(p)\cdot v(\alpha)$, for any $\alpha, \beta \in C$. The set of all tangent vectors to (M,C) at a point $p \in M$ we denote by T_pM and we call it the tangent space to (M,C) at the point p.

If $F: (M,C) \longrightarrow (N,D)$ is a smooth mapping then, for each point $p \in M$, the mapping $d_pF \equiv F_{*p}: T_pM \longrightarrow T_{F(p)}N$ is defined by $(F_{*p}v)(\beta) := v(F^*\beta)$, for any $\beta \in D$ and $v \in T_pM$.

Furthermore, by a smooth tangent vector field to (M,C) we mean an $\mathbb{R}$-linear mapping $X: C \longrightarrow C$ such that $X(\alpha\cdot\beta) = \alpha\cdot X(\beta) + \beta\cdot X(\alpha)$, for any $\alpha, \beta \in C$. The set of all smooth tangent vector fields to (M,C) we denote by $\mathfrak{X}(M)$.

Now, let (M,C) be a differential space and let C_0 be a subset of C, then C_0 is said to be a set of generators of the differential structure C on M if $C = (scC_0)_M$.

Let (M,C) and (N,D) be differential spaces. By $C\mathbf{x}D$ we denote the differential structure on $M\times N$ generated by the set

$$\{\alpha\circ pr_M: \alpha \in C\} \cup \{\beta\circ pr_N: \beta \in D\},$$

where $pr_M: M\times N \longrightarrow M$ and $pr_N: M\times N \longrightarrow N$ are the canonical projections on M and N, respectively. The pair $(M\times C, C\mathbf{x}D)$ is called the Cartesian product of differential spaces (M,C) and (N,D).

Evidently, pr_M and pr_N are smooth mappings. Similarly, for any point $q \in N$, the mapping $\iota_q: M \longrightarrow M\times N$, given by $\iota_q(p) := (p,q)$ for any $p \in M$, as well as the mapping $\iota_p: N \longrightarrow M\times N$ for any $p \in M$, given by $\iota_p(q) := (p,q)$ for any $q \in N$, are smooth.

2. Almost Poisson and Poisson differential spaces

Let (M,C) be a differential space.

Definition 2.1 A skew-symmetric $\mathbb{R}$-2-linear mapping $\{\cdot,\cdot\}: C\times C \longrightarrow C$ is said to be an almost Poisson structure on (M,C) if

$$\{f\cdot g,h\} = f\cdot\{g,h\} + g\cdot\{f,h\}, \tag{2.1}$$

for any $f,g,h \in C$.

If $\{\cdot,\cdot\}$ is an almost Poisson structure on (M,C) then the pair $((M,C),\{\cdot,\cdot\})$ is called an almost Poisson differential space. In such a case, the differential space (M,C) is called the phase space of $((M,C),\{\cdot,\cdot\})$ and $\{f,g\}$ is called the Poisson bracket of functions $f,g \in C$.

From (2.1) it follows that the mapping

$$X_h \equiv \{\cdot,h\}: C \longrightarrow C, \tag{2.2}$$

for any $h \in C$, is a smooth vector field on (M,C). Evidently, for arbitrary $f,h \in C$, we have

$$X_h(f) = -X_f(h). \tag{2.3}$$

It is easy to observe that if $\{\cdot.\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are almost Poisson structures on (M,C) and $f \in C$ then $f\cdot\{\cdot,\cdot\}_1$ as well as $\{\cdot,\cdot\}_1 + \{\cdot,\cdot\}_2$ are almost Poisson structures on (M,C). Thus we have

Proposition 2.1 The set of all almost Poisson structures on (M,C) constitutes a module over the ring C.

Definition 2.2 An almost Poisson structure $\{\cdot,\cdot\}$ on a differential space (M,C) is said to be a Poisson structure on (M,C) if, for any $f,g,h \in C$, there is

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0. \tag{2.4}$$

If $\{\cdot,\cdot\}$ is a Poisson structure on (M,C) then $((M,C),\{\cdot,\cdot\})$ is called the Poisson differential space.

Example 2.1 Let $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ be smooth vector fields on a differential space (M,C). Let us put

$$\{f,g\} := \sum_{i=1}^{n} \big(X_i(f)Y_i(g) - X_i(g)Y_i(f)\big), \tag{2.5}$$

for any $f,g \in C$. Evidently, the mapping $\{\cdot,\cdot\}: C\times C \longrightarrow C$ defined by (2.5) is an almost Poisson structure on (M,C).

Moreover, if we assume that

$$[X_i,X_j] = [X_i,Y_j] = [Y_i,Y_j] = 0,$$

for any $i,j = 1,\dots,n$, then by straightforward calculations we can show that $\{\cdot,\cdot\}$ is the Poisson structure on (M,C).

