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WHITNEY TOPOLOGY AND STRUCTURAL STABILITY OF SMOOTH MAPPINGS IN DIFFERENTIAL SPACES

The main part of this work is devoted to defining the differential structure on the space of k-jets in the category of differential spaces in the sense of Sikorski [21]. The proof of the consistency of the differential structure introduced here with the Ehresmann differential structure [4], in the case when all differential spaces are differentiable manifolds, is given. Then, with the help of the differential structure on the space of jets, the Whitney topology is defined on the set of smooth maps. The paper ends with a discussion of the structural stability of smooth mappings.

1. Introduction and notation

The notion of differential space as a natural generalization of the smooth manifold concept appeared independently in works of several authors, and the necessity of this generalization has become evident in many mathematical problems [1], [2], [3], [5], [9], [10], [20] - [27]. Some physical applications of this generalization are presented in [6], [7], [8], [11], [12], [17].

R. Sikorski introduced his concept of differential spaces in [21] and [22], and subsequently gave a beautiful exposition of modern differential geometry in book [23]. This approach has allowed him to substantially generalize many geometric problems so far reserved only for smooth manifolds.

Let M be any set, and C a set of real functions defined on M. The weakest topology on M, in which all functions of the family C are continuous, will be denoted by τ_C. For any subset $A \subset M$, let C_A denote the set of all real functions β on A such that, for any $p \in A$, there exist an open neighbourhood $U \in \tau_C$ of p and a function $\alpha \in C$ satisfying the condition $\beta|A\cap U = \alpha|A\cap U$. By scC we shall denote the family of all real functions on M which may be presented as $\omega\circ(\alpha_1,\dots,\alpha_n)$, where $\omega \in C^\infty(\mathbb{R}^n)$, $\alpha_1,\dots,\alpha_n \in C$, $n \in \mathbb{N}$. A family C is called the *differential structure* (in the sense of Sikorski, *d-structure*, for short) on M if $C = C_M = \mathrm{sc}C$, and its elements are called *smooth functions* on M [23]. The pair (M, C) is said to be a *differential space* (*d-space*, for short). For an arbitrary set C_o of real functions on M, the set $(\mathrm{sc}C_o)_M$ is the smallest differential structure on M containing C_o. A differential structure C on M is said to be *generated by* C_o iff $C = (\mathrm{sc}C_o)_M$. We shall use also the following shorter notation:

$$\mathrm{gen}(C_o) \equiv (\mathrm{sc}C_o)_M, \tag{1.1}$$

if M is well known from the context.

By $\mathcal{F}(M)$ we will denote a differential structure on a set M in the sense of Sikorski, i.e. $\mathcal{F}(M)$ is a set of all real functions on M which are assumed to be smooth. So, $\mathcal{F}(M)$ will always denote a chosen differential structure, the definition of which will be given explicitly or will be assumed clear from the context. For a differentiable manifold the differential structure is (traditionally) determined by its atlas.

All differentiable manifolds in this work are assumed to be smooth, i.e. of class C^∞, and by a smooth mapping of two manifolds we understand a map which is differentiable of class C^∞.

Let $(M, \mathcal{F}(M))$ and $(N, \mathcal{F}(N))$ be two differentiable manifolds, and $C^\infty(M, N)$ be the set of all smooth mappings from M to N. For any integer $k \geq 0$, one defines an equivalence relation in $C^\infty(M, N)$, the so called *k-jet relation* at $p \in M$;

for any f, g $\in C^\infty(M, N)$, $f \equiv_p^k g$ iff $f(p) = g(p)$ and there exist charts (U, ϕ) and (V, ψ) on M and N, respectively, such that

$$D_\alpha(\psi\circ f\circ\phi^{-1})(\phi(p)) = D_\alpha(\psi\circ g\circ\phi^{-1})(\phi(p)), \tag{1.2}$$

where $p \in U$, $f(p) \in V$, $D_\alpha := \partial_1^{\alpha 1}\circ\ldots\circ\partial_m^{\alpha m}$, $\alpha := (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}_o^m$, $\mathbb{N}_o := \{0, 1, 2, \ldots\}$, and $m = \dim M$, $n = \dim N$.

For future convenience, let us associate with the multiindex α the following two integers $|\alpha| := \alpha_1 + \ldots + \alpha_m$ and $\alpha! := \alpha_1!\cdot\ldots\cdot\alpha_m!$.

For a smooth mapping $f\colon M \longrightarrow N$, the equivalence class of f with respect to the relation $\equiv_p^k$ is said to be the *k-jet* of f at a point $p \in M$ and denoted by $j_p^k f$, i.e.

$$j_p^k f := [\ f\ ]_{\equiv_p^k}\ . \tag{1.3}$$

So, the set of all k-jets of smooth mappings from M to N is the quotient set

$$J_p^k(M, N) := C^\infty(M, N)/_{\equiv_p^k}\ . \tag{1.4}$$

The set of all k-jets of smooth mappings from $C^\infty(M, N)$ is given as the disjoint sum

$$J^k(M, N) := \bigcup_{p\in M} J_p^k(M, N). \tag{1.5}$$

For the set $J^k(M, N)$, one defines the following two mappings

$$\begin{aligned} &a\colon J^k(M, N) \ni j_p^k f \longmapsto p \in M,\\ &b\colon J^k(M, N) \ni j_p^k f \longmapsto f(p) \in N, \end{aligned} \tag{1.6}$$

which are known as *source* and *target* mappings.

