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# SOME REMARKS ON THE MOTION OF A RIGID BODY IN A SPACE OF CONSTANT CURVATURE WITHOUT EXTERNAL FORCES[1]

## 0. Introduction

Due to H. v. Helmholtz the foundations of classical mechanics do not require the "space" to satisfy the full Euclidean geometry, but only free movability of rigid bodies. This property of Euclidean spaces however is shared by all 3-dim. 1-connected Riemannian manifolds $\mathbb{M}^3_\kappa$ of constant curvature $\kappa \in \mathbb{R}$ and it characterizes these standard spaces in a large class of topological spaces (see e.g. [5],p.220, Corollaire 1; certain group actions can topologically model the "transport of rigid bodies"). All known experiments for detecting the value of $\kappa$ in "reality" will not give an exact value zero but only highly probable estimates for $\kappa$, close to zero. So it is interesting to study mechanical problems in $\mathbb{M}^3_\kappa$ for arbitrary $\kappa$ and to investigate, how the corresponding results depend on $\kappa$ and how "stable" these "laws of nature" are with respect to perturbations of the curvature, in particular when perturbing the value $\kappa=0$.

We want to present here some results on the motion of a rigid body in $\mathbb{M}^3_\kappa$ without external forces. The first results concern the equations of motion, which are reformulations of more or less known facts in new coordinates (see [4] for $\kappa \in \mathbb{R}^*$ and see [1] for an invariant interpretation in case of $\kappa=0$ and

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[1]This is an extract of main results of the doctoral thesis of the author (see [6]).

one point of the body fixed). These coordinates have the advantage of working uniformly for all values of $\kappa$ and of making geometric interpretations of the solutions easier. Computations in these coordinates make use of the Weierstraß model for $M^3_\kappa$ (see § 2.ii of the preceding article [2]). The procedure of proving the equations of motion suggests an idea, how to define the tensor of inertia and the center of mass for arbitrary $\kappa$, the latter definition being a modification of the corresponding concept in [3].

If the rigid body is a ball with a full rotational symmetric distribution of mass, then the equations of motion simplify and can be solved completely. We discuss the geometry of these solutions: When the ball moves without external forces, then the axis of rotation undergoes parallel transport (in the sense of Levi-Civita in $\mathbb{M}^3_\kappa$) along the path of its center of mass (which is of course the metric center of the ball), this path being a "normed helix" in $\mathbb{M}^3_\kappa$, i.e. a curve of constant speed, constant curvature and constant torsion. If $\kappa \neq 0$, then these constant values of curvature and torsion are different from zero on an open, dense subset of all possible initial conditions. In particular the path of the center is in general not a geodesic. On the other hand both the curvature and the torsion tend to 0 (like $O(\kappa)$) for $\kappa \longrightarrow 0$, thus explaining, that physical measurements in "reality" will behave "stable", when $\kappa$ varies around zero, whereas the quality of the mathematical solutions behaves "instable" in the sense, that it changes from the geodesic to the helix type.

If the rigid body has a distribution of mass, which allows a rotational symmetry only with respect to one fixed axis, then it seems, that most of the solutions of the equations of motion are accessible only through numerical methods. However, the few solutions, that can be given in closed form, contain all the solutions, where the axis of symmetry undergoes parallel transport (in the sense of Levi-Civita), and all the solutions, where the center of mass moves along a geodesic.

## 1. Notations and basic concepts

(i) We use the notations of the preceding article [2].

In particular, given a real number $\kappa \in \mathbb{R}$, let $\mathbb{M} := \mathbb{M}^3_\kappa$ denote the Weierstraß model of the 3-dim. 1-connected complete Riemannian $C^\omega$ manifold of constant curvature $\kappa$ (cf. [2], § 2.ii), which is a certain canonically oriented submanifold of $\mathbb{R}^4$, $g$ the Riemannian and $d := d_\kappa : \mathbb{M}\times\mathbb{M} \longrightarrow \mathbb{R}$ the intrinsic metric of $\mathbb{M}$, $G := G^3_\kappa$ the Lie group of all orientation preserving isometries of $\mathbb{M}$, considered as a Lie subgroup of $GL_+(4,\mathbb{R})$ (cf. [2], § 2.iv), $\mathfrak{g} := \mathfrak{g}^3_\kappa$ the Lie algebra of this group, considered as a subalgebra of the matrix Lie algebra $\mathfrak{M}(4,\mathbb{R})$ (cf. [2], § 2.iv), and $e := e_0 = = (1,0,0,0) \in \mathbb{M}$, which is a point of first order contact between all members of the family $(\mathbb{M}^3_\kappa)_{\kappa\in\mathbb{R}}$.

Furthermore, let $x_i : \mathbb{R}^4 \longrightarrow \mathbb{R}$ and $x_{ij} : \mathfrak{M}(4,\mathbb{R}) \longrightarrow \mathbb{R}$ for $i,j \in \{0,\ldots,3\}$ denote the canonical coordinates, $L_f = f\cdot\ldots : G \longrightarrow G$ resp. $R_f = \ldots\cdot f : G \longrightarrow G$ for $f\in G$ the left resp. right translation and $\ldots_L : TG \longrightarrow \mathfrak{g}$ the *left parallelism* of $G$, given by $\upsilon_L := (L_f^{-1})_*\upsilon$, if $f\in G$ and $\upsilon \in T_f G$. In particular, if $\gamma : J \longrightarrow G$ is any smooth curve and $t\in J$, then $\dot\gamma(t)_L = \gamma(t)^{-1}\cdot\gamma'(t)$, where $\gamma'(t) = \lim\limits_{h\to 0}\frac{\gamma(t+h)-\gamma(t)}{h}$ is the derivative in the vector space $\mathfrak{M}(4,\mathbb{R})$ $(\supseteq G)$.

In what follows, we identify the points of $\mathbb{M}$ resp. the tangent vectors of $T_p\mathbb{M}$ with the corresponding points in $\mathbb{R}^4$ resp. with the tangent vectors in $T_p\mathbb{R}^4$ under the inclusion map $i : \mathbb{M} \longrightarrow \mathbb{R}^4$ resp. under its differential, thereby giving rise to the canonical map $\vec{\ldots} : T\mathbb{M} \longrightarrow \mathbb{R}^4$ (the composition of $i_* : T\mathbb{M} \longrightarrow T\mathbb{R}^4$ and the standard map $\vec{\ldots} : T\mathbb{R}^4 \longrightarrow \mathbb{R}^4$, cf. [2],(2.3)).

