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DIFFERENTIAL GROUPS AND THEIR LIE ALGEBRAS

1. Introduction

A differential group is an object which is simultaneously a differential space and a group with the group operation and its inverse which are smooth with respect to a given differential structure. It is clear that this notion is a generalization of the concept of a Lie group and that the definition of differential group depends on the definition of differential space. The first generalization of this type has been considered in [14] for differential spaces in the sense of Spallek. The concept of a differential group based on the Sikorski theory of differential spaces has been introduced and investigated in [6]. Independently it has also appeared in [5]. Main results of [6] are given (without proofs) in [8] and some of them are closely examined in [7]. However, the majority of results has not been published as yet. The main purpose of this paper is to present the further part of these results. Here we restrict our interest only to the basic definitions, examples and some elementary facts concerning the theory of differential groups and their Lie algebras. The more advanced problems concerning this theory will be presented in subsequent papers.

The work is divided into five sections. In Section 2 we recall definitions and facts from the theory of differential spaces which are used in the next sections. The notion of differential group, differential subgroup, direct and skew-symmetric products of differential groups, and Hausdorff differential group associated with a given differential group are introduced in Section 3. Here we give also several examples. Section 4 is devoted to the investigation of the tangent space and the Lie algebra of all left-invariant vector fields on a differential group. In Section 5 it is shown that for any differential group there exists exactly one covariant derivative with respect to which all left-invariant vector fields are parallel.

For all basic definitions and more detailed considerations concerning the topics of the study we refer to [5], [6], [12], [13] and also to other papers contained in this volume.

2. Preliminaries

We shall use the following notation. If $\mathcal{F}$ is a non-empty family of real functions on a set M then $\tau_{\mathcal{F}}$ denotes the weakest topology on M in which all functions of the family $\mathcal{F}$ are continuous. For any subset $A \subset M$, $\mathcal{F}_A$ is the set of all real functions β on A such that, for any point $p \in A$, there exist an open neighborhood $U \in \tau_{\mathcal{F}}$ of p and a function $\alpha \in \mathcal{F}$ such that $\beta_{|A \cap U} = \alpha_{|A \cap U}$. By $sc\mathcal{F}$ we denote the family of all real functions on M which are of the form $\omega \circ (\alpha_1, \dots, \alpha_n)$, where $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in \mathcal{F}$ and $\omega \in C^\infty(\mathbb{R}^n)$. A family C of real functions on M is said to be a *differential structure* on M if $(scC)_M = C$ (see [11] [12] or [13]). It is easy to see that, for any $\mathcal{F} \subset \mathbb{R}^M$, the family $sc\mathcal{F}_M$ is a differential structure on M (see [12], [13]). It is called a *differential structure generated by* $\mathcal{F}$.

If C is a differential structure on M then the pair (M, C) is called a *differential space* (d-space, for short). A smooth map f on a d-space (M, C) into a d-space (M, D) is denoted by

$$f\colon (M, C) \longrightarrow (N, D),$$

(for the definition of smooth maps and diffeomorphisms see [12] or [13]).

Let (N, D) be a d-space and $f\colon M \longrightarrow N$. Then the differential structure C generated by the family $f^*D := \{\alpha \circ f\colon \alpha \in D\}$ on M is called a differential structure *induced from* D *by* f. In this case $C = (f^*D)_M$. Moreover, if f maps M onto N, then $C = f^*D$. A map g: X $\longrightarrow$ M is smooth, with respect to a differential structure $\mathcal{F}$ on X, iff $f\circ g : (X, \mathcal{F}) \longrightarrow (N, D)$ (see [6], and [15]).

Let (M, C) be a d-space. For an arbitrary mapping f: M $\longrightarrow$ N, we may consider a family $D := (f^*)^{-1}(C)$ of real functions defined on N. In [15] it was shown that D is a differential structure on N. This is the greatest differential structure on N with respect to which f is a smooth mapping. A mapping g: N $\longrightarrow$ X is smooth with respect to a differential structure $\mathcal{F}$ on X iff $g\circ f: (M, C) \longrightarrow (X, \mathcal{F})$. A function $\alpha: N \longrightarrow \mathbb{R}$ is an element of D iff $f^*(\alpha) = \alpha\circ f \in C$. The differential structure $(f^*)^{-1}(C)$ is called the differential structure *coinduced from* C *by* f.

If (M, C) is a d-space and $A \subset M$, $A \neq \emptyset$, then C_A is a differential structure on A and a d-space (A, C_A) is called a *differential subspace* of (M, C). We have $C_A = (i^*C)_A$, where i is the inclusion mapping of A in M.

Let $\{(M_i, C_i)\}_{i\in I}$ be an indexed family of d-spaces. Then the differential structure $\underset{i\in I}{X} C_i$ generated on the Cartesian product $\underset{i\in I}{X} M_i$ by the family $\{f_i\circ pr_i: i \in I, f_i \in C_i\}$ (pr_j is the natural projection of $\underset{i\in I}{X} M_i$ onto M_j) is called the *differential structure of the Cartesian product* of d-spaces $\{(M_i, C_i)\}_{i\in I}$. The d-space $(\underset{i\in I}{X} M_i, \underset{i\in I}{X} C_i)$ is said to be the *Cartesian product of the family* $\{(M_i, C_i)\}_{i\in I}$.

For a finite set of indices $I = \{i_1,\dots,i_k\}$, we write $C_{i_1} x\dots x C_{i_k}$ instead of $\underset{i\in I}{X} C_i$. It is easy to verify that the topology $\tau_{\underset{i\in I}{X} C_i}$ coincides with the standard topology of the Cartesian product of topological spaces. A function f on $\underset{i\in I}{X} M_i$ is an element of $\underset{i\in I}{X} C_i$ iff, for any $p = (p_i)_{i\in I} \in \underset{i\in I}{X} M_i$, there exist a finite subset $I_o = \{i_1,\dots,i_k\} \subset I$, sets $U_{i_j} \in \tau_{C_{i_j}}$, for $j = 1,\dots,k$, and functions $\alpha_j \in C_j$, $\omega \in C^\infty(\mathbb{R}^k)$ such that $f_{|U} = \omega\circ(\alpha_1\circ pr_{i_1},\dots,\alpha_k\circ pr_{i_k})_{|U}$, where $U = \underset{i\in I}{X} U_i$ and $U_j = M_j$, for $j \in I \setminus I_o$.