Following C. Ehresmann [4], one defines an atlas on the set $J^k(M, N)$ of k-jets which contains all charts

$$\langle\phi,\psi\rangle\colon (a,\ b)^{-1}(U\times V) \longrightarrow E_{mn}^k, \tag{1.7}$$

defined by

$$(1.8)\quad \langle\phi,\psi\rangle(j_p^k f) := \Big(\phi(p),\ \big(D_\alpha(\psi^j\circ f\circ\phi^{-1})(\phi(p));\ |\alpha|\le k,\ j\le n\big)\Big),$$

where (U, ϕ), (V, ψ) are charts on M and N, respectively, $m = \dim M$, $n = \dim N$, and

$$(1.9)\quad \mathbb{E}_{mn}^k := \Big\{\big((x^1,\dots,x^m),\ (x_\alpha^j;\ |\alpha|\le k,\ j\le n)\big);\ x^i,\ x_\alpha^j \in \mathbb{R}\Big\}$$

is the Euclidean $\left(m + n\cdot\binom{m+k}{m}\right)$-dimensional space, i.e. $\mathbb{E}_{mn}^k = \mathbb{R}^L$, where $L = \left(m + n\cdot\binom{m+k}{m}\right)$.

For future convenience, let us denote by ρ the projection of the space $\mathbb{E}_{mn}^k$ onto its subspace $\mathbb{R}^m$, i.e.

$$(1.10)\quad \begin{aligned}\rho\colon \mathbb{E}_{mn}^k \ni x \equiv \big((x^1,\dots,x^m),\ (x_\alpha^j\colon |\alpha|\le k,\ j\le n)\big) &\longmapsto \\ \longmapsto (x^1,\dots,x^m) &\in \mathbb{R}^m .\end{aligned}$$

Let $\langle\phi,\psi\rangle\colon (a, b)^{-1}(U\times V) \longrightarrow \mathbb{E}_{mn}^k$ be the Ehresmann chart determined by charts (U, ϕ), (V, ψ) of the manifolds M and N, respectively. One can easily see that if

$$\theta := \langle\phi,\psi\rangle^{-1}(x) \in J^k(M, N),$$

for some $x \in \mathbb{E}_{mn}^k$, then

$$(1.11)\quad a(\theta) = \phi^{-1}\circ\rho(x) \equiv \phi^{-1}(x^1,\dots,x^m).$$

The problem arises: how to choose a suitable mapping representing the k-jet θ in a simple way? Let us denote by

$$(1.12)\quad \theta_x^j\colon \phi(U) \longrightarrow \mathbb{R},$$

the real functions given by

$$(1.13)\quad \theta_x^j(\xi^1,\dots,\xi^m) := \sum_{|\alpha|\le k} \frac{x_\alpha^j}{\alpha!}(\xi - x)^\alpha,$$

$j = 1,\dots,n$, where $(\xi - x)^\alpha := (\xi^1 - x^1)^{\alpha_1}\cdot\ldots\cdot(\xi - x^m)^{\alpha_m}$, and

$$(1.14)\quad \theta_x := (\theta_x^1,\dots,\theta_x^n)\colon \phi(U) \longrightarrow \mathbb{R}^n.$$

It turns out that the mapping $\psi^{-1}\circ\theta_x\circ\phi$ is a suitable (and simple enough) local representative of the k-jet θ, i.e.

$$\theta \equiv \langle\phi,\psi\rangle^{-1}(x) = j^k_{\phi^{-1}\circ\rho(x)}(\psi\circ\theta_x\circ\phi). \tag{1.15}$$

The proof is reduced to a simple algebraic manipulations.

The mapping $\psi^{-1}\circ\theta_x\circ\phi$ is actually defined on the open subset $U \subset M$ and formally, according to the definition assumed here, should not appear in the form $j^k_p(\psi^{-1}\circ\theta_x\circ\phi)$. However, we can always replace $\psi^{-1}\circ\theta_x\circ\phi$ by a (globally defined) mapping $f_\theta\colon M \longrightarrow N$, such that both mappings $\psi^{-1}\circ\theta_x\circ\phi$ and f_θ have the same germ at p and, of course, the resulting k-jet is independent of the chosen representative of the germ. (Equivalently, one might base one's considerations on local smooth mappings from M to N and define k-jets by the corresponding equivalence relation in the set of all such local smooth mappings.) So, we should not fix up any global representative of $\psi^{-1}\circ\theta_x\circ\phi$.

2. The differential space of k-jets

It is evident that to define k-jets for smooth mappings of differential spaces there is no possibility, in general, to use charts, simply because charts may not be available (such a case is common for differential spaces).

The concept of a k-jet of smooth mappings in the case of differential spaces was originally introduced in [29] and then in [11]. The definitions presented in [29] and [11] are formally different and the corresponding concepts of k-jets slightly differ when tangent vectors to a d-space M have no smooth prolongations. The definition introduced in [11] and presented in this work is formulated entirely in terms of the two d-spaces M and N in question, which seems to be simpler and more natural. Let us remind this definition in a concise way.