(ii) We define $\pi := (x_1,x_2,x_3)|\mathbb{M} : \mathbb{M} \longrightarrow \mathbb{R}^3$.

Then $(1,\pi) : \mathbb{M} \longrightarrow \mathbb{R}^4$ is (extrinsically interpreted) the $(\ldots|\ldots)_\kappa$-orthogonal projection of $\mathbb{M}$ onto $e+\overrightarrow{T_e\mathbb{M}}=\mathbb{M}^3_0$. (use [2],(2.1),(2.5)).

The restrictions of the canonical coordinates

$x_0,\dots,x_3:\mathbb{R}^4 \longrightarrow \mathbb{R}$ to the submanifold $\mathbb{M}$ of $\mathbb{R}^4$ are related to the intrinsic geometry of $\mathbb{M}$ as follows (cf. [2], § 1.i,iv, § 2.i,ii):

$$(0,n) = (\frac{\sin_\kappa}{x}\circ d(e,..))\cdot(\exp_e^{-1})^{\rightarrow} \quad on\ \mathbb{M}\setminus\{-e\},$$

where $\exp_e^{-1}$ abbreviates $(\exp_e|U_{\pi_\kappa}(o))^{-1}$ and $\frac{\sin_\kappa}{x}$ is $C^\omega$ even at 0, and $x_0|\mathbb{M} = \cos_\kappa\circ d(e,..)$ , $\langle n,n\rangle = \sin_\kappa^2\circ d(e,..)$ ,

$$\langle n,n\rangle - \langle n,u\rangle^2 = \sin_\kappa^2\circ \mathrm{dist}(..,c_u(\mathbb{R})) \quad for\ all\ u\in S^2,$$

where $c_u:\mathbb{R} \longrightarrow \mathbb{M}$ denotes the geodesic of $\mathbb{M}$ with $c_u(0)=e$ and $\dot{c}_u(0)^{\rightarrow}=(0,u)$.

A proof can be obtained using [2],(2.13) and – for the last equation – in addition by some straightforward computations.

**(iii)** We introduce the $\mathbb{R}^3$-valued left invariant Pfaffian forms $\omega,v$ on G by requiring $\omega^o := \omega|_{id}$ , $v^o := v|_{id} : \mathfrak{g} \longrightarrow \mathbb{R}^3$ to be the linear projections given by $\omega_i^o := x_{i+2,i+1}|\mathfrak{g}$ with indices mod 3 in {1,2,3} and $v_i^o := x_{i0}|\mathfrak{g}$ for $i\in\{1,2,3\}$ (cf. (i)).

Then $(\omega_1^o,\omega_2^o,\omega_3^o,v_1^o,v_2^o,v_3^o)$ is a basis for $\mathfrak{g}^*$ (cf. [2],(2.28)) and from [2],$(2.28_0)$ follows by straightforward computations:

$$\omega([\xi,\eta]) = \omega(\xi)\times\omega(\eta) + \kappa\cdot v(\xi)\times v(\eta) \quad and$$
$$v([\xi,\eta]) = \omega(\xi)\times v(\eta) + v(\xi)\times\omega(\eta) \quad for\ all\ \xi,\eta\in\mathfrak{g},$$

in particular $\omega|\mathfrak{h}:\mathfrak{h} \longrightarrow (\mathbb{R}^3,..\times..)$ is a Lie algebra isomorphism, where $\mathfrak{h}$ (=ker($v^o$), cf. [2],$(2.29_2)$) denotes the Lie subalgebra of $\mathfrak{g}$ corresponding to the isotropy subgroup of G for the point $e\in\mathbb{M}$.

For geometric interpretations see below (iv),(v) and § 2.iii.

**(iv)** For $\xi\in\mathfrak{g}$ let $\omega_e(\xi)\in T_e\mathbb{M}$ denote the vector with $\omega_e(\xi)^{\rightarrow} = (0,\omega(\xi))$ . Then one checks easily, that $\omega_e(\xi)\times..:T_e\mathbb{M} \longrightarrow T_e\mathbb{M}$ is the velocity field of the one-parameter group of rotations of (the 3-dim. oriented

Euclidean vector space) $T_e\mathbb{M}$ generated by the Killing-orthogonal projection of $\xi$ onto $\mathfrak{d}$ (cf. (iii)). More precisely use the projection of the splitting $(2.29_2)$ in [2].

**(v)** For $\xi\in\mathfrak{g}$ let $v_e(\xi)\in T_e\mathbb{M}$ denote the vector with $v_e(\xi)^{\rightarrow} = (0,v(\xi))$ . Then again it follows easily, that $v_e(\xi)$ is the velocity vector at time 0 of the orbit of the point e under the one-parameter subgroup of G generated by $\xi$.

**(vi)** For $\xi\in\mathfrak{g}$ and $q\in\mathbb{M}$ the action of $\mathfrak{g}$ on $\mathbb{R}^4$ by matrix multiplication gives (cf. (ii),(iii) and use [2],(2.28)):

$$\xi\cdot q = (-\kappa\cdot\langle v(\xi),n(q)\rangle,(\omega(\xi)\times n(q))+x_0(q)v(\xi)) \in \mathbb{R}\times\mathbb{R}^3 = \mathbb{R}^4 .$$

## 2. A model for the rigid body in $\mathbb{M} := \mathbb{M}^3_\kappa$

**(i)** We describe the body in its initial position by its *distribution of mass* m. This is a (non-negative) Borel measure m on $\mathbb{M}$ with $0 < m(\mathbb{M}) < \infty$ , which has to satisfy additional conditions (e.g. for getting a well defined center of mass resp. definiteness of $\mathcal{I}_m$ , see below § 4.i,ii resp. § 2.iv and § 5.vi). In order to avoid longer discussions, depending mainly on too general "admissible measures", we shall restrict ourselves to measures m satisfying the following two conditions:

a) *Let* m *be* 3-*dim. extended, i.e.* $m(\mathbb{M}\setminus E)>0$ *for all* 2-*dim. totally geodesic submanifolds* E *of* $\mathbb{M}$.

b) *There exists* $p_0\in\mathbb{M}$ *and* $\varepsilon\in]0,\frac{1}{8}\cdot\pi_\kappa[$ (cf. [2], § 1.iv), *such that* m *is concentrated in the ball* $U_\varepsilon(p_0)$, *i.e* $m(\mathbb{M}\setminus U_\varepsilon(p_0))=0$.