Let ρ be an equivalence relation on M, where (M, C) is a d-space. By M/ρ we denote the quotient set of all equivalence classes $[p]_\rho$, for $p \in M$, and by π_ρ - the canonical projection $M \ni p \longrightarrow [p]_\rho \in M/\rho$. The differential structure coinduced from C on M/ρ by π_ρ will be denoted by $C|\rho$. A pair $(M/\rho, C|\rho)$ will be called the *quotient differential space* with respect to the equivalence relation ρ (see [9] or [15]).

Let $\equiv$ be a relation on M given as follows:

for any $p, q \in M$, $p \equiv q$ iff, for any $f \in C$, $f(p) = f(q)$.

It is easy to see that $\equiv$ is an equivalence relation. It can also be proved that the topology $\tau_{C|\equiv}$ is the standard quotient space topology on $M/\equiv$ (with respect to the topology τ_C on M, see [9] Th.1.1) and that it is a Hausdorff topology (see [13]). We call $(M/\equiv, C|\equiv)$ a *Hausdorff differential space associated with* (M, C).

By a *tangent vector* to a d-space (M, C) at a point $p \in M$ we mean any linear mapping $v: C \longrightarrow \mathbb{R}$ satisfying the so-called Leibniz condition (chain rule)

$$v(\alpha\cdot\beta) = \alpha(p)v(\beta) + v(\alpha)\beta(p),$$

for $\alpha, \beta \in C$.

It can be easily seen (see [13]) that the set T_pM of all tangent vectors to a d-space (M, C) at $p \in M$ is, in a natural way, a linear space over the field of real numbers $\mathbb{R}$. The linear space T_pM is called the *tangent space* to a d-space (M, C) at a point $p \in M$. For any $\alpha \in C$, we define the *differential* of α at $p \in M$ as a linear mapping $T_pM \longrightarrow \mathbb{R}$ given by the formula $d_p\alpha(v) := v(\alpha)$, where $v \in T_pM$.

Let us denote by TM the disjoint sum of all tangent spaces to (M, C), i.e.

$$TM := \bigcup_{p\in M} T_pM.$$

We define the *tangent mapping* or the *differential* of a smooth function $\alpha \in C$ as a mapping $d\alpha: TM \longrightarrow \mathbb{R}$ satisfying the condition $d\alpha_{|T_pM} = d_p\alpha$. By TC we denote the differential structure on TM defined as

$$TC := sc(\{\alpha\circ\pi\colon \alpha \in C\} \cup \{d\alpha\colon \alpha \in C\})_{TM},$$

where $\pi\colon TM \longrightarrow M$ is the natural projection, satisfying $\pi(v) = p$, for any $v \in T_pM$. The pair (TM, TC) is called the *tangent space* to (M, C). The triple $((TM, TC), \pi, (M, C))$ is called the *tangent bundle* of a d-space (M, C).

By a *tangent vector field* on a d-space (M, C) we mean any mapping V which associates with every point $p \in M$ a tangent vector $V(p) \in T_pM$. A tangent vector field V on (M, C) is said to be *smooth* iff $V\colon (M, C) \longrightarrow (TM, TC)$. We can also say that a (smooth) vector field is a (smooth) section of the tangent bundle. In other words (global interpretation), a smooth tangent vector field on (M, C) is any linear mapping $V\colon C \longrightarrow C$, satisfying the Leibniz condition $V(\alpha\cdot\beta) = \alpha V(\beta) + V(\alpha)\beta$. The correspondence between local and global interpretation of smooth vector fields is clear from the formula $(V(\alpha))(p) := V(p)(\alpha)$.

A d-space (M, C) is said to be of *constant differential dimension* iff

(i) $\dim T_pM = \dim T_qM$, for any $p, q \in M$,

(ii) for every tangent vector $v \in TM$, there exists a smooth tangent vector field V on (M, C) such that $v = V(p)$, where $p = \pi(v)$.

Remark. The corresponding definition formulated in the original monograph (see [13]) slightly differs from the above one. Namely, in the condition (i) Sikorski additionally assumes that $\dim T_pM = n < \infty$, for $p \in M$.

For any mapping $f\colon (M, C) \longrightarrow (N, D)$ one defines its *tangent mapping* (or *differential*) $d_pf\colon T_pM \longrightarrow T_{f(p)}N$, at $p \in M$, which is given by the formula

$$[d_pf(v)](\beta) := v(\beta\circ f),$$

for $\beta \in D$. By df we denote the mapping $TM \longrightarrow TN$, defined as $df_{|T_pM} = d_pf$, and we call it the *tangent mapping* (or *differential*) to the mapping f.

A mapping $f\colon (M, C) \longrightarrow (N, D)$ is said to be an *immersion* if, for any $p \in M$, the differential $d_p f$ is a monomorphism, and we call f an *imbedding* if f is an injective immersion.

The pair ((M, C), f) is said to be an *f-differential subspace* of a d-space (N, D) (f-d-subspace, for short) if (M, C) is a d-space and $f: (M, C) \longrightarrow (N, D)$ is an imbedding. Then, in the image $f(M) \subset N$, one can define a differential structure $\mathcal{F}$ coinduced from C by f, the so-called *image structure*. Then $f: M \longrightarrow f(M)$ is a diffeomorphism. One can easily see that the image structure $\mathcal{F}$ can, in general, be stronger than the differential structure $D_{f(M)}$ induced on f(M) from (N, D); i.e. $D_{f(M)} \subset \mathcal{F}$. If $D_{f(M)} = \mathcal{F}$, f-d-subspace will be called the *regular* f-d-subspace.

Notice that, if $A \subset M$ and $f = i_A: A \longrightarrow M$ is the inclusion mapping, the image structure $\mathcal{F}$ coincides with C_A, i.e. (A, C_A) is the regular i_A-d-subspace of (M, C).

Let us denote by $\mathfrak{X}(M)$ the linear C-module of all smooth vector fields on (M, C). The module $\mathfrak{X}(M)$ can be considered as the Lie algebra with the natural commutator

$$[V, W] := V \circ W - W \circ V, \tag{2.1}$$

for $V, W \in \mathfrak{X}(M)$.

For a diffeomorphism $f: (M, C) \longrightarrow (N, D)$ one defines the mapping $f_*: \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N)$ by the formula

$$f_*(V) := df \circ V \circ f^{-1},$$

for $V \in \mathfrak{X}(M)$.

From the above definitions we see that the tangent mapping df is well defined for any smooth mapping f but df induces the mapping f_* of vector fields only in the case when f is a diffeomorphism.

One can prove that, for any diffeomorphism $f: (M, C) \longrightarrow (N, D)$, the mapping $f_*: \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N)$ is an isomorphism of the Lie algebras. In addition, for a vector field $V \in \mathfrak{X}(M)$ and a function $\alpha \in C$ we have

$$f_*(\alpha V) = (\alpha \circ f^{-1}) f_* V. \tag{2.2}$$

Indeed, $\mathbb{R}$-linearity, inverseability of f_* and formula (2.2) follow simply from definition (2.1).