It is not difficult to observe that, if $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are Poisson structures on (M,C) then $\{\cdot,\cdot\}_1 - \{\cdot,\cdot\}_2$ is a almost Poisson structure (but not necessarily a Poisson structure) on (M,C).

Now we prove

Lemma 2.2 Let $\{\cdot,\cdot\}$ be an almost Poisson structure on (M,C). Then, for arbitrary $f,g,h \in C$, the identity is satisfied

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = \tag{2.6}$$
$$= \left(-X_{\{f,g\}} + [X_g,X_f]\right)(h).$$

Proof. For any $f,g,h \in C$, we have the identities

$$\{\{f,g\},h\} = -X_{\{f,g\}}(h), \quad \{\{h,f\},g\} = X_g(X_f(h))$$

and
$$\{\{g,h\},f\} = -X_f(X_g(h)).$$

Hence we get

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = \left(-X_{\{f,g\}} + [X_g,X_f]\right)(h).$$

Corollary 2.3 Let $\{\cdot,\cdot\}$ be an almost Poisson structure on (M,C). Then for any $f,g,h \in C$ we have

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$$

if and only if $[X_f,X_g] = X_{\{g,f\}}$.

Lemma 2.4 Let $((M,C),\{\cdot,\cdot\})$ be a Poisson differential space and C_0 be a set of generators of the differential structure C on M. Then, $[X_f,X_\alpha] = X_{\{\alpha,f\}}$, for any $f \in C$, $\alpha \in C_0$, implies $[X_f,X_g] = X_{\{g,f\}}$, for any $f,g \in C$.

Proof. Assume that $[X_f,X_\alpha] = X_{\{\alpha,f\}}$, for any $f \in C$, $\alpha \in C_0$, and suppose that $[X_f,X_g] \neq X_{\{g,f\}}$, for certain $f,g \in C$. Then there exist $\beta \in C_0$ such that $[X_f,X_g](\beta) \neq X_{\{g,f\}}(\beta)$. Hence we get

$$X_f(X_g(\beta)) - X_g(X_f(\beta)) \neq -X_\beta(\{g,f\}),$$

or eqivalently

$$-X_f(X_\beta(g)) - X_g(\{\beta,f\}) \neq -X_\beta(X_f(g)),$$

and consequently

$$-X_g(\{\beta,f\}) \neq [X_f,X_\beta](g)$$

or equivalently

$$X_{\{\beta,f\}}(g) \neq [X_f,X_\beta](g).$$

But the last inequality contradicts the assumption. Therefore the lemma is true.

Similarly, we prove

Lemma 2.5 Let $((M,C),\{\cdot,\cdot\})$ be a Poisson differential space and C_0 be a set of generators of the differential structure C on M. If, for any $\alpha,\beta \in C_0$, $[X_\alpha,X_\beta] = X_{\{\beta,\alpha\}}$ then, for any $f \in C$ and $\beta \in C_0$, $X_{\{\beta,f\}} = [X_f,X_\beta]$ and, in the strength of Lemma 2.4, $X_{\{g,f\}} = [X_f,X_g]$, for any $f,g \in C$.

From the last lemma we obtain

Corollary . Let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space and C_0 be a set of generators of the differential structure C on M. If, for any $\alpha,\beta,\gamma \in C_0$,

$$\{\{\alpha,\beta\},\gamma\} + \{\{\gamma,\alpha\},\beta\} + \{\{\beta,\gamma\},\beta\} = 0,$$

then, for any $f,g,h \in C$,

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0,$$

and consequently $((M,C),\{\cdot,\cdot\})$ is the Poisson differential space.

Evidently, every Poisson differential space is an almost Poisson differential space; the inverse of this statement is not true.

One can prove

Lemma 2.7 Let (M,C) be a differential space, $\psi: N \longrightarrow M$ be a mapping of a set N into a set M, and $f,g \in C$. Then $\psi^*f|A = \psi^*g|A$, for some $A \subset N$, if and only if $f|\psi(A) = g|\psi(A)$.

Let (M,C) be a differential space, N be a set and let $\psi: N \longrightarrow M$ be a mapping. Let us put $\bar{D} = \psi^*C$ and $f = \omega\circ(\alpha_1,\dots,\alpha_n)$, where $\alpha_1,\dots,\alpha_n \in C$ and $\omega \in C^\infty(\mathbb{R}^n)$. Then

$$\psi^*f = \psi^*\omega\circ(\alpha_1,\dots,\alpha_n) = \omega\circ(\psi^*\alpha_1,\dots,\psi^*\alpha_n).$$

Hence $\bar{D} = sc\bar{D}$. Now, let us put $D = (\bar{D})_N$. Then D is a differential structure on N induced from (M,C) by ψ [12].