Note, that $\pi_\kappa=\mathrm{diam}(\mathbb{M})$, and physics tells us that classical mechanics does not apply to objects of a size comparable to the diameter of the universe. So condition b) is not a serious restriction.

The condition a) is easily seen to be equivalent to the following one:

ā) *If* $\varphi:\mathbb{M}\longrightarrow\mathbb{R}$ *is any* m-*measurable non-negative function on* $\mathbb{M}$, *the zeros of which are contained in a* 2-*dim. totally geodesic submanifold of* $\mathbb{M}$, *then* $\int_{\mathbb{M}}\varphi(q)dm(q) > 0$ .

(ii) Any given position of the body can be obtained by exactly one $f \in G$ (cf. § 1.i) transporting the body from the initial position to the given one. This identifies (as usual) the *configuration space* of the rigid body with the Lie group G of all orientation preserving isometries.

A "resting" observer describes the body by the mass distribution $f^*m$ (with $(f^*m)(B):=m(f(B))$ for all Borel subsets $B \subseteq \mathbb{M}$) after it was transported with $f \in G$ from the initial position. If we denote by $\mathfrak{B}\mathbb{M}$ the set of all non-negative Borel measures on $\mathbb{M}$ satisfying the conditions a) and b) from (i), then this actually allows the "resting" observer – from a more sophisticated point of view – to identify the "rigid body in $\mathbb{M}$ with mass distribution m" itself with the subset $\{(f,f^*m) \mid f \in G\}$ of $G \times \mathfrak{B}\mathbb{M}$, which is the orbit of (id,m) under the the action of the group G from the right on $G \times \mathfrak{B}\mathbb{M}$, and the different elements $(f,f^*m)$ of that orbit are then viewed as the different possible "positions" of the rigid body in $\mathbb{M}$. The usage of the group action of G on G from the right can be motivated by (iii).

An observer moving along with the body describes the body in every possible position by the mass distribution m introduced in (i). Therefore we will most of the time use descriptions relative to an observer moving along with the body rather than that moving relative to a "resting" observer.

(iii) Let $\gamma : J \longrightarrow G$ be a smooth curve in the Lie group of motions (:= orientation preserving isometries) of $\mathbb{M}$. Using (ii) we consider $\gamma$ as a *process of motion* of the rigid body in $\mathbb{M}$. Correspondingly the tangent vector field $\dot{\gamma} : J \longrightarrow TG$ of $\gamma$ (which allows to recapture $\gamma$ as $\pi \circ \dot{\gamma}$, using the canonical projection $\pi : TG \longrightarrow G$ ) is called the *(evolution-) path of states* of the given process of motion $\gamma$.

The so-called *body representation* on $T\mathbb{M}$ of the process of motion $\gamma$ (resp. of its path of states $\dot{\gamma}$) is the following time-dependent vector field $\mathbb{V}^{\gamma}$ on $\mathbb{M}$:

For $(t,p) \in J \times \mathbb{M}$ let $\mathbb{V}^{\gamma}_t(p)$ be the vector in $T_p\mathbb{M}$, which is mapped by the differential of the isometry $\gamma(t)\cdot \ldots : \mathbb{M} \longrightarrow \mathbb{M}$ onto the velocity vector at time t of the orbit $\gamma \cdot p : J \longrightarrow \mathbb{M}$ of the point p under $\gamma$, i.e.

$$\mathbb{V}_t^\gamma(p) := (\gamma(t)^{-1}\cdot\ldots)_*(\gamma\cdot p)^\cdot(t) \in T_p\mathbb{M} \quad \textit{for all } (t,p)\in J\times\mathbb{M}.$$

Using the notations from § 1.i, this vector field can be computed by $\mathbb{V}_t^\gamma(p)^{\rightarrow} = \dot\gamma(t)_L\cdot p \in \mathbb{R}^4$ *for all* $(t,p)\in J\times\mathbb{M}$.

Now we consider the ball $U := U_\varepsilon(p_0)$ from (i)b) as an open neighborhood of the rigid body and, if $(A_1,A_2,A_3)$ is an orthonormal frame field on U (the bundle $T\mathbb{M}$ is trivial over the cell U !), then we define the *$\gamma$-moving frame* $(E_1,E_2,E_3)$ along the orbit path $J\times U \longrightarrow \mathbb{M}$ ( $(t,p) \mapsto \gamma(t)\cdot p$ ) of the body by

$$E_i(t,p) := (\gamma(t)\cdot\ldots)_*A_i(p) \quad \textit{for all } (t,p)\in J\times U,\ i\in\{1,2,3\}.$$[2]

Then evidently (since $\gamma(t)\cdot\ldots:\mathbb{M}\longrightarrow\mathbb{M}$ is an isometry)

$$g((\gamma\cdot p)^\cdot(t),E_i(t,p)) = g(\mathbb{V}_t^\gamma(p),A_i(p))$$
$$\textit{for all } (t,p)\in J\times U \textit{ and } i\in\{1,2,3\}.$$

This allows us to interpret, for a fixed time $t\in J$, the vector field $\mathbb{V}_t^\gamma$ on $\mathbb{M}$ as the distribution at the moment t of the velocity (of the various orbits under the action of $\gamma$) in $\mathbb{M}$ relative to an observer moving along with the body under $\gamma$.