Let us check only that

$$f_*[V, W] = [f_*V, f_*W], \tag{2.3}$$

for any $V, W \in \mathfrak{X}(M)$.

$$[(f_*V)\beta](q) = [(df\circ V\circ f^{-1})\beta](q) = [df\circ V\circ f^{-1}(q)]\beta =$$

$$= df(V(f^{-1}(q)))\beta = V(f^{-1}(q))(\beta\circ f) = [V(\beta\circ f)](f^{-1}(q)) =$$

$$= (V(\beta\circ f)\circ f^{-1})(q),$$

$q \in N$, i.e.

$$f_*V(\beta) = V(\beta\circ f)\circ f^{-1}, \tag{2.4}$$

for any $\beta \in D$. From formula (2.4) we obtain a

$$f_*[V, W](\beta) = [V, W](\beta\circ f)\circ f^{-1} =$$

$$= V(W(\beta\circ f))\circ f^{-1} - W(V(\beta\circ f))\circ f^{-1} =$$

$$= V(W(\beta\circ f)\circ f^{-1}\circ f)\circ f^{-1} - W(V(\beta\circ f)\circ f^{-1}\circ f)\circ f^{-1} =$$

$$= (f_*V\circ f_*W)(\beta) - (f_*W\circ f_*V)(\beta) = [f_*V, f_*W](\beta),$$

which ends the proof of (2.3).

Let $f: (M, C) \longrightarrow (M, C)$ be a diffeomorphism. A vector field $V \in \mathfrak{X}(M)$ is said to be *f-invariant* if

$$f_*V = V. \tag{2.5}$$

From the above statements it follows that all f-invariant vector fields $V \in \mathfrak{X}(M)$ form a subalgebra of the Lie algebra $\mathfrak{X}(M)$.

By a *smooth curve* in a d-space (M, C) one means any smooth mapping

$$\gamma: (I, \mathcal{E}_I) \longrightarrow (M, C), \tag{2.6}$$

where $(I, \mathcal{E}_I)$ is a d-subspace of the Euclidean manifold $(\mathbb{R}, \mathcal{E})$ such that I my be one of the intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$, for $a, b \in [-\infty, \infty]$.

For the case of a differential manifold (M, C), we know that, for any field $V \in \mathfrak{X}(M)$ and $p \in M$, there exists an open neighborhood U of p, an open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ and a smooth mapping

$$\gamma: (-\varepsilon, \varepsilon) \times U \longrightarrow M, \tag{2.7}$$

such that, for any $q \in U$, the curve

(2.8) $$\gamma_q : (-\varepsilon, \varepsilon) \ni t \longrightarrow \gamma_q(t) := \gamma(t, q) \in M,$$

is the unique *integral curve* for the field V defined on $(-\varepsilon, \varepsilon)$ and satisfying the initial condition

(2.9) $$\gamma_q(0) = q.$$

Unfortunately, the analogous fact is no longer valid for the general case of differential spaces. For example, it fails to be true if we consider the differential space (also d-group) $(\mathbb{Q}, \mathcal{E}_{\mathbb{Q}})$, where $\mathbb{Q}$ denotes the set of all rational numbers. One can easily see that in this case no non-constant smooth curve can exist because there is no non-constant continuous mapping from $\mathbb{R}$ into $\mathbb{Q}$. The unique character of integral curves, which existed for vector fields on differentiable manifolds, is often lost in the general case of differential spaces. In general, more than one integral curve for a vector field on a differential space is possible. However, for the case of locally finitely generated differential spaces (manifolds, in particular), there is always no more than one integral curve of every vector field .

3. Differential groups

Definition 3.1 A pair $(G, \mathcal{G})$ is said to be a *differential group* iff

1° G is a group;

2° $(|G|, \mathcal{G})$ is a d-space, where $|G|$ denotes the set of elements of G;

3° a map σ: $(G \times G, \mathcal{G} \times \mathcal{G}) \longrightarrow (G, \mathcal{G})$, defined by the formula

(3.1) $$\sigma(g, h) := gh^{-1},$$

is a smooth map.

It is evident that a differential group is automatically a topological group (with the topology $\tau_{\mathcal{G}}$ in G) because the smoothness of the mapping σ implies its continuity (see [5], [6] or [7]). In [7] it is proved that the group multiplication and the inverse mapping in G are smooth.

Example 3.1 Let G be an arbitrary group. If $\mathcal{F}_o$ denotes the differential structure of all constant functions on G then (G, $\mathcal{F}_o$) is a differential group. Similarly, if $\mathbb{R}^G$ is the differential structure of all real-valued functions on G then (G, $\mathbb{R}^G$) is also a differential group. In the last case the topology $\tau_{\mathbb{R}^G}$ is the discrete topology on G.

Proposition 3.1 Let H be a group, (G, $\mathcal{F}$) - a differential group and ϕ - (an algebraic) homomorphism on H into G. Then (H, $\phi^*(\mathcal{F})_H$) is a differential group and ϕ: (H, $\phi^*(\mathcal{F})_H$) $\longrightarrow$ (G, $\mathcal{F}$) is a smooth map.

Proof. The smoothness of ϕ follows directly from the definition of the differential structure $\phi^*(\mathcal{F})_H$ (see Section 2). This implies that the map

$$H\times H \ni (g, h) \longrightarrow \eta(g, h) := \phi(g)\phi(h)^{-1} \in G$$

is smooth with respect to the differential structures $\phi^*(\mathcal{F})_H \times \phi^*(\mathcal{F})_H$ and $\mathcal{F}$, respectively.

Let us consider the map

$$H\times H \ni (g, h) \longrightarrow \sigma_H(g, h) := gh^{-1} \in H.$$

Since $\phi\circ\sigma_H = \eta$ is smooth on G×G, σ_H is also smooth (see Section 2, for the properties of the differential structure induced from $\mathcal{F}$ by ϕ).

Example 3.2 If H is a Lie group then (H, $C^\infty(H)$) is a differential group. If G is an arbitrary subgroup of H, ϕ is the natural embedding of G into H and $\mathcal{F} := \phi^*((C^\infty(H))_G = C^\infty(H)_G$ then (G, $\mathcal{F}$) is a differential group.