For any two differential spaces $(M, \mathcal{F}(M))$ and $(N, \mathcal{F}(N))$, let us denote by $C(M, N)$ the set of all smooth mappings from M to N. Then, we define the k-jet equivalence relation at $p \in M$ in the set $C(M, N)$, namely

Definition 2.1 For any $f, g \in C(M, N)$, we assume

$$f \equiv_p^k g \quad \text{iff} \quad d_p^k f = d_p^k g, \tag{2.1}$$

where the symbol $d_p^k f$ stands for the k-th differential of the mapping f at the point $p \in M$ (for the definition see Appendix I). So, by the k-jet of a mapping $f \in C(M, N)$ at $p \in M$ we shall understand the equivalence class

$$j_p^k f := \{g \in C(M, N): d_p^k g = d_p^k f\}. \tag{2.2}$$

We shall denote (as it is the use in the case of differentiable manifolds)

$$J_p^k(M, N) := \{j_p^k f: f \in C(M, N)\} \equiv {}^{C(M, N)}/_{\equiv_p^k}, \tag{2.3}$$

$$J^k(M, N) := \bigcup_{p \in M} J_p^k(M, N). \tag{2.4}$$

On the set of k-jets $J^k(M, N)$ we define the differential structure (in the sense of Sikorski)

$$\mathcal{F}(J^k(M, N)) := \text{gen} \Big(\{\alpha \circ a: \alpha \in \mathcal{F}(M)\} \cup \{\beta \circ b: \beta \in \mathcal{F}(N)\} \cup \cup \{\Gamma_\beta^X: X \in \mathfrak{X}^k(M), \beta \in \mathcal{F}(N)\} \Big), \tag{2.5}$$

where $\mathfrak{X}^k(M)$ denotes the $\mathcal{F}(M)$-module of all smooth k-vector fields defined on $(M, \mathcal{F}(M))$ (see Appendix I and also [11] or [15]), and

$$\Gamma_\beta^X(j_p^k f) := X(p)(\beta \circ f) \equiv [d_p^k f(X(p))](\beta). \tag{2.6}$$

Definition 2.2 The pair $(J^k(M, N), \mathcal{F}(J^k(M, N)))$ will be called the *differential space of k-jets* of smooth mappings from M to N.

Further we can define fiber bundles of k-jets (a suitable concept of the fiber bundle in the category of d-spaces was introduced and investigated in [28])as, for example, $(J^k(M, N), a, M)$, $(J^k(M, N), b, N)$, $(J^k(M, N), (a, b), M \times N)$.

Of course, the differential structure $\mathcal{F}(J^k(M, N))$ is required to coincide with the Ehresmann structure for differentiable manifolds. This is actually stated by the following

Proposition 2.3 Let $(M, \mathcal{F}(M))$ and $(N, \mathcal{F}(N))$ be differentiable manifolds. Then the differential structure $\mathcal{F}(J^k(M, N))$, defined by (2.5), coincides with the Ehresmann differential structure on $J^k(M, N)$.

Proof. Starting from Ehresmann differential structure on $J^k(M, N)$, determined by charts (1.7), for the case when M and N are differentiable manifolds, we shall show the smoothness of the functions generating the d-structure $\mathcal{F}(J^k(M, N))$, defined by (2.5), which is enough for the proof of the smoothness of all functions of $\mathcal{F}(J^k(M, N))$.

For any $\alpha \in \mathcal{F}(M)$, we obtain

$$\alpha\circ a\circ\langle\phi,\psi\rangle^{-1}(x) = \alpha\circ a\left(j^k_{\phi^{-1}\circ\rho(x)}(\psi\circ\theta_x\circ\phi)\right) = \alpha\circ\phi^{-1}\circ\rho(x).$$

The function $\alpha\circ\phi^{-1}$ is differentiable because $\alpha \in \mathcal{F}(M)$ and (U, ϕ) is a chart on M. The projection ρ is also differentiable. This means the differentiability of $\alpha\circ a\circ\langle\phi,\psi\rangle^{-1}$ and proves the smoothness of $\alpha\circ a$ (locally, but it is enough).

In turn, for any $\beta \in \mathcal{F}(N)$, we get

$$\beta\circ b\circ\langle\phi,\psi\rangle^{-1}(x) = \beta\circ b\left(j^k_{\phi^{-1}\circ\rho(x)}(\psi\circ\theta_x\circ\phi)\right) = \alpha\circ\psi^{-1}\circ\phi(\phi^{-1}\circ\rho(x)) =$$

$$= \beta\circ\psi^{-1}\circ\theta_x\circ\rho(x).$$

Let us notice that $(\theta_x)_{x\in\mathbb{E}^k_{mn}}$ is a smooth family of mappings (see Appendix II for the definition and properties of smooth families). Indeed, the mapping

$$\Psi^\theta\colon \phi(U)\times\mathbb{E}^k_{mn} \ni (\xi, x) \longmapsto \Psi^\theta(\xi, x) := \theta_x(\xi) \in \mathbb{R}^n,$$

defined as the composition of polynomials

$$\Psi^\theta(\xi, x) := (\Psi^\theta_1,\ldots,\Psi^\theta_n)(\xi, x) \equiv (\theta^1_x(\xi),\ldots,\theta^n_x(\xi)) \equiv \theta_x(\xi),$$

where

$$\Psi_j^\theta(\xi, x) := \sum_{|\alpha| \le k} \frac{x_\alpha^j}{\alpha!}(\xi - x)^{\bar{\alpha}},$$

is differentiable of class C^∞. Obviously, $\beta \circ \psi^{-1}$ is differentiable of class C^∞ since $\beta \in \mathcal{F}(N)$ and (V, ψ) is a chart on N. Thus, as it follows from Proposition A.II.1(ii), $(\beta \circ \psi^{-1} \circ \theta_x)_{x \in \mathbb{E}^k_{mn}}$ is a smooth family and the mapping $x \longmapsto \beta \circ \psi^{-1} \circ \theta_x \circ \rho(x)$ is smooth (see Proposition A.II.2(i)). (The smoothness of $\beta \circ \&$ may be also demonstrated in a shorter way. Namely, one can easily see that the function $\beta \circ \& \circ \langle\phi, \psi\rangle^{-1}$ is a constant function.)