Let now $f \in D$. By definition of the differential structure, for any point $p \in N$ there exist an open neighbourhood A of p and a function $g \in \psi^* C$ such that $f|A = g|A = \psi^* \alpha|A$, for some $\alpha \in C$.

Similarly, as in the theory of differentiable manifolds, we prove

Lemma 2.8 Let (M,C) be a differential space and $f \in C$. If $f|A = 0$, for some $A \in \tau_C$, then for any $X \in \mathfrak{X}(M)$ and $p \in A$,

$$X(f)(p) = X(p)(f|A) = 0,$$

or equivalently $X|_A(f|A) = 0$.

Hence we get

Corollary 2.9 Let (M,C) be a differential space and let $f,g \in C$. If $f|A = g|A$, for some $A \in \tau_C$, then for any $X \in \mathfrak{X}(M)$ and $p \in A$

$$X(f)(p) = X(g)(p),$$

or equivalently $X|_A(f|A) = X|_A(g|A)$.

Now, we shall prove

Lemma 2.10 Let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space and let $f,g \in C$. If $f|A = g|A$, for some $A \in \tau_C$, then $\{f,h\}|A = \{g,h\}|A$, for any $h \in C$.

Proof. $\{f,h\}|A = X_h(f)|A = X_h|_A(f|A) = X_h|_A(g|A) =$

$$= X_h(g)|A = \{g,h\}|A.$$

Let now $\alpha,\beta \in C$. Then, for every point $p \in M$, there exist $A \in \tau_C$, $p \in A$ and $f,g \in \bar{C}$, where $\bar{C}_M = C$, such that $\alpha|A = f|A$ as well as $\beta|A = g|A$. Hence, by Lemma 2.10,

$$\{\alpha,\beta\}|A = \{f,g\}|A.$$

Thus we get

Corollary 2.11 The Poisson structure $\{\cdot,\cdot\}$ on (M,C) is uniquely determined by its values $\{f,g\}$ for $f,g \in \bar{C}$, where $\bar{C}_M = C$.

Now, let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space. Let $\Phi\colon N \longrightarrow M$ be a mapping such that $\Phi(N)$ is an open subset of M. By putting $D = (\Phi^* C)_M$ we get the differential space (N,D) with the induced differential structure D on N.

Now let us put

(2.7) $$\{\Phi^*f,\Phi^*g\}_N = \Phi^*\{f,g\},$$

for $f,g \in C$. One can easily show that the formula (2.7) defines an almost Poisson structure on (N,D) (or Poisson structure, if $((M,C),\{\cdot,\cdot\})$ is a Poisson differential space, respectively).

Hence we get

Corollary 2.12 Let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space (Poisson differential space) and let A be an open subset of M. Then $((A,C_A),\{\cdot,\cdot\}_A)$ is an almost Poisson differential space (Poisson differential space), where $\{\cdot,\cdot\}_A$ is defined by

(2.8) $$\{\iota^*\alpha,\iota^*\beta\}_A = \iota^*\{\alpha,\beta\},$$

for any $\alpha,\beta \in C$, where ι is the natural imbedding of A into M.

$((A,C_A),\{\cdot,\cdot\}_A)$ is said to be an almost Poisson differential subspace (Poisson differential subspace) of $((M,C),\{\cdot,\cdot\})$.

Corollary 2.13 Let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space (Poisson differential space) and A be an open subset of M. Then, for any $f,g \in C$,

$$\{f,g\}|A = \{f|A,g|A\} .$$

Now, let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space (Poisson differential space) and let $\Phi: M \longrightarrow N$ be a mapping of M onto N. Let D be the differential structure on N coinduced from (M,C) by Φ. Then (N,D) is a differential space. Let us put

(2.9) $$\{\alpha,\beta\}_N\circ\Phi = \{\alpha\circ\Phi,\beta\circ\Phi\}.$$

We shall prove

Proposition 2.14 The mapping $\{\cdot,\cdot\}_N$: $D\times D \longrightarrow D$, defined by (2.9), is an almost Poisson structure (a Poisson structure, respectively) on (N,D).

Proof. $\mathbb{R}$-2-linearity and skew-symmetry of $\{\cdot,\cdot\}_N$ are evident immediately from (2.9). The Jacobi identity follows from the identities

$$\{\{\alpha,\beta\}_N,\gamma\}_N\circ\Phi = \{\{\alpha,\beta\}_N\circ\Phi,\gamma\circ\Phi\} =$$
$$= \{\{\alpha\circ\Phi,\beta\circ\Phi\},\gamma\circ\Phi\} = \{\{\bar{\alpha},\bar{\beta}\},\bar{\gamma}\},$$

where $\bar{\alpha} = \alpha\circ\Phi$, $\bar{\beta} = \beta\circ\Phi$ and $\bar{\gamma} = \gamma\circ\Phi$. Evidently, $\bar{\alpha},\bar{\beta},\bar{\gamma} \in C$, for $\alpha,\beta,\gamma \in D$. Similarly, condition (2.1) follows from

$$\{\alpha\cdot\beta,\gamma\}_N\circ\Phi = \{\alpha\cdot\beta\circ\Phi,\gamma\circ\Phi\} = \{(\alpha\circ\Phi)\cdot(\beta\circ\Phi),\gamma\circ\Phi\} = \{\bar{\alpha}\cdot\bar{\beta},\bar{\gamma}\},$$

for $\alpha,\beta,\gamma \in D$.