Together with § 1.iv,v this motivates the following terminology (cf. § 1.i,iii):

$\omega(\dot\gamma) = \omega^o(\dot\gamma_L):J\longrightarrow\mathbb{R}^3$ is called the *angular velocity* of $\gamma$ *in body coordinates* with respect to e, $v(\dot\gamma) = v^o(\dot\gamma_L):J\longrightarrow\mathbb{R}^3$ is called the *translation velocity* of $\gamma$ *in body coordinates*

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[2] Oppositely, if we are given a point $p\in\mathbb{M}$, a positively oriented orthonormal frame $(e_1,e_2,e_3)$ of $T_p\mathbb{M}$ and a positively oriented orthonormal $C^\infty$ frame field $(E_1,E_2,E_3)$ along a $C^\infty$ curve $c:J\longrightarrow\mathbb{M}$, then these data determine (G operates simply transitive on the oriented frame bundle of $\mathbb{M}$) a unique $C^\infty$ curve $\gamma:J\longrightarrow G$, such that $\gamma(t)\cdot p = c(t)$ and $(\gamma(t)\cdot\ldots)_*e_i = E_i(t)$ for all $t\in J$. This explains, why so often in the classical literature a process of motion of a rigid body in $\mathbb{E}^3$ (with the distinguished $p:=o$ and the canonical frame $(e_1,e_2,e_3)$ in $o$) is simply described by a certain positively oriented frame field $(E_1,E_2,E_3)$ along some $C^\infty$ curve in $\mathbb{E}^3$ as the moving reference system ("Gang-Bezugssystem" in German).

with respect to e.

The so called *space representation* on TM of the process of motion $\gamma$ (resp. of its path of states $\dot{\gamma}$) is the following time-dependent vector field $W^{\gamma}$ on M:

For $(t,p)\in J\times M$ let $W_t^{\gamma}(p)\in T_pM$ be the velocity vector at time t of the orbit under $\gamma$ of that point of M, whose orbit passes at time t just the point p, i.e.

$$W_t^{\gamma}(p) := (\gamma\cdot(\gamma(t)^{-1}\cdot p))^{\cdot}(t) \in T_pM \quad \textit{for all} \ (t,p)\in J\times M.$$

For fixed $t\in J$ we can interpret the vector field $W_t^{\gamma}$ on M as the distribution, at the moment t in time, of the velocity (of various orbits under the action of $\gamma$) in M relative to a *"resting" observer*. Using the notations from § 1.i, this vector field can be computed by

$$W_t^{\gamma}(p)^{\rightarrow} = \dot{\gamma}(t)_R\cdot p \in \mathbb{R}^4 \quad \textit{for all} \ (t,p)\in J\times M,$$

where $.._R:TG\longrightarrow\mathfrak{g}$ denotes the right parallelism of G, given by $\upsilon_R := (R_f^{-1})_*\upsilon$, if $f\in G$ and $\upsilon\in T_fG$.

(iv) We define the function $\mathcal{I}_m:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathbb{R}$ (cf. § 1.i, § 2.i) by

$$\mathcal{I}_m(\xi,\eta) := \int_M g((\exp(x\xi)\cdot q)^{\cdot}(0),(\exp(x\eta)\cdot q)^{\cdot}(0))dm(q) =$$
$$= \int_M (\xi\cdot q|\eta\cdot q)_{\kappa} dm(q) \quad \textit{for all} \ \xi,\eta\in\mathfrak{g}.$$

The integrals exist because of the compact support of m and the second equation follows from $(\exp(x\xi)\cdot q)^{\cdot}(0)^{\rightarrow} = \xi\cdot q$ for $\xi\in\mathfrak{g}$ and $q\in M$ and the definition of g. Then $\mathcal{I}_m$ is obviously a symmetric positive semidefinite $\mathbb{R}$-bilinear form on $\mathfrak{g}$ and we will see later in § 5.vi, that the conditions imposed in (i) on m imply the definiteness of $\mathcal{I}_m$. Hence $\mathcal{I}_m$ is an *Euclidean inner product* on the Lie algebra $\mathfrak{g}$ of G, induced by the rigid body.

(v) Let $\mathcal{L}_m:TG\longrightarrow\mathbb{R}$ denote one half times the square of the norm, given by the left-invariant Riemannian metric on G induced by $\mathcal{I}_m$ (cf. (iv)). Then for a smooth curve $\gamma:J\longrightarrow G$ one obtains

$$\mathcal{L}_m\circ\dot{\gamma} = \frac{1}{2}\cdot\mathcal{I}_m(\dot{\gamma}_L,\dot{\gamma}_L) = \frac{1}{2}\cdot\int_{\mathbb{M}}(\dot{\gamma}_L\cdot q|\dot{\gamma}_L\cdot q)_\kappa dm(q) =$$

$$= \frac{1}{2}\cdot\int_{\mathbb{M}}(\gamma\cdot\dot{\gamma}_L\cdot q|\underbrace{\gamma\cdot\dot{\gamma}_L\cdot q}_{=\gamma'\cdot q = ((\gamma\cdot q)^\cdot)^\rightarrow})_\kappa dm(q)$$

$$= \frac{1}{2}\cdot\int_{\mathbb{M}}g((\gamma\cdot q)^\cdot,(\gamma\cdot q)^\cdot)dm(q) \ .$$

So, interpreting $\gamma$ as a process of motion of the rigid body, $\mathcal{L}_m\circ\dot{\gamma}$ is the *kinetic energy* of the rigid body (with mass distribution m) moving under $\gamma$, and therefore $\mathcal{L}_m$ is the *Lagrangian* of the rigid body in the absence of external forces.

**(vi)** According to (iv),(v) and [1],3.7.1 a curve $\gamma:J \longrightarrow G$ is a process of motion of the rigid body without external forces iff $\gamma$ is a geodesic in G with respect to the left-invariant Riemannian metric on G induced by $\mathcal{I}_m$.