In the proof of smoothness of functions Γ^X_β, for $X \in \mathfrak{X}^k(M)$ and $\beta \in \mathcal{F}(N)$, we shall denote by the same symbol X the restriction of the global k-field X to the open subspace $U \subset M$, where U is the domain of the chart ϕ. We obtain

$$\Gamma^X_\beta \circ \langle\phi, \psi\rangle^{-1}(x) = \Gamma^X_\beta(j^k_{\phi^{-1}\circ\rho(x)}(\psi \circ \theta_x \circ \phi)) =$$

$$= X(\phi^{-1} \circ \rho(x))(\beta \circ \psi^{-1} \circ \theta_x \circ \phi) = [d^k\phi(X(\phi^{-1} \circ \rho(x)))](\beta \circ \psi^{-1} \circ \theta_x) =$$

$$= [[(\phi)^k_* X](\rho(x))](\beta \circ \psi^{-1} \circ \theta_x) = [[(\phi)^k_* X](\beta \circ \psi^{-1} \circ \theta_x)] \circ \rho(x).$$

Obviously, the k-field $(\phi)^k_* X$ is smooth, and therefore $([(\phi)^k_* X](\beta \circ \psi^{-1} \circ \theta_x))_{x \in \mathbb{E}^k_{mn}}$ is a smooth family. From Appendix II it follows that $\Gamma^X_\beta \circ \langle\phi, \psi\rangle^{-1}$ is a differentiable function, which means that Γ^X_β is smooth.

Now, let us assume the smoothness of all functions of the family $\mathcal{F}(J^k(M, N))$. We have to show the smoothness of all functions λ on $J^k(M, N)$ for which the composition $\Lambda_\lambda := \lambda \circ \langle\phi, \psi\rangle^{-1}$ is differentiable, i.e. Λ_λ is a (local) C^∞-differentiable function on $\mathbb{E}^k_{mn}$. The differentiability of Λ_λ means that $\lambda \overset{*}{=} \Lambda_\lambda \circ \langle\phi, \psi\rangle$ is smooth ($\overset{*}{=}$ denotes the local equality on the domain of $\langle\phi, \psi\rangle$) provided the components of $\langle\phi, \psi\rangle$ are smooth with respect to the differential structure $\mathcal{F}(J^k(M, N))$. Let $\theta \in J^k(M, N)$, then

$$<\phi,\psi>(\theta) = (\phi(a(\theta)), (D_\alpha(\psi^j\circ f_\theta\circ\phi^{-1})(\phi(a(\theta))); |\alpha|\le k, j\le n)).$$

The components $<\phi,\psi>^i = \phi^i\circ a$ are smooth since they are generators of the differential structure $\mathcal{F}(J^k(M, N))$.
For $|\alpha| = 0$ we get the components

$$<\phi,\psi>_o^j(\theta) = \psi^j\circ f_\theta(a(\theta)) = \psi^j\circ b(\theta), \qquad \text{i.e.} \qquad <\phi,\psi>_o^j = \psi^j\circ b,$$

$j = 1,\ldots,n$, which are smooth since they are also generators of $\mathcal{F}(J^k(M, N))$. For the case when $|\alpha| \ge 1$ we obtain

$$<\phi,\psi>_\alpha^j(\theta) = D_\alpha(\psi^j\circ f_\theta\circ\phi^{-1})(\phi(a(\theta))) =$$

$$= [d^k\phi^{-1}(D_\alpha(\phi\circ a(\theta)))](\psi^j\circ f_\theta) = [[d^k\phi^{-1}\circ D_\alpha\circ\phi](a(\theta))](\psi^j\circ f_\theta) \equiv$$

$$\equiv [((\phi)_*^k D_\alpha)(a(\theta))](\psi^j\circ f_\theta) = \Gamma_{\psi^j}^{(\phi^{-1})_*^k D_\alpha}(\theta),$$

i.e.

$$<\phi,\psi>_\alpha^j(\theta) = \Gamma_{\psi^j}^{(\phi^{-1})_*^k D_\alpha},$$

$j = 1,\ldots,n$, which generate (locally on the domains of charts) the structure $\mathcal{F}(J^k(M, N))$ and therefore they are smooth.

This ends the proof of the compatibility of the differential structure $\mathcal{F}(J^k(M, N))$ with the Ehresmann structure in the case of differentiable manifolds.

Let us end this section with stating one more fact.

Proposition 2.4 For any $f \in C(M, N)$, the mapping

$$j^k f\colon M \longrightarrow J^k(M, N), \tag{2.7}$$

defined by $j^k f(p) := j_p^k f$, $p \in M$, is smooth.

Proof. We shall show that $\kappa\circ j^k f \in \mathcal{F}(M)$, for any $\kappa \in \mathcal{F}(J^k(M, N))$. Of course, it is enough to take elements κ from the set of generators of $\mathcal{F}(J^k(M, N))$. So, if $\kappa = \alpha\circ a$, $\alpha \in \mathcal{F}(M)$, we get

$$\alpha\circ a\circ j^k f(p) = \alpha\circ a(j_p^k f) = \alpha(p),$$

i.e. $\alpha\circ a\circ j^k f \equiv \alpha \in \mathcal{F}(M)$. Next, for $\kappa = \beta\circ b$, $\beta \in \mathcal{F}(N)$, we obtain

$$\beta\circ\hat{s}\circ j^kf(p) = \beta\circ\hat{s}(j^k_pf) = \beta(f(p)) = \beta\circ f(p),$$

i.e. $\beta\circ\hat{s}\circ j^kf \equiv \beta\circ f \in \mathcal{F}(M)$ because f is smooth. For $\kappa = \Gamma^X_\beta$, $X \in \mathfrak{X}^k(M)$, $\beta \in \mathcal{F}(N)$, we have

$$\Gamma^X_\beta\circ j^kf(p) = \Gamma^X_\beta(j^k_pf) = X(p)(\beta\circ f) = (X(\beta\circ f))(p),$$

i.e. $\Gamma^X_\beta\circ j^kf = X(\beta\circ f) \in \mathcal{F}(M)$, which is the consequence of the facts $\beta\circ f \in \mathcal{F}(M)$ and $X \in \mathfrak{X}^k(M)$. This ends the proof of this fact.