Corollary 2.15 Let $\{\cdot,\cdot\}$ be an almost Poisson structure (a Poisson structure) on the Cartesian product $(M\times N, C\times D)$ of differential spaces (M,C) and (N,D). Then the formulae

$$\mathrm{pr}_M^*\{f,g\}_M = \{\mathrm{pr}_M^*f,\mathrm{pr}_M^*g\}, \tag{2.10}$$

for any $f,g \in C$, and

$$\mathrm{pr}_N^*\{\alpha,\beta\}_N = \{\mathrm{pr}_N^*\alpha,\mathrm{pr}_N^*\beta\}, \tag{2.11}$$

for any $\alpha,\beta \in D$, define almost Poisson structures (Poisson structures) on (M,C) and (N,D), respectively.

Now, we shall prove

Proposition 2.16 Let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space (Poisson differential space) and let C_0 be a set of generators of the differential structure C on M. Then, for any $f,g \in C$, the Poisson bracket $\{f,g\}$ of f and g can be locally (in a neighbourhood U_p of each point $p \in M$) represented by a linear combination of the Poisson brackets of functions from C_0 with the coefficients from C_{U_p}.

Proof. Let $f,g \in C$ and $p \in M$. There exist an open neighbourhood U_p of p and functions $\alpha_1,\dots,\alpha_k, \beta_1,\dots,\beta_l \in C_0$ as well as $\omega_1 \in C^\infty(\mathbb{R}^k)$ and $\omega_2 \in C^\infty(\mathbb{R}^l)$ such that $f|U_p = \omega_1\circ(\alpha_1,\dots,\alpha_k)|U_p$ and $g|U_p = \omega_2\circ(\beta_1,\dots,\beta_l)|U_p$. Hence we get the following equalities

$$\{f,g\}|U_p = \{f|U_p,g|U_p\}_{U_p} = X_g|_{U_p}(f|U_p) =$$
$$= X_g|_{U_p}(\omega_1\circ(\alpha_1,\dots,\alpha_k)|U_p) =$$
$$= \sum_{i=1}^{k} \omega'_{1|i}\circ(\alpha_1,\dots,\alpha_k)|U_p\cdot X_g|_{U_p}(\alpha_i|U_p) =$$
$$= -\sum_{i=1}^{k} \omega'_{1|i}\circ(\alpha_1,\dots,\alpha_k)|U_p\cdot X_{\alpha_i}|_{U_p}(g|U_p) =$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} \left(\omega'_{1|i}\circ(\alpha_1,\dots,\alpha_k)\cdot\omega'_{2|j}(\beta_1,\dots,\beta_l)\right)|U_p\cdot\{\alpha_i,\beta_j\}U_p =$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} \lambda^{ij}\cdot\{\alpha_i,\beta_j\}|U_p,$$

where, for $i = 1,\dots,k$ and $j = 1,\dots,l$, we have put $\lambda^{ij} = \left(\omega'_{1|i}\circ(\alpha_1,\dots,\alpha_k)\cdot\omega'_{2|j}\circ(\beta_1,\dots,\beta_l)\right)|U_p$.

From Proposition 2.12 it follows

Corollary 2.17 Let C_0 be a set of generators of the differential structure C on M, and let $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ be almost Poisson structures on (M,C). Then, if for any $\alpha,\beta \in C_0$, $\{\alpha,\beta\}_1 = \{\alpha,\beta\}_2$ then $\{\cdot,\cdot\}_1 = \{\cdot,\cdot\}_2$.

Proposition 2.18 Let $((M,C),\{\cdot,\cdot\}_M)$ and $((N,D),\{\cdot,\cdot\}_N)$ be almost Poisson differential spaces (Poisson differential spaces). Then the mapping

$$\{\cdot,\cdot\}: (C\text{x}D)\times(C\text{x}D) \longrightarrow C\text{x}D,$$

defined by

$$(2.12)\quad \{f,g\}(x,y) = \{f(\cdot,y),g(\cdot,y)\}_M(x) + \{f(x,\cdot),g(x,\cdot)\}_N(y)$$

for any $f,g \in C\text{x}D$ and $(x,y) \in M\times N$, is an almost Poisson structure (a Poisson structure, respectively) on the differential space $(M\times N, C\text{x}D)$. Moreover, $\{\alpha\circ pr_M,\beta\circ pr_N\} = 0$, for any $\alpha \in C$ and $\beta \in D$.