**(vii)** The group action of G on G from the left and the Lagrangian $\mathcal{L}_m$ satisfy the hypotheses of E. Noether's theorem (see [1],4.2.14). This gives us an Ad*-equivariant

*momentum mapping* $\mathcal{J}^m:TG \longrightarrow \mathfrak{g}^*$ $(v \mapsto \mathcal{J}^m_v)$

defined by $\mathcal{J}^m_v(\xi) := \mathbb{F}_v\mathcal{L}_m(\xi_G|_f)$ for $f\in G$, $v\in T_fG$, $\xi\in\mathfrak{g}$, where according to (iv),(v) and [1],3.6.10,4.1.25.a we have $\mathbb{F}_v\mathcal{L}_m = \mathcal{I}_m(v_L,\cdot\cdot_L)$ and $\xi_G|_f = (R_f)_*\xi$ . Therefore we obtain

$$\mathcal{J}^m_v = \mathcal{I}_m(v_L,Ad_f^{-1}\cdot\cdot):\mathfrak{g} \longrightarrow \mathbb{R} \quad \textit{for } f\in G \textit{ and } v\in T_fG,$$

and the Ad*-equivariance of $\mathcal{J}^m$ means (cf. [1],4.2.6)

$$\mathcal{J}^m_{(L_f)_*v}(\xi) = \mathcal{J}^m_v(Ad_f^{-1}\xi) \quad \textit{for all } v\in TG,\ f\in G,\ \xi\in\mathfrak{g}.$$

If $\gamma:J \longrightarrow G$ is a process of motion of the rigid body without external forces, then

$$\mathcal{J}^m_{\dot{\gamma}} = \mathcal{I}_m(\dot{\gamma}_L,Ad_\gamma^{-1}\cdot\cdot):J \longrightarrow \mathfrak{g}^* \ (t \mapsto \mathcal{J}^m_{\dot{\gamma}(t)}) \quad \textit{is constant on } J.$$

**(viii)** We define the matrix $\theta^m\in\mathfrak{M}(3,\mathbb{R})$ implicitly by

$$\langle\omega(\xi),\theta^m\cdot\omega(\eta)\rangle = \mathcal{I}_m(\xi,\eta) \quad \textit{for all } \xi,\eta\in\mathfrak{h},$$

where $\mathfrak{h}$ denotes the Lie algebra of the isotropy subgroup of the point $e\in\mathbb{M}$ in G. $\omega|\mathfrak{h}:\mathfrak{h} \longrightarrow \mathbb{R}^3$ is a vector space isomorphism (cf. § 1.iii). Together with (iv) follows, that $\theta^m$

is well defined, symmetric and positive. We call $\theta^m$ the *tensor of inertia* of m with respect to e. More precisely this tensor is an endomorphism of $T_e\mathbb{M}$ and $\theta^m$ is its representation matrix (with respect to the basis of $T_e\mathbb{M}$ mapped by $..^{\to}$ onto $(e_1,e_2,e_3)$).

## 3. Geometry of orbits under a process of motion

We use the notations introduced in § 1. Let $\gamma:J \longrightarrow G$ denote a $C^3$ curve in the Lie group of motions of the standard space $\mathbb{M}$. We study the geometry of the orbit $\gamma\cdot e:J \longrightarrow \mathbb{M}$ of the point $e\in\mathbb{M}$ under $\gamma$.

(i) Let $(a_1,a_2,a_3)$ denote the orthonormal frame of $(T_e\mathbb{M},g_e)$ with $a_i^{\to} = e_i$ for $i\in\{1,2,3\}$. For $t\in J$ and $i\in\{1,2,3\}$ we define $E_i(t) := \gamma(t)_* a_i \in T_{\gamma(t)\cdot e}\mathbb{M}$. Then $(E_1,E_2,E_3)$ is a $C^3$ positively oriented orthonormal frame field in $\mathbb{M}$ along $\gamma\cdot e$ (classically called “moving reference frame”, in German: “Gang-Bezugssystem”, see footnote 2) and (note $G\subseteq GL(4,\mathbb{R})$)

$$E_i(t)^{\to} = \gamma(t)\cdot e_i \quad \textit{for all } t\in J \textit{ and } i\in\{1,2,3\}.$$

(ii) The velocity field of the orbit $\gamma\cdot e$ is given by

$$(\gamma\cdot e)^{\cdot} = \sum_{i=1}^{3} v_i(\dot\gamma)E_i \ .$$

[Because

$$((\gamma\cdot e)^{\cdot})^{\to} = \gamma'\cdot e \underset{\S\ 1.i}{=} \gamma\cdot\dot\gamma_L\cdot e \underset{[2],(2.28)}{=} \gamma\cdot(0,v(\dot\gamma)) = \sum_{i=1}^{3} v_i(\dot\gamma)\gamma\cdot e_i \ .]$$

(iii) **Lemma.** For every $C^1$ map $z:J \longrightarrow \mathbb{R}^3$ one has

$$\nabla_\partial\Big(\sum_{i=1}^{3} z_iE_i\Big) = \sum_{i=1}^{3} [z'+(\omega(\dot\gamma)\times z)]_iE_i \ ,$$

where $\nabla$ is the Levi-Civita covariant derivative in $\mathbb{M}$ and $\partial := \frac{d}{dx}$ the canonical vector field on $\mathbb{R}$ ($\supseteq J$).

**Proof.** $$g\Big(\nabla_\partial\Big(\sum_{j=1}^{3} z_jE_j\Big),E_i\Big) = \Big(\Big[\sum_{j=1}^{3} z_j\gamma\cdot e_j\Big]' \,\Big|\, \gamma\cdot e_i\Big)_\kappa =$$

$$\underset{\S\ 1.i}{=} \Big(\gamma\cdot\sum_{j=1}^{3}[z_j'e_j + z_j\dot\gamma_L\cdot e_j] \,\Big|\, \gamma\cdot e_i\Big)_\kappa =$$

$$= z_i' + \sum_{j=1}^{3}(\dot\gamma_L\cdot e_j|e_i)_\kappa\cdot z_j =$$

$= [z' + (\omega(\dot\gamma)\times z)]_i$ *for all* $i\in\{1,2,3\}$.

**(iv) Remark.** The results (ii),(iii) shed some light on the relation between the two $\mathbb{R}^3$-valued left invariant Pfaffian forms v resp. $\omega$ on G on the one side and the basis forms $\theta_i$ resp. the connection forms $\omega_{ij}$ of $\mathbb{M}$ ($i,j\in\{1,2,3\}$) on the other side.