Example 3.3 Let θ: G $\longrightarrow$ Gl(n, $\mathbb{R}$) be an n-dimensional matrix representation of a group G, n $\in \mathbb{N}$. By Proposition 3.1 the pair (G, $\theta^*(C^\infty(Gl(n, \mathbb{R})))_G$) is a differential group. It can be easily seen that the differential structure $\theta^*(C^\infty(Gl(n, \mathbb{R})))_G$ is generated by the family $\{\theta_{ij}\}_{1\le i,j\le n}$ of all matrix elements of the representation θ.

Example 3.4 Let G be a locally compact, connected topological group. Let U be an arbitrary neighborhood of the identity element of G. Then there exists a normal subgroup N of G such that $N \subset U$ and G/N is a Lie group (see [2], II, §10).

Denote by ϕ the canonical map on G onto G/N. From Proposition 3.1 it follows that $(G, \phi^*(C^\infty(G/N))_G)$ is a differential group.

Theorem 3.1 Let $\mathcal{F}$ be a family of real-valued functions defined on a group G. Let $\mathcal{G} := sc\mathcal{F}_G$ be a differential structure generated by $\mathcal{F}$ on G. The pair $(G, \mathcal{G})$ is a differential group iff the following condition is satisfied

(GS) for any $f \in \mathcal{F}$ and any $(g, h) \in G \times G$, there exist a neighborhood $U \in \tau_{\mathcal{F}}$ of g, a neighborhood $V \in \tau_{\mathcal{F}}$ of h, mappings $\alpha \in \mathcal{F}^r$, $\beta \in \mathcal{F}^s$ and a function $\omega \in C^\infty(\mathbb{R}^{r+s})$ such that for each $(g', h') \in U \times V$

$$f(g'h'^{-1}) = \omega(\alpha(g'), \beta(h')).$$

Proof. Suppose that $(G, \mathcal{G})$ is a differential group. Since the map σ is smooth (see (3.1)) we obtain that, for any $f \in \mathcal{F} \subset \mathcal{G}$, the map $f\circ\sigma \in \mathcal{G} \times \mathcal{G}$, and the condition (GS) follows directly from the definition of $\mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$ (see Sec. 2).

Suppose now that $\mathcal{F}$ satisfies the condition (GS). Since any function of G is locally a function from $sc\mathcal{F}$, we obtain that $\mathcal{G}$ also fulfills (GS). Hence σ is a smooth map, and $(G, \mathcal{G})$ is a differential group.

Example 3.5 Let $G = \mathrm{Diff}(M)_p$ be a group of all smooth diffeomorphisms of some differentiable manifold M which leave a point $p \in M$ fixed. Let $\phi = (\phi^1,\ldots,\phi^n): U \longrightarrow \mathbb{R}^n$ be a smooth chart defined on a neighborhood U of p such that $\phi(p) = 0$. For any multiindex $(i_1,\ldots,i_n) \in \mathbb{N}^n$, we define a function $f^k_{i_1\ldots i_n}: U \longrightarrow \mathbb{R}$ in the following way

$$f^k_{i_1\ldots i_n}(g) := \frac{\partial^i g^k}{\partial x^{i_1}\ldots\partial x^{i_n}}(0),$$

where $i = i_1 + \ldots + i_n$ and $g^k = \phi^k\circ g\circ\phi^{-1}$. Denote by $\mathcal{G}$ the differential structure on G generated by the family of functions $\mathcal{F}_\phi := \{f^k_{i_1\ldots i_n} : 1 \le k \le n,\ (i_1,\ldots,i_n) \in \mathbb{N}^n\}$. By the differentiation rule of superposition of maps and differentiation of the inverse map, it follows that the family $\mathcal{F}_{\phi}$ satisfies the condition (GS) from Theorem 3.1 (in this case $U = V = G$ and ω is a rational function). Then $(G, \mathcal{G})$ is a differential group. It is easy to see that the differential structure $\mathcal{G}$ does not depend on the choice of the chart ϕ.

Analogously, we can define the differential group structure on the group of germs of diffeomorphisms preserving the point p, on the group of analytic diffeomorphisms or on the group of all germs of local analytic diffeomorphisms. In the last two cases the topology $\tau_{\mathcal{G}}$ on G is the Hausdorff topology.

Example 3.6 Let $f_n, g_n : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f_n(x) = \sin \frac{x}{n}, \quad g_n(x) = \cos \frac{x}{n}, \quad n \in \mathbb{N},\ x \in \mathbb{R}.$$

We have

$$f_n(x-y) = f_n(x)g_n(y) - f_n(y)g_n(x),$$

$$g_n(x-y) = g_n(x)g_n(y) + f_n(x)f_n(y),$$

where $x, y \in \mathbb{R}$. Let $\mathcal{G}$ be a differential structure generated on $\mathbb{R}$ by the family $\mathcal{F} := \{f_n, g_n\}_{n\in\mathbb{N}}$. By Theorem 3.1 $((\mathbb{R}, +), \mathcal{G})$ is a differential group. It can be proved (see [3] or [6]) that the topological group $((\mathbb{R}, +), \tau_{\mathcal{G}})$ is not complete.

Let $(G, \mathcal{G})$ be a differential group and H be any subgroup of G. It is an immediate corollary from Theorem 3.1 that the differential structure $\mathcal{G}$ satisfies the condition (GS). This implies that the family $\mathcal{F} := \{f_{|H}: f \in \mathcal{G}\}$ also satisfies (GS) and consequently the group H with the differential structure $\mathcal{H} = \mathcal{F}_H$ generated by $\mathcal{F}$ on H is a differential group. It is called a *differential subgroup* of the differential group $(G, \mathcal{G})$. Let us notice that $\mathcal{H} = \phi^*(\mathcal{G})_G$, where ϕ is the natural embedding of H into G (see Prop. 3.1 and Ex. 3.2). Thus, the category of differential groups turns out to be closed with respect to the operation of taking a subgroup, which is not true in the category of Lie groups.

Suppose now that $\{(G_i, \mathcal{G}_i)\}_{i\in I}$ is a family of differential groups, where I is an arbitrary set of indices. Let, for any

$j \in I$, $pr_j : \underset{i \in I}{X} G_i \longrightarrow G_j$ be the natural projection of the Cartesian product $\underset{i \in I}{X} G_i$ onto G_j. Since, for any $j \in I$, $\mathcal{G}_j$ satisfies (GS), the family $\mathcal{F} = \{f_j \circ pr_j : f_j \in \mathcal{G}_j,\ j \in I\}$ of functions on the product $\underset{i \in I}{X} G_i$ satisfies the condition (GS). Since $\mathcal{F}$ generates the differential structure $\underset{i \in I}{X} \mathcal{G}_i$ on $\underset{i \in I}{X} G_i$, $(\underset{i \in I}{X} G_i, \underset{i \in I}{X} \mathcal{G}_i)$ is a differential group. It is called the *direct product* of the family $\{(G_i, \mathcal{G}_i)\}_{i \in I}$ of differential groups.