Proof. From (2.12) it follows immediately that $\{\cdot,\cdot\}$ is an $\mathbb{R}$-2-linear and skew-symmetric mapping. It is easy to observe that $\{\cdot,\cdot\}$, defined by (2.12), satisfies condition (2.1). (In the case of Poisson differential spaces, by using Corollary 2.6, one can easily show that the Jacobi identity is satisfied.)

Let $((M,C),\{\cdot,\cdot\}_M)$ and $((N,D),\{\cdot,\cdot\}_N)$ be almost Poisson differential spaces (Poisson differential spaces).

Definition 2.3 A smooth mapping $F: (M,C) \longrightarrow (N,D)$ is said to be a morphism of almost Poisson differential spaces (Poisson differential spaces) $((M,C),\{\cdot,\cdot\}_M)$ and $((N,D),\{\cdot,\cdot\}_N)$ if

$$(2.13)\quad \{F^*f,F^*g\}_M = F^*\{f,g\}_N,$$

for any $f,g \in D$.

Evidently, the composition of morphisms of almost Poisson differential spaces (Poisson differential spaces) is a morphism in the respective category. If F is a diffeomorphism of differential spaces satisfying condition (2.13), then F is called the isomorphism of almost Poisson differential spaces (Poisson differential spaces).

Therefore the class of all almost Poisson differential spaces (Poisson differential spaces), together with the above morphisms, forms a category, called the category of almost Poisson differential spaces (Poisson differential spaces). Obviously, the category of Poisson differential spaces is a subcategory of almost Poisson differential spaces.

Definition 2.4 Let $((M,C),\{\cdot,\cdot\})$ be an almost Poisson differential space (Poisson differential space) and let $f,g \in C$. A function f is said to be in involution with a function g if $\{f,g\} = 0$.

Evidently, this relation is symmetric.

Definition 2.5 A smooth function f on an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$ is called the Casimir function if $\{f,g\} = 0$ for any $g \in C$.

Let us observe that if f is a Casimir function on an almost Poisson differential space then the vector field $X_f = \{\cdot,f\}$ is the zero vector field.

Let us denote by C_C the set of all Casimir functions on an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$. One can easily prove

Proposition 2.19 The set C_C of all Casimir functions on $((M,C),\{\cdot,\cdot\})$ is

(i) a differential substructure of the differential structure C on M,

(ii) a module over C as well as a Lie subalgebra of the Lie algebra $(C,\{\cdot,\cdot\})$, if $((M,C),\{\cdot,\cdot\})$ is a Poisson differential space.

Now, let (M,C) be a differential space and let X be a smooth vector field on (M,C). Similarly as in the theory of differentiable manifolds, we accept

Definition 2.6 A function $f \in C$ is said to be a first integral of a smooth vector field X on (M,C) if $X(f) = 0$.

Similarly, we also prove

Proposition 2.20 If $\gamma: (-\varepsilon,\varepsilon) \longrightarrow M$ is an integral curve (a trajectory) of a smooth vector field X on (M,C) then any first integral f of X is a constant function along γ; this means that $f\circ\gamma = \text{const}$.

Let us observe that in the case of differentiable manifolds the assertion inverse to the Proposition 1.10 is satisfied. However, one can prove that, in general, such proposition is not true for differential spaces.

One can easily prove

Lemma 2.21 A smooth function f is a first integral of the vector field $X_h = \{\cdot,h\}$, if $\{f,h\} = 0$.

Proposition 2.22 The set of all first integrals of a vector field X_h on $((M,C),\{\cdot,\cdot\})$ forms a differential substructure of the differential structure C on M as well as a Lie subalgebra of the Lie algebra $(C,\{\cdot,\cdot\})$, if $((M,C),\{\cdot,\cdot\})$ is a Poisson differential space, where $X_h = \{\cdot,h\}$.

Now, we shall prove

Proposition 2.23 Let $F: M \longrightarrow N$ be an isomorphism of almost Poisson differential spaces $((M,C),\{\cdot,\cdot\}_M)$ and $((N,D),\{\cdot,\cdot\}_N)$. For an arbitrary $h \in D$, let us put $Y_h = \{\cdot,h\}_N$ and $X_{F^*h} = \{\cdot,F^*h\}_M$. Then the following equation holds

$$F_*X_{F^*h} = Y_h.$$

Proof. Let $f,h \in D$ and let h be fixed. Then we have

$$Y_h(f) = \{f,h\}_N = \{f,h\}_N\circ F\circ F^{-1} = F^*\{f,h\}_N\circ F^{-1} =$$

$$= \{F^*f,F^*h\}_M\circ F^{-1} = X_{F^*h}(f\circ F)\circ F^{-1} =$$

$$= dF(X_{F^*h}(f)\circ F^{-1} = F_*X_{F^*h}(f).$$

3. General Hamiltonian dynamical systems on differential spaces

Definition 3.1 A smooth vector field X on an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$ is said to be a

Hamiltonian vector field if there exists a function $H \in C$ such that

(3.1) $$X = \{\cdot,H\} \equiv X_H.$$

The function H is called the Hamiltonian function of X or shortly the hamiltonian. The triple $((M,C),\{\cdot,\cdot\},H)$ is called the Hamiltonian dynamical system on (M,C).