Here the Pfaffian forms $\theta_i$ resp. $\omega_{ij}$ ($i,j\in\{1,2,3\}$) are understood to be defined on the principal bundle $\pi:F\mathbb{M}\longrightarrow\mathbb{M}$ of the Riemannian manifold $\mathbb{M}$ (consisting of all orthonormal frames tangent to $\mathbb{M}$), as usual. If $\mathbb{E}_i:F\mathbb{M}\longrightarrow T\mathbb{M}$ for $i\in\{1,2,3\}$ are the canonical projections ($\mathbb{E}_i$ assigning to each orthonormal frame tangent to $\mathbb{M}$ its $i^{th}$ component vector), then the definitions of the $\theta_i$ resp. $\omega_{ij}$ amount to the following equations: For any differentiable manifold N, any $C^1$ map $\ell:N\longrightarrow F\mathbb{M}$ and any vector field $X\in\mathcal{X}(N)$ one has for all $i,j\in\{1,2,3\}$:

$\theta_i(\ell_*X) = g(\mathbb{E}_i\circ\ell,(\pi\circ\ell)_*X)$ *resp.* $\omega_{ij}(\ell_*X) = g(\mathbb{E}_i\circ\ell,\nabla_X(\mathbb{E}_j\circ\ell))$.

Therefore, if $E := (E_1,E_2,E_3):J\longrightarrow F\mathbb{M}$ is the orthonormal frame field along $\gamma\cdot e$ introduced in (i), then $\pi\circ E = \gamma\cdot e$ and the results (ii),(iii) are equivalent to saying, for all $i\in\{1,2,3\}$:

$$\theta_i(\dot E) = g(E_i,(\gamma\cdot e)^\cdot) = v_i(\dot\gamma) \quad \textit{resp.}$$

$$\omega_{i+1,i+2}(\dot E) = g(E_{i+1},\nabla_\partial E_{i+2}) = -\,\omega_i(\dot\gamma) \quad \textit{with indices } \bmod 3.$$

**(v)** The acceleration field of the orbit $\gamma\cdot e$ is given by

$$\nabla_\partial(\gamma\cdot e)^\cdot = \sum_{i=1}^{3}[(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma))]_i E_i$$

(use (ii),(iii)).

**(vi)** The orbit $\gamma\cdot e:J\longrightarrow\mathbb{M}$ is a (constant speed) geodesic iff $(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma)) = o$ .

**(vii)** The scalar invariants – velocity $\upsilon_{\gamma\cdot e}$, curvature $\kappa_{\gamma\cdot e}$ and torsion $\tau_{\gamma\cdot e}$ – of the orbit $\gamma\cdot e$ can be computed by

$$\upsilon_{\gamma\cdot e} = \|v(\dot\gamma)\| \ ,$$

$$\kappa_{\gamma\cdot e} = (\upsilon_{\gamma\cdot e})^{-3}\cdot\|v(\dot\gamma)\times[(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma))]\| \ ,$$

$$\tau_{\gamma\cdot e} = (\kappa_{\gamma\cdot e}\cdot\upsilon^3_{\gamma\cdot e})^{-2}\cdot\det(v(\dot\gamma),(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma)),u) \ ,$$

where

$$u := [(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma))]' + \omega(\dot\gamma)\times[(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma))]$$

(use (i),(ii),(v) and $u_i = g(\nabla_\partial\nabla_\partial(\gamma\cdot e)^{\cdot}, E_i)$ , cf. (iii)).

**(viii)** If any tangential vector $a\in T_e\mathbb{M}$ at $\mathbb{M}$ in **e** is given and $a\in\mathbb{R}^3$ characterized by $a^{\rightarrow} = (0,a)$ , then the translation of $a$ by $\gamma$ is $\nabla$-parallel, i.e. the vector field along $\gamma\cdot e$ defined by $t \mapsto \gamma(t)_* a$ is a parallel field (in the sense of Levi-Civita in $\mathbb{M}$), iff $\omega(\dot\gamma)\times a = o$ (use (i),(iii)).

4. The center of mass of the rigid body

**(i) Theorem.** Let m be given as in § 2.i and define the function $F:\mathbb{M} \longrightarrow \mathbb{R}$ by $F(p) := \int_{\mathbb{M}} \sin_\kappa^2(d(p,q))\,dm(q)$ for $p\in\mathbb{M}$ (in German literature called the "second order moment" of m with respect to p). Then one has:

a) F is a $C^\infty$ function.

b) $\operatorname{grad}_p F = -2\cdot\int_{\mathbb{M}} \frac{\cos_\kappa\cdot\sin_\kappa}{x}(d(p,q))\cdot \exp_p^{-1}(q)\,dm(q)$ for all $p\in\mathbb{M}$,
where $\exp_e^{-1}$ stands for $(\exp_e | U_{\pi_\kappa}(o))^{-1}$.

c) If $p_0$ and $\varepsilon$ are chosen as in § 2.i, then the ball $U_\varepsilon(p_0)$ contains exactly one critical point S of F and F attains its absolute minimum at S.

d) If S is chosen as in c), $c:\mathbb{R} \longrightarrow \mathbb{M}$ is any unit-speed geodesic with $c(0) = S$ and if we define

$$\mathcal{M}(\dot c(0)) := \int_{\mathbb{M}} [\cos_\kappa^2(d(S,q)) - \kappa\cdot(\dot c(0))^{\rightarrow}|q)_\kappa^2]\,dm(q) \ ,$$

then $(F\circ c)(t) = F(S) + \mathcal{M}(\dot c(0))\cdot\sin_\kappa^2(t)$ *for all* $t\in\mathbb{R}$
and $\mathcal{M}(\dot c(0)) > 0$ .

**(ii) Definition.** The point S of (i)c) is called the *center of mass* of m.

**(iii) Corollary** (use § 1.ii). If the center of mass of m is the point $e\in\mathbb{M}$, then

$$\int_{\mathbb{M}} x_0(q)\,\mathfrak{r}(q)\,dm(q) = o \ .$$

**Remark.** In [3] similar results were proved for the function $p \mapsto \int_{\mathbb{M}} \psi_\kappa(d(p,q))dm(q)$ , and like there, usage of the embedding of $\mathbb{M}$ in $\mathbb{R}^4$ (cf. [2], § 2.ii) shortens the proof.