The mappings j^kf, introduced above, will play an essential role in the next section.

3. Whitney C^∞-topology on the set C(M, N)

The differential structure $\mathcal{F}(J^k(M, N))$, introduced in the previous section, endows the set $J^k(M, N)$ with the topology $\tau_{\mathcal{F}(J^k(M, N))}$; this is the weakest topology on $J^k(M, N)$ in which all functions of $\mathcal{F}(J^k(M, N))$ are continuous.

For any set $\mathcal{B} \in \tau_{\mathcal{F}(J^k(M, N))}$, we define the set $G_\mathcal{B} \subset C(M, N)$ by the formula

(3.1) $$G_\mathcal{B} := \{f \in C(M, N):\ j^kf(M) \subset \mathcal{B}\}.$$

Obviously, for any $\mathcal{B}, \mathcal{B}' \in \tau_{\mathcal{F}(J^k(M, N))}$,

(3.2) $$G_\mathcal{B} \cap G_{\mathcal{B}'} = G_{\mathcal{B}\cap\mathcal{B}'}.$$

Thus, the family $\{G_\mathcal{B} : \mathcal{B} \in \tau_{\mathcal{F}(J^k(M, N))}\}$ forms a base of some topology on the set C(M, N); we shall denote it by $\tau_k(C(M, N))$. This topology, for the differentiable manifolds, is known as the *strong Whitney C^k-topology.*

Definition 3.1 By *Whitney C^∞-topology* on C(M, N) we shall understand, in analogy with the manifold case, the topology determined by the base $\bigcup_{k=0}^{\infty} \tau_k(C(M, N))$; i.e. the sum $\bigcup_{k=0}^{\infty} \tau_k(C(M, N))$ is the base of the Whitney C^∞-topology introduced here for the case of differential spaces.

Remark. Having the C^∞-topology we may consider every differential structure, every family of smooth (pseudo)Riemannian metrics, the space of smooth k-fields or smooth forms, etc., defined in the category of d-spaces, as some topological spaces. In particular, this enables us to consider the continuity of (almost) Poisson structures in d-spaces [17].

4. Structural stability of smooth mappings

Let M, N be any differential spaces, and $f_1, f_2 : M \longrightarrow N$ smooth mappings. We shall introduce the concept of the structural stability of smooth mappings of d-spaces in full analogy with the standard case of differentiable manifolds. However, this construction would not be possible without having the differential structure on $J^k(M, N)$ which enabled us to define the Whitney C^∞-topology on C(M, N). As practice shows, this topology is the best one for the stability consideration purposes.

Definition 4.1 We shall say that $f_1, f_2 \in C(M, N)$ are C^∞-equivalent iff there exist diffeomorphisms $\phi: M \longrightarrow M$ and $\Psi: N \longrightarrow N$, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f_1} & N \\ \phi \downarrow & & \downarrow \psi \\ M & \xrightarrow{f_2} & N \end{array}$$

commutes, i.e. $f_2 = \psi \circ f_1 \circ \phi^{-1}$.

Remark. Following the classical approach, we shall assume that ϕ and ψ preserve any additional structures on M or N, e.g. a linear or partial ordering. If, for example, $N = \mathbb{R}$, we assume that ψ preserves the natural linear ordering of the real line $\mathbb{R}$.

Definition 4.2 A mapping $f \in C(M, N)$ is said to be *structurally stable* iff the equivalence class [f] of f, with respect to the C^∞-equivalence relation, is an open set in the Whitney C^∞-topology on the set C(M, N).

Definition 4.3 A subset $G \subset C(M, N)$ will be called the *generic subset* iff G is open and dense in the Whitney

C^{∞}-topology.

If a generic set $G \subset C(M, N)$ is the collection of all elements which possess some property $\mathcal{P}$, determining G uniquely, we shall speak of $\mathcal{P}$ as a *generic property* of all elements of the set G.

Generic properties of objects are of primary importance from the stability theory point of view. As an example of an advantage we may have from these concepts, let us notice that any property of elements of the completion of some generic set never refers to structurally stable elements because such a completion does not contain any open set.

5. Final remarks

It remains an open question concerning a natural differential structure on the topological space C(M, N) with the Whitney C^{∞}-topology. My suggestion would be to call the *admissible differential structure* each of the differential structures on C(M, N) that determines (not merely agrees with) the Whitney C^{∞}-topology on C(M, N). This is not obvious for me whether such admissible d-structures always exists, and - if they exist - which of them should be chosen as the most natural one.