Definition 3.2 A smooth vector field X on an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$ is said to be a locally Hamiltonian vector field if, for any point $p \in M$, there exists a neighbourhood U of p such that $X|_U$ is a Hamiltonian vector field on $((U,C_U),\{\cdot,\cdot\}_U)$.

The differential space (M,C) is called the phase space (or the space of states) of the dynamical system $((M,C),\{\cdot,\cdot\},H)$. The points of M are interpreted as different states of the system. In turn, the smooth functions of C are called the observables or the dynamical quantities.

Let us denote by $\mathcal{H}(M)$ the set of all Hamiltonian vector fields on an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$.

Proposition 3.1 The set $\mathcal{H}(M)$ has the natural structure of a module over the Casimir ring C_C as well as the structure of a Lie subalgebra of the Lie algebra $(\mathfrak{X}(M),[\cdot,\cdot])$.

Proof. Let X_H, $X_F \in \mathcal{H}(M)$ and $f \in C_C$. By definition we get

$$X_H + X_F = \{\cdot,H\} + \{\cdot,F\} = \{\cdot,H+F\} = X_{H+F}.$$

Similarly,

$$f\cdot X_H(g) = f\cdot\{g,H\} = f\cdot\{g,H\} + g\cdot\{f,H\} = \{g,fH\} = X_{fH}(g),$$

for any $g \in C$. Hence we get $f\cdot X_H = X_{fH}$, for $f \in C_C$.

Now, if we assume that $((M,C),\{\cdot,\cdot\})$ is a Poisson differential space, we also have the equality

(3.2) $$[X_H,X_F] = X_{\{F,H\}}.$$

Hence, for any $X_H,X_F \in \mathcal{H}(M)$, also $[X_H,X_F] \in \mathcal{H}(M)$.

From Proposition 3.1, or more exactly from (3.2), it follows

Corollary 3.2 For any Hamiltonian vector fields X_H, X_F on a Poisson differential space $((M,C),\{\cdot,\cdot\})$, $[X_H,X_F] = 0$ if and only if $\{F,H\} \in C_C$.

Corollary 3.3 The mapping $C \ni f \longmapsto X_f \in \mathcal{X}(M)$ is a homomorphism of linear space C over $\mathbb{R}$ onto the linear space $\mathcal{X}(M)$ over $\mathbb{R}$. Moreover, if $((M,C),\{\cdot,\cdot\})$ is a Poisson differential space, then this mapping is a homomorphism of the Lie algebra $(C,\{\cdot,\cdot\})$ into the Lie algebra $(\mathcal{X}(M),[\cdot,\cdot])$.

Now, we shall prove

Proposition 3.4 Let H_1 and H_2 be hamiltonians of the same Hamiltonian vector field X on an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$. Then $H_1 - H_2 \in C_C$.

Proof. By the assumption, $X = X_{H_1} = X_{H_2}$. Hence, for any $f \in C$, we have $X_{H_1}(f) = X_{H_2}(f)$, or equivalently $\{f,H_1 - H_2\} = 0$. Thus $H_1 - H_2 \in C_C$.

It is well known that in the classical mechanics on a connected differentiable manifold, H_1 and H_2 are hamiltonians of the same Hamiltonian vector field X if and only if $H_1 - H_2$ is a constant function.

Now, we want to study the behaviour of an observable F along a phase curve (a history of a dynamical system) γ of our Hamiltonian dynamical system. As it is well known, the behaviour of F along a phase curve γ in (M,C) can be described by the function $\dot{F}: M \longrightarrow \mathbb{R}$, given by

$$\dot{F}(\gamma(t)) = e_t(F\circ\gamma) = \frac{d}{dt}(F\circ\gamma). \tag{3.3}$$

But $\frac{d}{dt}(F\circ\gamma)(t) = (d(F\circ\gamma)(e_t) = dF(d\gamma(e_t)) = dF(X_H(\gamma(t)) =$
$= X_H(\gamma(t))(F) = X_H(F)(\gamma(t)) = \{F,H\}(\gamma(t))$.

Hence, by (2.3) we get

$$\dot{F} = \{F,H\}. \tag{3.4}$$

Equation (3.4) is called the equation of evolution or the equation of motion of the observable F.