**Proof of (i):**

**Ad a).** Because m has compact support (cf. § 2.i), it suffices to prove that $\sin_\kappa^2 \circ d: \mathbb{M}\times\mathbb{M} \longrightarrow \mathbb{R}$ is a $C^\infty$ function. The case $\kappa=0$ is trivial (recall $\sin_0(x)=x$) and [2],(1.6),(2.14) imply in case of $\kappa\neq 0$:

$$\sin_\kappa^2 \circ d = \frac{1}{\kappa}\cdot(1-\cos_\kappa^2\circ d) = \frac{1}{\kappa}\cdot(1 - \langle ..,..\rangle_\kappa^2)\,|\mathbb{M}\times\mathbb{M} .$$

**Ad b).** It is sufficient to prove for $q\in\mathbb{M}$, for a unit-speed geodesic $c:\mathbb{R} \longrightarrow \mathbb{M}$ and $t\in\mathbb{R}$:

$(\sin_\kappa^2\circ d(..,q)\circ c)'(t) =$

$$= \begin{cases} 0 & \textit{if } (\kappa>0 \textit{ and}) \;\; d(c(t),q) = \pi_\kappa \\ -2\cdot\dfrac{\cos_\kappa\cdot\sin_\kappa}{x}(d(c(t),q))\cdot g(\exp_{c(t)}^{-1}(q),\dot{c}(t)) & \textit{otherwise} \end{cases}$$

In case $\kappa=0$ both sides are equal to $2\cdot(c(t)-q|\dot{c}(t)^{\rightarrow})_\kappa$ . In case $d(c(t),q) = \pi_\kappa$ the function $\sin_\kappa^2\circ d(..,q)\circ c$ attains its absolute minimum (of value 0) at t. In the other cases the assertion follows from (cf. [2],(2.2),(2.14))

$$[\sin_\kappa^2(d(c,q))]' = -\frac{2}{\kappa}\cdot\langle c,q\rangle_\kappa\cdot\langle\dot{c}^{\rightarrow},q\rangle_\kappa = -2\cdot\cos_\kappa(d(c,q))\cdot(\dot{c}^{\rightarrow}|q)_\kappa$$

and using [2],(2.2),$(2.8_3)$ and the identity (cf. [2],(2.13))

$$q = \cos_\kappa(d(c(t),q))\cdot c(t) + \frac{\sin_\kappa}{x}(d(c(t),q))\cdot\exp_{c(t)}^{-1}(q)^{\rightarrow} .$$

**Ad c).** For all $p,q\in\mathbb{M}$ with $d(q,p_0) < d(p,p_0) < \frac{1}{2}\cdot\pi_\kappa$ one can show $g(\exp_p^{-1}(q),\exp_p^{-1}(p_0)) > 0$ (see e.g. [3],p.90,92). Together with b) and (cf. § 2.i) $\varepsilon < \frac{1}{4}\cdot\pi_\kappa$ this implies for all $p\in\partial U_\varepsilon(p_0)$:

$g(\mathrm{grad}_p F,\exp_p^{-1}(p_0)) < 0$ , $\exp_p^{-1}(p_0)$ pointing "inwards" $U_\varepsilon(p_0)$.

Therefore, $F|\overline{U_\varepsilon(p_0)}$ must attain its minimum at an interior point S of $\overline{U_\varepsilon(p_0)}$ and d) will complete the proof of c), since every point $q\in U_\varepsilon(p_0)\setminus\{S\}$ can be joined with S by a geodesic and $\varepsilon < \frac{1}{4}\cdot\pi_\kappa$ implies $d(q,S) < \frac{1}{2}\cdot\pi_\kappa$ , hence $(\sin_\kappa^2)'(d(q,S)) > 0$ .

**Ad d).** We shall only use the fact that S is any critical point of F contained in $U_\varepsilon(p_0)$. Choose $q\in M$ and any unit-speed geodesic $c:\mathbb{R} \longrightarrow \mathbb{M}$. Then one has

$$[\sin_\kappa^2(d(c,q))]'' = 2\cdot\cos_\kappa^2(d(c,q)) - 2\kappa\cdot(\dot{c}^{\rightarrow}|q)_\kappa^2 .$$

[Because, in case $\kappa=0$ both sides are equal to 2. Otherwise one computes $[\sin_\kappa^2(d(c,q))]'' = -\frac{2}{\kappa}\cdot\langle\dot{c}^{\rightarrow},q\rangle_\kappa^2 - \frac{2}{\kappa}\cdot\langle c,q\rangle_\kappa\cdot\langle c'',q\rangle_\kappa$, which implies the desired result, using that $c''$, the second order derivative of c in $\mathbb{R}^4$, equals $-\kappa c$.]

If $d(c(0),q) < \pi_\kappa$ and $t\in\mathbb{R}$, then

$$(\sin_\kappa^2\circ d(..,q)\circ c)''(t) =$$
$$= 2\cdot\cos_\kappa(2t)\cdot[\cos_\kappa^2(d(c(0),q)) - \kappa\cdot(\dot{c}(0)^{\rightarrow}|q)_\kappa^2] +$$
$$+ 4\kappa\cdot\sin_\kappa(2t)\cdot\frac{\cos_\kappa\cdot\sin_\kappa}{x}(d(c(0),q))\cdot g(\exp_{c(0)}^{-1}(q),\dot{c}(0)) .$$

[Because, in case $\kappa=0$ both sides are equal to 2. Otherwise we have $(\sin_\kappa^2\circ d(..,q)\circ c)''(t) = 2\kappa^2\cdot(c(t)|q)_\kappa^2 - 2\kappa\cdot(\dot{c}(t)^{\rightarrow}|q)_\kappa^2$. Inserting $c(t) = \cos_\kappa(t)\cdot c(0) + \sin_\kappa(t)\cdot\dot{c}(0)^{\rightarrow}$ (cf. [2], (2.13)) and $\dot{c}(t)^{\rightarrow} = -\kappa\cdot\sin_\kappa(t)\cdot c(0) + \cos_\kappa(t)\cdot\dot{c}(0)^{\rightarrow}$ (cf. [2],(1.1')) and using the bilinearity of $(..|..)_\kappa$ and [2],(1.7) we get the proposed equation.]