Example 3.7 Let G be a group and $\{\theta^i\}_{i \in I}$ be an arbitrary family of matrix representations of G. For any $i \in I$, the map $\theta^i : G \longrightarrow Gl(n_i, \mathbb{R})$ is a homomorphism of groups. Define the map $\theta : G \longrightarrow \underset{i \in I}{X} Gl(n_i, \mathbb{R})$ in the following way

$$\theta(g) := (\theta^i(g))_{i \in I} \in \underset{i \in I}{X} Gl(n_i, \mathbb{R}), \quad g \in G.$$

It is obvious that θ is a homomorphism of G into the direct product $\underset{i \in I}{X} Gl(n_i, \mathbb{R})$. By Proposition 3.1 the pair $(G, \mathcal{G})$, where $\mathcal{G} := \theta^*[\underset{i \in I}{X} C^\infty(Gl(n_i, \mathbb{R}))]_G$, is a differential group. It is also easy to see that the differential structure $\mathcal{G}$ is generated by the family $\{\theta^i_{k,l}\}_{i \in I, l \le k, l \le n_i}$ of all matrix elements of all representations θ^i, $i \in I$.

Especially interesting is the case when G is a compact topological group and $\{\theta^i\}_{i \in I}$ is the family of all irreducible representations of G. From the Weyl approximation theorem (see [4] IV, 884) it follows that any continuous function f on G is a limit of an uniformly convergent sequence of linear combinations of functions of the family $\{\theta^i_{k,l}\}_{i \in I, l \le k, l \le n_i}$. Now, by the Urysohn lemma it follows that the topology $\tau_{\mathcal{G}}$ coincides with the initial topology.

Proposition 3.2 Let $(G, \mathcal{G})$ and $(H, \mathcal{H})$ be differential groups and denote by Aut(H) the set of all smooth automorphisms of the group H. Let $\omega : G \ni g \longmapsto \omega_g \in Aut(H)$ be such a homomorphism of groups that the mapping

$$\Omega : G\times H \ni (g, h) \longmapsto \Omega(g, h) := \omega_g(h) \in H,$$

is smooth. In G × H we define the group multiplication in the following way

$$(g, h)\bullet(g', h') := (gg', h\omega_g(h')).$$

The triple $(G\times H, \bullet, \mathcal{G}\times\mathcal{H})$ is a differential group, called *Ω-skew-product* of G and H.

Proof. We shall show the smoothness of the mapping

$$\sigma: (G\times H)\times(G\times H) \longrightarrow G\times H,$$

where $\sigma((g,h),(g',h')) := (g,h)\bullet(g'^{-1},\omega_{g'^{-1}}(h'^{-1}))$.

We see that

$$\sigma((g,h),(g',h')) = (gg'^{-1},h\omega_g(\omega_{g'^{-1}}(h'^{-1}))) =$$

$$= (gg'^{-1},h\omega_{gg'^{-1}}(h'^{-1})).$$

It is enough to show the smoothness of the components σ_1, σ_2 of $\sigma \equiv (\sigma_1, \sigma_2)$. Indeed, these components are smooth since they are composed out of smooth mappings

$$\sigma_1 = \sigma_G\circ(\pi_1\circ pr_1, \pi_1\circ pr_2),$$

$$\sigma_2 = \mu_H\circ [\pi_2\circ pr_1, \Omega\circ(\sigma_G\circ(\pi_1\circ pr_1, \pi_1\circ pr_2), inv_H\circ\pi_2\circ pr_2),$$

where μ_H is the (smooth) multiplication in H and the mappings π_1, π_2, pr_1, pr_2 are the natural projections.

Notice that the concept of a direct product of two groups is evidently a special case of the concept of a skew-product defined above.

Let e be the identity element of the topological group G. Denote by N the closure of the set {e}. It is known (see [1], III, §2) that N is a normal subgroup of G and that the quotient topology on G/N is the Hausdorff topology. In the theory of topological groups the quotient group G/N is called the Hausdorff topological group associated with G. We are going to introduce an analogous notion for differential groups.

Proposition 3.3 If $(G, \mathcal{F})$ is a differential group and N is the closure of $\{e\}$, where e is the neutral element of G, then

$$(3.2) \qquad gN = \{x \in G: \forall f \in \mathcal{F} \ f(x) = f(g)\},$$

for any $g \in G$.

Proof. Let $N_1 := \bigcap_{f \in \mathcal{F}} f^{-1}(\{f(e)\})$. Since, for any $f \in \mathcal{F}$, the set $f^{-1}(\{f(g)\})$ is closed in G, we obtain that N_1 is closed, and consequently $N \subset N_1$.

Suppose now that there exists $g \in N_1 \setminus N$. Since $G \setminus N \in \tau_{\mathcal{F}}$ there exist $k \in \mathbb{N}$, $\alpha \in \mathcal{F}^k$ and an open non-empty set $D \subset \mathbb{R}^k$ such that $g \in \alpha^{-1}(D) \subset G \setminus N$. On the other hand, $g \in N_1$, which implies $\alpha(e) = \alpha(g) \in D$. Then $e \in \alpha^{-1}(D) \subset G \setminus \{e\}$ which is an evident contradiction. Consequently, $N_1 \setminus N = \emptyset$, and (3.2) is proved for $g = e$.

If g is an arbitrary element of G, then $x \in gN$ iff $g^{-1}x \in N$. By the first part of the proof, this is equivalent to the following condition: if $f \in \mathcal{F}$ then

$$f(x) = (f \circ L_g)(g^{-1}x) = (f \circ L_g)(e) = f(g),$$

where L_g denotes the left translation in G (by Proposition 4.1 L_g is a diffeomorphism on G onto G). This completes the proof.

Corollary 3.1 If "$\equiv$" is the equivalence relation on the group G, defined in Sec. 2, then $G/N = G/\equiv$.

Theorem 3.2 Let $(G, \mathcal{F})$ be a differential group. Then $(G/\equiv, \mathcal{F}|\equiv)$, where $\mathcal{F}|\equiv$ is defined in Sec. 2, is a differential group.

Proof. Denote by ϕ the canonical map on G onto $G/\equiv$ and choose the map $R: G/\equiv \longrightarrow G$ satisfying the following condition $\phi \circ R = id_{G/\equiv}$. By the definition of $\mathcal{F}|\equiv$, any function $f \in \mathcal{F}$ is of the form $f = \overline{f} \circ \phi$, where $\overline{f} \in \mathcal{F}|\equiv$.