Let $((M,C),\{\cdot,\cdot\}_M,H)$ and $((N,D),\{\cdot,\cdot\}_N,F)$ be Hamiltonian dynamical systems, and let X_H and Y_F be Hamiltonian vector fields associated with the hamiltonians H and F, respectively.

Let us put

$$(3.5) \qquad Z(x,y) = \iota_{y*}X_H(x) + \iota_{x*}Y_F(y),$$

for any $(x,y) \in M\times N$, where ι_y and ι_x are natural imbeddings of M and N into M×N. Evidently, formula (3.5) defines a smooth vector field on the differential space $(M\times N, C\times D)$. By Proposition 2.14 we have the canonical almost Poisson structure (Poisson structure, if $((M,C),\{\cdot,\cdot\}_M)$ and $((N,D),\{\cdot,\cdot\}_N)$ are Poisson differential spaces) on $(M\times N, C\times D)$, defined by (2.12). We shall show that the vector field Z defined by (3.5) is a Hamiltonian vector field on $((M\times N, C\times D),\{\cdot,\cdot\})$ with the hamiltonian $G = H\circ pr_M + F\circ pr_N$ and, consequently, $((M\times N, C\times D),\{\cdot,\cdot\},G)$ is a Hamiltonian dynamical system. Indeed, from (2.12) and (3.5) we get the equalities

$$\{f, H\circ pr_M + F\circ pr_M\}(x,y) = Z_{H\circ pr_M + F\circ pr_N}(x,y)(f) =$$

$$= \{f(\cdot,y), (H\circ pr_M + F\circ pr_N)(\cdot,y)\}_M(x) +$$

$$+ \{f(x,\cdot), (H\circ pr_M + F\circ pr_N)(x,\cdot)\}_N(y) =$$

$$= \{f\circ\iota_y, (H\circ pr_M + F\circ pr_N)\circ\iota_y\}_M(x) + \{f\circ\iota_x, (H\circ pr_M + F\circ pr_N)\circ\iota_x\}_N(y) =$$

$$= X_H(x)(f\circ\iota_y) + Y_F(y)(f\circ\iota_x) = \iota_{y*}X_H(x)(f) + \iota_{x*}Y_F(y)(f) =$$

$$= (\iota_{y*}X_H(x) + \iota_{x*}Y_F(y))(f) = Z(x,y)(f),$$

for an arbitrary $f \in C\times D$.

Thus we get

Proposition 3.5 Let $((M,C),\{\cdot,\cdot\}_M,H)$ and $((N,D),\{\cdot,\cdot\}_N,F)$ be Hamiltonian dynamical systems. Then

$$((M\times N, C\times D),\{\cdot,\cdot\}, H\circ pr_M + F\circ pr_N)$$

is a Hamiltonian dynamical system, where $\{\cdot,\cdot\}$ is an almost Poisson structure (Poisson structure) on $(M\times N, C\times D)$, defined by (2.12).

Now, let α and β be observables of Hamiltonian dynamical systems $((M,C),\{\cdot,\cdot\}_M,H)$ and $((N,D),\{\cdot,\cdot\}_N,F)$, respectively. Of course, the function $\alpha\circ pr_M + \beta\circ pr_N$ is an observable of the

Hamiltonian dynamical system $((M\times N, C\times D), \{\cdot,\cdot\}, H\circ pr_M + F\circ pr_N)$.

By putting $E = \alpha\circ pr_M + \beta\circ pr_N$ and by using (3.4), we get

$$\dot{E} = \{E, H\circ pr_M + F\circ pr_N\} = \{\alpha\circ pr_M + \beta\circ pr_N, H\circ pr_M + F\circ pr_N\} =$$

$$= pr_M^*\{\alpha, H\}_M + pr_N^*\{\beta, F\}_N = \dot{\alpha}\circ pr_M + \dot{\beta}\circ pr_N.$$

Now, let E be an arbitrary observable of the Hamiltonian dynamical system $((M\times N, C\times D), \{\cdot,\cdot\}, H\circ pr_M + F\circ pr_N)$. Then of course, $E_M := E\circ i_y$ and $E_N := E\circ i_x$ are observables of $((M,C), \{\cdot,\cdot\}_M, H)$ and $((N,D), \{\cdot,\cdot\}_N, F)$, respectively, for any $(x,y) \in M\times N$. One can show that in this case, for the equation of evolution, we obtain the following formula

$$\dot{E} = \dot{E}_N\circ pr_{\tilde{M}} + \dot{E}_N\circ pr_N.$$

Let $((M,C), \{\cdot,\cdot\}, H)$ be a Hamiltonian dynamical system and let X_H be a corresponding Hamiltonian vector field.