There is one differential structure on the set C(M, N), suggested to me by Professor A. Frölicher, which is also natural in a certain sense. Namely, let us denote by DS the category of differential spaces. Then, let $M, N \in DS$ be d-spaces. It is a well known fact that the sets $\left(N^M\right)^X$ and $N^{X\times M}$ are of the same cardinality, for any d-space $X \in DS$. Indeed, the mapping

$$\Phi\colon \left(N^M\right)^X \ni f \longmapsto \Phi(f) \in N^{X\times M},$$

defined by the formula

$$\Phi(f)(x,p) := (f(x))(p),$$

is a bijection. If X, M and N are d-spaces, the bijection Φ allows us to select all mappings $f\colon X \longrightarrow C(M, N)$ such that

$\Phi(f)$: $X \times M \longrightarrow N$ is smooth; let us denote by $C_o(X, C(M, N))$ the set of all such mappings f. Then, on the set C(M, N) we may define the differential structure $\mathcal{F}_\Phi(C(M, N))$ coinduced by all mappings from the sets $C_o(X, C(M, N))$, $X \in DS$ [30], i.e. the strongest differential structure on C(M, N) with respect to which all mappings $f \in C_o(X, C(M, N))$ are smooth, for any $X \in DS$.

As it seems, the d-structure defined above, should hardly be expected to agree with the Whitney topology on C(M, N).

There is also one more interesting and open question closely related to the above definition of $\mathcal{F}_\Phi(C(M, N))$. Namely, one may be interested whether the category of differential spaces is Cartesian closed or not [5]. Let us remind that the category of differential spaces would (by definition) be Cartesian closed iff, for any smooth mapping $h: (X, \mathcal{F}(X)) \longrightarrow (C(M, N), \mathcal{F}_\Phi(C(M, N)))$, the mapping $\Phi(h)$ is also smooth, for all differential spaces $X, M, N \in DS$.

Let us end this section with the proposal of the partial differential equation concept in the category of differential spaces. Namely, by a *partial differential equation of order k* in the category of differential spaces we may understand a subbundle

$$((S, \mathcal{F}(J^k(M, N)_S), a_{|S}, (M, \mathcal{F}(M)))$$

of the bundle

$$((J^k(M, N), \mathcal{F}(J^k(M, N)), a, (M, \mathcal{F}(M))).$$

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Appendix

I. For all definitions mentioned here, see [11] and [15].

Let $(M, \mathcal{F}(M))$ be a d-space. By $\mathfrak{a}_p$, $p \in M$, we denote the linear subspace of $\mathcal{F}(M)$ defined as

(A I.1) $$\mathfrak{a}_p := \{f - f(p) \colon f \in \mathcal{F}(M)\}$$

Next, let $\mathfrak{a}_p^k$ denote the linear subspace of $\mathcal{F}(M)$ spanned by all products of k functions of $\mathfrak{a}_p$, i.e.

(A I.2) $$\mathfrak{a}_p^k = \operatorname{span}\{\alpha_1 \cdot \ldots \cdot \alpha_k \colon \alpha_1, \ldots, \alpha_k \in \mathfrak{a}_p\}.$$

By the *k-th tangent vector* to $(M, \mathcal{F}(M))$ at $p \in M$, where $k \geq 1$ is an integer, we understand any linear mapping $v\colon \mathcal{F}(M) \longrightarrow \mathbb{R}$, such that

(i) $v(\mathbf{k}) = 0$, for any constant function $\mathbf{k} \in \mathcal{F}(M)$,

(ii) $v_{|\mathfrak{a}_p^k} \equiv 0$.

One can find an analogous definition, for the manifold case, in [31]. Obviously, this is a generalization of the usual concept of tangent vector specified with the help of the well known Leibniz rule; we get the standard case when $k = 1$.

Since, for any $k \geq 1$, $\mathfrak{a}_p^{k+1} \subset \mathfrak{a}_p^k$, we get $T_p^{(k)}M \subset T_p^{(k+1)}$, where $T_p^{(k)}M$ denotes the set of all k-th tangent vectors to M at p. This means that, for any $k \geq 1$, $T_p^{(k)} = \bigcup_{=}^{k} T_p^{(n)}$, which allows for the following definition of the set of all tangent vectors of any order

$$T_p^{(\infty)} = \bigcup_{n=1}^{\infty} T_p^{(n)}.$$

Remark. Of course, $T_p^{(\infty)} = \bigcup_{n=1}^{\infty} T_p^{(n)}$, for any $i = 1, 2, \ldots$.

Naturally, $T_p^{(k)}M$ turns out to form a linear space, for any $k = 1, 2, \ldots, \infty$.

For any smooth mapping $f\colon (M, \mathcal{F}(M)) \longrightarrow (N, \mathcal{F}(N))$, we define the so-called *k-th differential* of f, as the linear mapping

(A I.3) $$d_p^k f\colon T_p^{(k)}M \longrightarrow T_{f(p)}^k N,$$

given by the formula

(A I.4) $$[d_p^k f(v)](\beta) := v(\beta\circ f),$$

for any $v \in T_p^{(k)}M$, $k = 1, 2, \ldots, \infty$, and $\beta \in \mathcal{F}(N)$; $d_p^0 f := f(p)$. In turn, by $d^k f$ we denote the mapping

(A I.5) $$d^k f\colon T^{(k)}M \equiv \bigcup_{\in} T_p^{(k)}M \longrightarrow T^{(k)}N \equiv \bigcup_{\in} T_q^{(k)}N,$$

such that $d^k f_{|T_p^{(k)}M} \equiv d_p^k f$, $k = 1, 2, \ldots, \infty$; $d^k f$ is said to be the *k-th differential* of the mapping f or the *k-th tangent mapping* to f. Directly from the definition, we get the rule

(A I.6) $$d^k(g\circ f) = d^k g\circ d^k f,$$

for any smooth mappings $f\colon (M, \mathcal{F}(M)) \longrightarrow (N, \mathcal{F}(N))$ and $g\colon (N, \mathcal{F}(N)) \longrightarrow (P, \mathcal{F}(P))$. (The chain rule for k-th vectors can be found in [11].)