Since the family $\{f^{-1}(\theta) = \phi^{-1}(\overline{f}^{-1}(\theta) \subset G: \theta$ is an open interval in $\mathbb{R}\}$ forms a subbasis of the topology $\tau_{\mathcal{F}}$, any $U \in \tau_{\mathcal{F}}$ has the following property: if $g \in U$ then $gN \subset U$, where N is the closure of $\{e\}$. Hence, for any $U \in \tau_{\mathcal{F}}$, $R(\phi(U)) \subset U$.

Suppose now that $\overline{f} \in \mathcal{F}|\equiv$ and put $f := \overline{f} \circ \phi$. By Theorem 3.1, for any $x, y \in G/\equiv$ there exist neighborhoods U and V of $R(x)$

and R(y), respectively, mappings $\alpha \in \mathcal{G}^r$, $\beta \in \mathcal{G}^s$ and a function $\omega \in C^\infty(\mathbb{R}^{r+s})$ such that, for any $(g', h') \in U \times V$,

$$f(g'h'^{-1}) = \omega(\alpha(g'), \beta(h')).$$

Then, for each $(x', y') \in \phi(U) \times \phi(V)$,

$$\bar{f}(x'y'^{-1}) = \bar{f}(\phi(R(x'))\phi(R(y'))^{-1}) = \bar{f}(\phi(R(x')R(y')^{-1})) =$$
$$= f(R(x')R(y')^{-1}) = \omega\circ(\alpha(R(x')), \beta(R(y'))).$$

Since $\alpha = \bar{\alpha}\circ\phi$ and $\beta = \bar{\beta}\circ\phi$, where $\bar{\alpha} \in (\mathcal{G}|\equiv)^r$ and $\bar{\beta} = (\mathcal{G}|\equiv)^s$, we obtain

$$\bar{f}(x'y'^{-1}) = \omega(\bar{\alpha}(x'), \bar{\beta}(y')).$$

Consequently $\mathcal{G}|\equiv$ satisfies the condition (GS), which implies that $(G/\equiv, \mathcal{G}|\equiv)$ is a differential group.

Obviously, the topology $\tau_{\mathcal{G}|\equiv}$ is the Hausdorff topology. We shall call $(G/\equiv, \mathcal{G}|\equiv)$ a *Hausdorff differential group associated with* $(G, \mathcal{G})$.

4. The tangent space and the Lie algebra of a differential group

For any $g \in G$, by the symbols L_g and R_g we shall denote the so-called *left* and *right translations* in the group G, which are defined as mappings $G \longrightarrow G$ such that

$$L_g(h) := gh,$$

$$R_g(h) := hg,$$

and the automorphism $ad_g(h) := ghg^{-1}$. It is obvious that

$$ad_g \equiv L_g\circ R_{g^{-1}}.$$

Proposition 4.1 If $(G, \mathcal{G})$ is a differential group then, for any $g \in G$, the translations L_g, R_g and the automorphism ad_g are diffeomorphisms.

The proof can be found in [7].

From this we have the following

Corollary 4.1 Let $(G, \mathcal{G})$ be a differential group. Then, for any $g \in G$,

$$\dim T_gG = \dim T_eG,$$

where e is the neutral element of G.

Proof. Since L_g and R_g are diffeomorphisms, the tangent mapping (differential) d_eL_g: $T_eG \longrightarrow T_gG$ proves to be an isomorphism of linear spaces.

Lemma 4.1 For any $\eta \in T_eG$ and $\alpha \in \mathcal{G}$, the function $f: G \longrightarrow \mathbb{R}$, given by the formula

$$f(g) := [d_eL_g(\eta)](\alpha) = \eta(\alpha \circ L_g),$$

is a smooth function, i.e. $f \in \mathcal{G}$.

Before we prove this lemma let us make one point clear.

Remark. This lemma does not seem to be obvious because it does not follow from the assumption that $\alpha \in \mathcal{G}$ and L_g is a diffeomorphism $G \longrightarrow G$. Indeed, if we consider a differential space $(\mathbb{R}, C)$, with the differential structure C generated by the set $C_o := \{f_a: \mathbb{R} \longrightarrow \mathbb{R},\ a \in \mathbb{R}\}$, where $f_a(x) := |x-a|$, we see that $L_x: \mathbb{R} \longrightarrow \mathbb{R}$, defined as $L_x(y) := x+y$, is a diffeomorphism in $(\mathbb{R}, C)$. Nonetheless, if we take a non-zero tangent vector $\eta \in T_o\mathbb{R}$, we see that the function f defined as $f(x) := \eta(f_a \circ L_x)$ is not a smooth function. The translations L_x and R_x are diffeomorphisms if the mapping (3.1) is smooth with respect to each of its arguments separately. However it is not enough for f to be smooth.

By Theorem 3.1, for any $\alpha \in \mathcal{G}$,

$$\alpha \circ \sigma \overset{*}{=} \omega \circ (\alpha_1 \circ \pi_1,\ \alpha_2 \circ \pi_2),$$

for some $\omega \in C^\infty(\mathbb{R}^{2n})$, $\alpha_1, \alpha_2 \in \mathcal{G}^n$, $n \in \mathbb{N}$. Let $j_G: G \longrightarrow G\mathrm{x}G$, $j_g(h) := (g, h)$. It is well known that, for any $g \in G$, j_g is a smooth mapping [13]. The translation L_g can be written in the form

$$L_g = \sigma \circ j_g \circ \sigma_e.$$

From the smoothness of L_g we see that, for any $\alpha \in \mathcal{G}$,

$$\alpha \circ L_g = \alpha \circ \sigma \circ j_g \circ \sigma_e \in \mathcal{G}. \tag{4.1}$$

Let $\eta \in T_eG$, and $f(g) := \eta(\alpha \circ L_g)$. We shall show that $f \in \mathcal{G}$. From (4.1) we have

$$\alpha \circ L_g = \alpha \circ \sigma \circ j_g \circ \sigma_e = \omega \circ (\alpha_1 \circ \pi_1 \alpha_2 \circ \pi_2) \circ j_g \circ \sigma_e.$$