Definition 3.3 An automorphism f of an almost Poisson differential space $((M,C), \{\cdot,\cdot\})$ is said to be a symmetry of the Hamiltonian system $((M,C), \{\cdot,\cdot\}, H)$ if

$$f_* X_H = X_H. \tag{3.6}$$

From Proposition 3.4 and (3.6) it follows

Corollary 3.6 If f is a symmetry of a Hamiltonian dynamical system $((M,C), \{\cdot,\cdot\}, H)$ then $f^*H = H + g$, where $g \in C$ is a Casimir function of the system.

It is easy to see that the set of all symmetries of a Hamiltonian dynamical system on a differential space is a group with respect to composition of mappings. However, in general, this group is not a Lie group.

Now, let $((M,C), \{\cdot,\cdot\})$ be an almost Poisson differential space and let $f_1, \dots, f_n \in C$ be such that, for any $p \in M$ and $i,j = 1,\dots,n$,

$$\det(\{f_i, f_j\}(p)) \neq 0. \tag{3.7}$$

From (3.7) it follows that $n = 2m$, $m \in \mathbb{N}$, as well as

$d_p f_i \neq 0$, or equivalently $X_{f_i}(p) \neq 0$, for any $p \in M$ and $i = 1,\dots,n$.

Let us put

$$\{f,g\}' = \{f,g\} - \sum_{ij} A^{ij}\cdot\{f,f_i\}\cdot\{f_j,g\}, \tag{3.8}$$

for any $f,g \in C$, where A^{ij} denotes the inverse matrix to the matrix $(\{f_i,f_j\})$. It is easy to observe that formula (3.8) defines an almost Poisson structure $\{\cdot,\cdot\}'$ on (M,C). Moreover, one can easily show that, for any $i = 1,\dots,n$, f_i is a Casimir function with respect to $\{\cdot,\cdot\}'$.

Similarly as it is in the classical case, we accept

Definition 3.4 A function $\phi \in C$ is said to be a constraint in an almost Poisson differential space $((M,C),\{\cdot,\cdot\})$ if $d_p\phi \neq 0$ for any $p \in \phi^{-1}(0)$. Then the equation

$$\phi(p) = 0,$$

is called the constraint equation.

Let us consider the set $\{\phi_1,\dots,\phi_n\}$ of constraints satisfying the condition

$$\det(\{\phi_i,\phi_j\})(p) \neq 0, \tag{3.9}$$

for any $p \in W := \bigcap_{i=1}^{n} \phi_i^{-1}(0) \subset M$, $i,j = 1,\dots,n$. Of course, (W,C_W) is a differential subspace of (M,C) and the number of constraints must be even.

Now, by using the functions $\bar{\phi}_i := \phi_i|W$, for $i = 1,\dots,n$, we define an almost Poisson structure on (W,C_W) by the formula

$$\{f,g\}' = \{f,g\}_W - \sum_{ij} \Phi^{ij}\cdot\{f,\bar{\phi}_i\}_W\cdot\{\bar{\phi}_j,g\}_W, \tag{3.10}$$

for any $f,g \in C_W$. Let us observe that, in this way, the constraints have been built into the almost Poisson structure $\{\cdot,\cdot\}'$ on the differential space (W,C_W). Consequently, the constraints $\phi_1,\dots,\phi_n$ do not enforce any restrictions on solutions of dynamical problems in almost Poisson differential space $((W,C_W),\{\cdot,\cdot\}')$.

REFERENCES

[1] **R. Sikorski:** An Introduction to Differential Geometry, Warsaw 1972, in Polish.

[2] **R. Sikorski:** Abstract covariant derivative, Colloq. Math. 18 (1967), 251-272.

[3] **R. Sikorski:** Differential modules, Colloq. Math. 24 (1971), 45-79.

[4] **P.A.M. Dirac:** Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University New York, (1964).

[5] **P.A.M. Dirac:** Can. Journ. of Math. 2 (1950) p.147.

[6] **P.G. Bergmann, I. Goldberg:** Phys. Rev. (1955) p.531.

[7] **R. Abraham, J.E. Marsden:** Foundations of Mechanics, Benjamin, New York, (1967).

[8] **J.E. Marsden:** The Hamiltonian Formulation of Classical Field Theory, Contemp. Math. 71 (1988), 221-235.

[9] **W. Sasin, Z. Zekanowski:** Some relations between almost symplectic, pseudoriemannian and almost product structures on differential spaces, Demonstratio. Math. 21 (1988), 1139-1152.

[10] **J. Gruszczak, M. Heller, P. Multarzyński:** A generalization of manifolds as space-time models, J. Math. Phys 29 (1988), 2576-2580.

[11] **P. Multarzyński:** An application of differential spaces to classical and quantum field theory, Doctoral thesis, Cracow, (1989) in Polish.

[12] **W. Waliszewski:** Regular and coregular mappings of differential spaces, Ann. Polon. Math., 30 (1975), 263-281.

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