By the *k-th vector field* on $(M, \mathcal{F}(M))$ (k-th field, shortly) we mean any mapping $V\colon M \longrightarrow T^{(k)}M$, such that $V(p) \in T_p^{(k)}M$, $k = 1, \ldots, \infty$, $p \in M$. V is said to be a *smooth k-th field* if, for any $f \in \mathcal{F}(M)$, the function $M \ni p \longmapsto V(p)f \in \mathbb{R}$ is smooth. Equivalently, we can understand the concept of a smooth k-th field as the smooth mapping $V\colon (M, \mathcal{F}(M)) \longrightarrow (T^{(k)}M, \mathcal{F}(T^{(k)}M))$, where by $\mathcal{F}(T^{(k)}M)$ we mean the weakest differential structure on $T^{(k)}M$ with respect to which the mappings $\alpha\circ\pi\colon T^{(k)}M \longrightarrow \mathbb{R}$ and $d^k\alpha\colon T^{(k)}M \longrightarrow \mathbb{R}^k$ are smooth, for any $\alpha \in \mathcal{F}(M)$, where $\pi\colon T^k M \longrightarrow M$ is the natural projection, i.e. $\pi(T_p^{(k)}M) = \{p\}$ [11]. For the set $\mathbb{R}^k$ we assume the natural (Euclidean) differential structure $\mathcal{E}_k$ generated by the projections $\rho_1, \ldots, \rho_k$. The pair $(T^{(k)}M, \mathcal{F}(T^{(k)}M))$ is called the *tangent space of k-th order* to $(M, \mathcal{F}(M))$. Obviously, π is smooth. We denote

(A I.7) $$T^{(k)}(M, \mathcal{F}(M)) := (T^{(k)}M, \mathcal{F}(T^{(k)}M)).$$

The triple $((T^{(k)}M, \mathcal{F}(T^{(k)}M)), \pi, (M, \mathcal{F}(M)))$ is said to be the *tangent bundle of k-th order*.

II. The main reference concerning the topic considered here is [19].

Let $(M, \mathcal{F}(M))$ and $(N, \mathcal{F}(N))$ be d-spaces.

Definition II.1 (i) A family $(\alpha_q)_{q\in N}$ of smooth functions $\alpha_q \in \mathcal{F}(M)$, indexed by points of N, is said to be a *smooth family of functions* if the function $\Psi^{\alpha}: M\times N \longrightarrow \mathbb{R}$, defined by $\Psi^{\alpha}(p, q) := \alpha_q(p)$, for $(p, q) \in M\times N$, is smooth, i.e. $\Psi^{\alpha} \in \mathcal{F}(M\times N) \equiv \mathcal{F}(M)\mathrm{x}\mathcal{F}(N)$.

(ii) Analogously, a family $(\mu_q)_{q\in N}$ of smooth mappings $\mu_q: M \longrightarrow L$, where M, N, L are d-spaces, is said to be a *smooth family of mappings* if the mapping $\Psi^{\mu}: M\times N \ni (p,q) \longmapsto \mu_q(p) \in L$ is smooth.

(iii) Next (in analogy with Ref.[19]), the family $(X_q)_{q\in N}$ of smooth k-fields on M, indexed by points of N, is said to be a *smooth family of k-th fields* if the mapping

$$\Psi^{X}: M\times N \ni (p, q) \longmapsto \Psi^{X}(p, q) := X_q(p) \in T^{(k)}M,$$

is smooth.

We shall prove some useful properties of smooth families.

Proposition II.1 Let $(\mu_q)_{q\in N}$, $\mu_q: M \longrightarrow L$, be a smooth family. Moreover, let $\xi: P \longrightarrow M$, $\eta: L \longrightarrow Q$, $\zeta: R \longrightarrow N$ be smooth mappings, where M, N, P, Q and R are considered to be d-spaces. Then the following families are smooth:

(i) $(r_q)_{q\in N}$, where $r_q := \mu_q\circ\xi: P \longrightarrow L$,

(ii) $(s_q)_{q\in N}$, where $s_q := \eta\circ\mu_q: M \longrightarrow Q$,

(iii) $(t_z)_{z\in R}$, where $t_z := \mu_{\zeta(z)}: M \longrightarrow L$.

Proof. (i) The smoothness of $(\mu_q)_{q\in N}$ requires the mapping $\Psi^{\mu}: M\times N \ni (p, q) \longmapsto \Psi^{\mu}(p, q) := \mu_q(p) \in L$ to be smooth. To prove the smoothness of $(r_q)_{q\in N}$ we have to check the smoothness of the mapping

$$\Psi^{r}: P\times N \ni (m, q) \longmapsto \Psi^{r}(m, q) := r_q(m) = \mu_q(\xi(m)) =$$

$$= (\mu_q\circ\xi)(m) \equiv \Psi^{\mu}(\xi(m), q) \in L.$$

Let us denote by pr_P and pr_N the projections of the Cartesian product P×N on its axes, of course both these projections are smooth mappings. Hence, the mapping Ψ^r is smooth as the composition of smooth mappings, $\Psi^r = \Psi^\mu \circ (\xi \circ pr_P,\ pr_N)$.