If we denote $\mu_g := (\alpha_1 \circ \pi_1,\ \alpha_2 \circ \pi_2) \circ j_g \circ \sigma_e$, we can write f in the form

$$f(g) = \eta(\alpha \circ L_g) = \eta(\omega \circ \mu_g).$$

Therefore

$$f(g) = \omega_1'(\mu_g(e)) \cdot \eta(\alpha_1 \circ \pi_1 \circ j_g \circ \sigma_e) + \omega_2'(\mu_g(e)) \cdot \eta(\alpha_2 \circ \pi_2 \circ j_g \circ \sigma_e) =$$

$$= \omega_2'(\mu_g(e)) \cdot \eta(\alpha_2 \circ \sigma_e).$$

Since $\omega \in C^\infty(\mathbb{R}^{2n})$, we obtain that also $\omega_2' \in C^\infty(\mathbb{R}^{2n})$. Therefore

$$f(g) = \eta(\alpha \circ \sigma_e)\omega_2'(\alpha_1(g), \alpha_2(e)),$$

or

$$f \overset{*}{=} \eta(\alpha \circ \sigma_e)\omega_2' \circ (\alpha_1, \alpha_2(e)).$$

Let us denote by Ω the smooth mapping $\mathbb{R} \longrightarrow \mathbb{R}$, defined as

$$\Omega(x) := \omega'(x, \alpha\ (e)).$$

Thus,

$$f \overset{*}{=} a \cdot \Omega \circ \alpha_1,$$

where $a := \eta(\alpha \circ \sigma_e) \in \mathbb{R}$, $\Omega \in C^\infty(\mathbb{R}^n)$, $\alpha_1 \in \mathcal{G}$. This ends the proof of smoothness of the function f.

From the above lemma we have the following

Corollary 4.2 For any $\eta \in T_eG$, the tangent vector field V_η, given by

(4.2) $$V(g) := d_eL_g(\eta),$$

is a smooth tangent vector field.

Proof. Indeed, for any $\alpha \in \mathcal{G}$, $(V_\eta\alpha)(g) = V_\eta(g)\alpha = f(g)$, where f is a function from the proof of Lemma 4.1. Hence $V_\eta\alpha \in \mathcal{G}$, which means that V is smooth.

As a consequence of the above statements, we obtain the following important

Theorem 4.1 Every differential group is of constant differential dimension.

Proposition 4.2 For any $\eta \in T_eG$, the (smooth) tangent vector field V_η, given by (4.2) is left-invariant, i.e. it is invariant with respect to all left translations L_g, for $g \in G$,

$$L_{g*}V_\eta = V_\eta.$$

The proof of this fact is simple.

On the strength of formula (2.5), the module of all left invariant vector fields is a subalgebra of the Lie algebra $\mathfrak{X}(M)$.

Now, we are ready to formulate the following

Definition 4.1 Let $(G, \mathcal{G})$ be a differential group. The vector space $\mathcal{L}(G)$, over $\mathbb{R}$, of all left-invariant and smooth vector fields on G, together with the Lie multiplication $[\cdot,\cdot]$ defined by (2.3), is said to be the *Lie algebra* of a differential group $(G, \mathcal{G})$; i.e. the Lie algebra of $(G, \mathcal{G})$ is the pair $(\mathcal{L}(G). [\cdot,\cdot])$.

Proposition 4.3 For any differential group $(G, \mathcal{G})$, the linear spaces T_eG and $\mathcal{L}(G)$ are isomorphic.

Proof. By Corollary 4.1, the map

$\jmath_G\colon T_eG \ni \eta \longmapsto V_\eta \in \mathcal{L}(G)$, where $V_\eta(g) = d_eL_g(\eta)$, for $\eta \in T_eG$, $g \in G$, is an isomorphism of these linear spaces.

Proposition 4.4 Let $(G, \mathcal{G})$ and $(H, \mathcal{H})$ be d-groups. For any smooth homomorphism $f\colon G \longrightarrow H$ (f is said to be a *homomorphism of d-groups*), the mapping $\mathcal{L}(f)\colon \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$, defined by

$$\mathcal{L}(f) := \jmath_H \circ d_e f \circ \jmath_G^{-1}, \tag{4.3}$$

is a homomorphism of Lie algebras. Moreover, for any smooth homomorphisms $f\colon G \longrightarrow H$ and $g\colon H \longrightarrow Z$,

$$\begin{aligned} \mathcal{L}(g\circ f) &= \mathcal{L}(g)\circ\mathcal{L}(f) \\ \mathcal{L}(id_G) &= id_{\mathcal{L}(G)}. \end{aligned} \tag{4.4}$$

Proof. The linearity of $\mathcal{L}(f)$ is obvious. We shall show only that

$$\mathcal{L}(f)[V, W] = [\mathcal{L}(f)V, \mathcal{L}(f)W], \tag{4.5}$$

for $V, W \in \mathcal{L}(G)$.

For any element $z \in G$, we see that

$$\begin{aligned} L_h^H\circ f\circ L_g^G(z) &= L_h^H\circ f(gz) = hf(gz) = hf(g)f(z) = \\ &= L_{hf(g)}^H f(z) = (L_{hf(g)}^H\circ f)(z), \end{aligned}$$

which means that

$$(4.6) \qquad L^H_{hf(g)} \circ f = L^H_h \circ f \circ L^G_g.$$

On the other hand, we see that for any $X \in \mathcal{L}(G)$, $g \in G$ and $\beta \in \mathcal{H}$

$$X(\beta \circ L^H_h \circ f)(g) = X(g)\beta \circ L^H_h \circ f = d_e L^G_g X(e)\beta \circ L^H_h \circ f =$$
$$= X(e)\left(\beta \circ L^H_h \circ f \circ L^G_g\right) = X(e)\left(\beta \circ L^H_{hf(g)} \circ f\right) =$$
$$= \left(d_e(L^H_{hf(g)} \circ f)X(e)\right)\beta = \left((\mathcal{L}(f)X)(hf(g))\right)\beta =$$
$$= \left([\mathcal{L}(f)X]\beta\right)(hf(g)) = \left((\mathcal{L}(f)X)\beta \circ L^H_h \circ f\right)(g),$$

and consequently

$$(4.7) \qquad X(\beta \circ L^H_h \circ f) = \left([\mathcal{L}(f)X]\beta\right) \circ L^H_h \circ f.$$

In turn, for $h \in H$ and $X \in \mathcal{L}(G)$, we obtain

$$(4.8) \qquad (\mathcal{L}(f)X)(h) = d_\varepsilon L^H_h(d_e f\ X(e)) = d_e(L^H_h \circ f)X(e),$$

(see (4.3)), where $e \in G$ and $\varepsilon = f(e) \in H$ are neutral elements in G and H.