(ii) For $(s_q)_{q\in N}$, we focus on the mapping

$$\Psi^s\colon M\times N \ni (p,q) \longmapsto \Psi^s(p,q) := s_q(p) \equiv$$
$$\equiv \eta\circ\mu_q(p) = \eta\circ\Psi^\mu(p,q) \in Q.$$

Hence, $\Psi^s = \eta\circ\Psi^\mu$ which proves the smoothness of Ψ^s and, consequently, it means that $(s_q)_{q\in N}$ is smooth.

(iii) For $(t_z)_{z\in R}$, we have

$$\Psi^t\colon M\times R \ni (p,z) \longmapsto \Psi^t(p,z) := t_z(p) \equiv \mu_{\zeta(z)}(p) \in L.$$

However, $\mu_{\zeta(z)}(p) = \Psi^\mu(p,\ \zeta(z)) = \Psi^\mu\circ(pr_M,\ \zeta\circ pr_R)(p,\ z)$, where pr_M and pr_R are the natural projections of MxR onto the axes M and R, respectively. The smoothness of Ψ^t (and consequently of $(t_z)_{z\in R}$) follows from the formula $\Psi^t = \Psi^\mu\circ(pr_M,\ \zeta\circ pr_R)$.

Proposition II.2 Let $(\mu_q)_{q\in N}$ be a smooth family of smooth mappings $\mu_q\colon M \longrightarrow L$, and let $\rho\colon N \longrightarrow M$, $\lambda\colon M \longrightarrow N$ be smooth mappings. Then the following two mappings are smooth:

(i) $$F_\rho\colon N \ni q \longmapsto F_\rho(q) := \mu_q(\rho(q)) \in L,$$

(ii) $$F_\lambda\colon M \ni p \longmapsto F_\lambda(p) := \mu_{\lambda(p)}(p) \in L.$$

Proof. (i) $F_\rho(q) = \mu_q(\rho(q)) = \Psi^\mu(\rho(q),q) = \Psi^\mu\circ(\rho,id_N)(q)$, i.e. $F_\rho = \Psi^\mu\circ(\rho,id_N)$, which is evidently a smooth mapping.

(ii) $F_\lambda(p) = \mu_{\lambda(p)}(p) = \Psi^\mu(p,\ \lambda(p)) = \Psi^\mu\circ(id_M,\ \lambda)(p)$, hence $F_\lambda = \Psi^\mu\circ(id_M,\ \lambda)$, which ends the proof.

Proposition II.3 Let $(\alpha_q)_{q\in N}$ be a smooth family of real functions on M, $\alpha_q \in \mathcal{F}(M)$, and let X be a smooth k-field on M. Then the family $(\gamma_q)_{q\in N}$, where $\gamma_q\colon := X\alpha_q$, is smooth.

Proof. Let $i_q\colon M \longrightarrow M\times N$ be the inclusion mapping, which is smooth.

$$\Psi^\gamma(p,\ q) \equiv \gamma_q(p) = (X\alpha_q)(p) = X(p)\alpha_q = X(p)(\Psi^\alpha\circ i_q) =$$

$$= [d_p^k(i_q)X(p)]\Psi^\alpha \equiv [(i_q)_*^k X(p)]\Psi^\alpha.$$

Of course, the field $\overline{X}$, $\overline{X}(p, q) := (i_q)_{*p}^k X(p)$, is smooth on the Cartesian product M×N. Indeed,

$$\overline{X}(p, q)(f\circ pr_M) = [(i)_{*p}^k X(p)](f\circ pr_M) \equiv [d_p^k(i_q)X(p)](f\circ pr_M) =$$

$$= X(p)(f\circ pr_M\circ i_q) = X(p)f = (Xf)(p) = (Xf)\circ pr_M(p, q),$$

which means that $\overline{X}(f\circ pr\ ) =(Xf)\circ pr\ $, where $f \in \mathcal{F}(M)$.

In turn,

$$[\overline{X}(g\circ pr_N)](p, q) = \overline{X}(p, q)(g\circ pr_N) =$$

$$= (i_q)_*^k X(p)(g\circ pr_N) = X(p)(g\circ pr_N\circ i_q) = 0,$$

because $pr_N\circ i_q$ is a constant mapping, $g \in \mathcal{F}(N)$. This means that $\overline{X}$ is a smooth k-field on M×N. Hence $\Psi^\gamma \equiv \overline{X}(\Psi^\alpha)$ is a smooth function on M×N, and therefore the family $(\gamma_q)_{q\in N}$ proves to be smooth.

Proposition II.4 Let $(X_q)_{q\in N}$ be a smooth family of smooth k-fields on $(M, \mathcal{F}(M))$, and $\alpha \in \mathcal{F}(M)$. Then, the family $(\alpha_q)_{q\in N}$, where $\alpha_q := X_q\alpha$, $q \in N$, is smooth.

Proof. From the assumption of smoothness of the family $(X_q)_{q\in N}$ we know that the mapping

$$\Psi^X: M\times N \ni (p, q) \longmapsto X_q(p) \in T^{(k)}M$$

is smooth. In turn, the mapping

$$\Psi^\alpha: M\times N \ni (p, q) \longmapsto \Psi^\alpha(p, q) := \alpha_q(p) \in \mathbb{R}$$

is also smooth because

$$\Psi^\alpha p, q) = \alpha_q(p) = X_q(p)\alpha =[\Psi^X(p, q)](\alpha) =$$

$$= d^k\alpha(\Psi^X(p, q)) = (d^k\alpha\circ\Psi^X)(p, q),$$

i.e. $\Psi^\alpha \equiv d^k\alpha\circ\Psi^X$, which is a composition of smooth mappings. The last formula shows the smoothness of the family $(\alpha_q)_{q\in N}$.

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