By replacing X with [V, W] in formula (4.8) and taking into account formulae (4.6) and (4.7), we obtain

$$\left(\mathcal{L}(f)[V, W]\beta\right)(h) = \left(\{\mathcal{L}(f)[V, W]\}(h)\right)\beta =$$
$$= \left(d_e(L^H_h \circ f)([V, W](e))\right)\beta = [V, W](e)\{\beta \circ L^H_h \circ f\} =$$
$$= V(e)\left(W(\beta \circ L^H_h \circ f)\right) - W(e)\left(V(\beta \circ L^H_h \circ f)\right) =$$
$$= V(e)\left([\mathcal{L}(f)W]\beta \circ L^H_h \circ f\right) - W(e)\left([\mathcal{L}(f)V]\beta \circ L^H_h \circ f\right) =$$
$$= \left(d_e(L^H_h \circ f)V(e)\right)[\mathcal{L}(f)W]\beta - \left(d_e(L^H_h \circ f)W(e)\right)[\mathcal{L}(f)V]\beta =$$
$$= \left([\mathcal{L}(f)V](k)\right)[\mathcal{L}(f)W]\beta - \left([\mathcal{L}(f)W](k)\right)[\mathcal{L}(f)V]\beta =$$
$$= \left([\mathcal{L}(f)V] \circ [\mathcal{L}(f)W]\beta\right)(h) - \left([\mathcal{L}(f)W] \circ [\mathcal{L}(f)V]\beta\right)(h) =$$
$$= \left([\mathcal{L}(f)V \circ \mathcal{L}(f)W - \mathcal{L}(f)W \circ \mathcal{L}(f)V]\beta\right)(h) =$$
$$= \left([\mathcal{L}(f)V, \mathcal{L}(f)W]\beta\right)(h),$$

which ends the proof of formula (4.8).

Now, let $f\colon G \longrightarrow H$ and $g\colon H \longrightarrow Z$ be smooth homomorphisms.

$$\mathcal{L}(g\circ f) = j_Z\circ d_e(g\circ f)\circ j_G^{-1} = j_Z\circ\{d_{f(e)}g \circ d_e f\}\circ j_G^{-1} =$$

$$= j_Z\circ d_{f(e)}g\circ j_H^{-1}\circ j_H\circ d_e f\circ j_G^{-1} = \mathcal{L}(g)\circ\mathcal{L}(f).$$

And finally we prove that

$$\mathcal{L}(id_G) = j_g\circ d_e id_G\circ j_G^{-1} = j_G\circ id_{T_eG}\circ j_G^{-1} = j_G\circ j_G^{-1} = id_{\mathcal{L}(G)}.$$

As an immediate consequence of the above proposition we have the following

Corollary 4.3 $\mathcal{L}$ is a covariant functor from the category of differential groups into the category of Lie algebras. With every d-group G the functor $\mathcal{L}$ associates its Lie algebra $\mathcal{L}(G)$, and with every homomorphism $f\colon G \longrightarrow H$ of d-groups the functor $\mathcal{L}$ associates the homomorphism $\mathcal{L}(f)\colon \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ of Lie algebras.

Let H be a d-subgroup of a d-group G. On the strength of the above statements we can identify the Lie algebra $\mathcal{L}(H)$ with the subalgebra of $\mathcal{L}(G)$; the identification is done with the help of the monomorphism $\mathcal{L}(\iota_H)$, where $\iota_H\colon H \longrightarrow G$ denotes the inclusion mapping.

Comment. For a Lie group G and $X \in \mathfrak{X}(G)$, exactly one (maximal) integral curve of X can be drawn through a chosen point $g \in G$, i.e. the equation $\frac{d}{dt}\,\phi = X(\phi(t)$ has exactly one solution satisfying the condition $\phi(0) = g$. This unique solution is denoted by $\Phi_{X,g}$ and it is defined on an interval $(-\varepsilon, \varepsilon)$, where ε depends on g. If X is a smooth left-invariant field, the integral curve $\phi_{X,g}$ is well defined on the entire line $\mathbb{R}$. The corresponding statements are, in general, no longer valid for differential groups since it can occure that there is no non-constant and smooth curve $\gamma\colon I \longrightarrow G$.

Example. Let us consider the d-group $(\mathbb{Q}, \mathcal{E}_{\mathbb{Q}})$, where $\mathbb{Q}$ is the set of rational numbers. It is very well known that there is no continuous and non-constant mapping $\mathbb{R} \longrightarrow \mathbb{Q}$. Since every smooth mapping is continuous, there is no smooth and non-constant curve in $\mathbb{Q}$.

For Lie groups the exponential mapping is defined with the help of the so-called one parameter group $\phi_{X,e}: \mathbb{R} \longrightarrow G$. The formal translation of the definition, in general case of d-groups, is impossible. Therefore a suitable definition for the case of d-groups must have quite a different form.

Lie groups are locally diffeomorphic to their Lie algebras while this is, in general, not true for d-groups.

5. The natural covariant derivative in a differential group

Before we prove the main statement of this section we remind the following

Lemma 5.1 Let (M, C) be a d-space of constant differential dimension and $\mathcal{W}$ a differential C-module [12]. Let $V_1,\dots,V_m$ be a vector basis on M, $W_1,\dots,W_n$ a C-basis of the C-module $\mathcal{W}$, and $\Gamma^k_{ij} \in C$ smooth functions on M. Then there exists exactly one covariant derivative ∇ in $\mathcal{W}$, such that Γ^k_{ij} are its coordinates with respect to the basis $V_1,\dots,V_m$ and $W_1,\dots,W_n$, i.e.

$$\nabla_{V_i} W_j = \Gamma^k_{ij} W_k. \tag{5.1}$$

For the proof see book [13].

On the strength of this lemma one can prove the following

Proposition 5.1 For any differential group $(G, \mathcal{G})$ of finite dimension, $\dim T_eG < \infty$, there exists exactly one covariant derivative ∇ on G, such that

$$\nabla_v j_G(\eta) = 0, \tag{5.2}$$

for any $v \in TG$ and $\eta \in T_eG$.

Proof. Let $\mathcal{B}_e := \{\eta_t\}_{t\in T}$ be a linear basis of the tangent space T_eG. Since, for any $g \in G$, the translation $L_g: G \longrightarrow G$ is a diffeomorphism, the set $\mathcal{B}_g := L_{g*}(\mathcal{B}_e) = \{L_{g*}\eta_t: t \in T\}$ forms a linear basis for the tangent space T_gG. Hence the vector fields W_t, $t \in T$, given by

$$W_t(g) := L_{g*}\eta_t, \tag{5.3}$$

$g \in G$, form a vector basis on the d-space $(G, \mathcal{G})$. From the above lemma we learn that there exists exactly one covariant derivative ∇ on G, the coordinates of which (the so-called Christoffel symbols) with respect to the vector basis $\{W_t\}_{t\in T}$ on G are equal to zero.

This unique covariant derivative defined by (5.2) is said to be the *natural covariant derivative* on a differential group $(G, \mathcal{G})$.

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