In case $c(0) = S$ we get from the last equation (using b) and $\mathrm{grad}_S F=o$):

$$(F\circ c)''(t) = \int_{\mathbb{M}}(\sin_\kappa^2\circ d(..,q)\circ c)''(t)dm(q) =$$
$$= 2\cdot\cos_\kappa(2t)\cdot\mathcal{M}(\dot{c}(0)) \quad \textit{for all } t\in\mathbb{R}.$$

This yields (observing $(F\circ c)'(0) = g(\mathrm{grad}_S F,\dot{c}(0)) = 0$)

$$(F\circ c)'(t) = \sin_\kappa(2t)\cdot\mathcal{M}(\dot{c}(0)) \quad \textit{for all } t\in\mathbb{R},$$

and thereby the first assertion of d).

It remains to show $\mathcal{M}(\dot{c}(0)) > 0$. This follows, in case $\kappa\le 0$, from $\cos_\kappa^2(d(S,..)) - \kappa\cdot(\dot{c}(0)^{\rightarrow}|..)_\kappa^2 \ge 1$ *on* $\mathbb{M}$ and in case $\kappa>0$ from $m(\mathbb{M}\setminus U_{2\varepsilon}(S)) = 0$ (cf. c) and § 2.i) and

$$\cos_\kappa^2(d(S,..))-\kappa\cdot(\dot{c}(0)^{\rightarrow}|..)_\kappa^2 =$$
$$= \cos_\kappa^2(d(S,..))-\cos_\kappa^2(d(c(\tfrac{1}{2}\cdot\pi_\kappa),..)) \ge$$
$$\ge \cos_\kappa^2(2\varepsilon) - \tfrac{1}{2} > 0 \quad \textit{on } U_{2\varepsilon}(S),$$

which depends on $\dot{c}(0)^{\rightarrow} = \sqrt{\kappa}\cdot c(\frac{1}{2}\cdot\pi_\kappa)$ (use [2],(2.13)) and our choice of $\varepsilon < \frac{1}{8}\cdot\pi_\kappa$ .

## 5. Preparations for the equations of motion

We use the notations introduced in § 1 and § 2.

**(i) Lemma.**

$$\langle w,\theta^m\cdot w\rangle = \langle w,w\rangle\cdot\int_{\mathbb{M}}\sin_\kappa^2(\mathrm{dist}(q,c_w(\mathbb{R})))dm(q) \quad \textit{for all } w\in\mathbb{R}^3,$$

where $c_w:\mathbb{R}\longrightarrow\mathbb{M}$ denotes the geodesic with $c_w(0) = e$ and $\dot{c}_w(0)^{\rightarrow} = (0,w)$ .

**Proof.** The case $w=o$ is trivial. Let $w\neq o$ and $\xi\in\mathfrak{b}$ with $\omega(\xi) = w$ (cf. § 1.iii). Let $\gamma:\mathbb{R}\longrightarrow G$ denote the one-parameter group generated by $\xi$. Then

$$\langle w,\theta^m\cdot w\rangle = \mathcal{T}_m(\xi,\xi) = \mathcal{T}_m(\dot{\gamma}_L,\dot{\gamma}_L) = \int_{\mathbb{M}} g((\gamma\cdot q)^{\cdot},(\gamma\cdot q)^{\cdot})dm(q)$$

(use § 2.v,viii) and the orbit $\gamma\cdot q$ traces a "plane" circle with (measured in $\mathbb{M}$) radius $\mathrm{dist}(q,c_w(\mathbb{R}))$ and constant angular velocity $\|w\|$, hence $\|(\gamma\cdot q)^{\cdot}\| = \|w\|\cdot\sin_\kappa(\mathrm{dist}(q,c_w(\mathbb{R})))$ .

**(ii) Lemma.**

$$\langle w,\theta^m\cdot\tilde{w}\rangle = \int_{\mathbb{M}}[\langle w,\tilde{w}\rangle\cdot\langle n(q),n(q)\rangle - \langle w,n(q)\rangle\cdot\langle\tilde{w},n(q)\rangle]dm(q) =$$

$$= \int_{\mathbb{M}}\langle w\times n(q),\tilde{w}\times n(q)\rangle dm(q) \quad \textit{for all } w,\tilde{w}\in\mathbb{R}^3.$$

**Proof.** The second equation follows by standard identities about the cross product in $(\mathbb{R}^3,\langle..,..\rangle)$. Both sides of the first equation are bilinear and symmetric in $w,\tilde{w}\in\mathbb{R}^3$ (cf. § 2.iv,viii). Hence it is sufficient, to consider $w=\tilde{w}\in\mathbb{R}^3$. The case $w=o$ is trivial and in case $w\neq o$ we conclude from § 1.ii

$$\langle w,w\rangle\cdot\sin_\kappa^2(\mathrm{dist}(q,c_w(\mathbb{R}))) = \langle w,w\rangle\cdot\langle n(q),n(q)\rangle - \langle n(q),w\rangle^2$$

for all $q\in\mathbb{M}$, where $c_w$ is the geodesic introduced in (i).

**(iii) Lemma.** For all $i,j\in\{1,2,3\}$ we have

$$\theta^m_{ij} = \int_{\mathbb{M}}[(x_1^2+x_2^2+x_3^2)\delta_{ij}-x_ix_j]dm ,$$

$$[m(\mathbb{M})\cdot I - \kappa\cdot\theta^m]_{ij} = \int_{\mathbb{M}}(x_0^2\delta_{ij}+\kappa x_ix_j)dm ,$$

$$[2\cdot\theta^m - \mathrm{trace}(\theta^m)\cdot I]_{ii} = -2\cdot\int_{\mathbb{M}}x_i^2dm < 0 ,$$

where I denotes the (3×3)-unit matrix. In particular if $\kappa\neq 0$:

$\theta^m_{ii} < \theta^m_{i+1,i+1} + \theta^m_{i+2,i+2}$ with indices mod 3 in $\{1,2,3\}$, which is a certain triangle inequality for the principal axes of the tensor of inertia in case $\kappa \neq 0$.

**Proof.** The first equation follows from $r := (x_1, x_2, x_3)|\mathbb{M}$ and (ii), the second from $(x_0^2|\mathbb{M}) + \kappa\cdot\langle r, r\rangle = 1$ and the first one. These two equations imply the third one and in case $\kappa \neq 0$

$$\theta^m_{ii} = \frac{1}{\kappa}\cdot[m(\mathbb{M}) - \int_{\mathbb{M}} x_0^2 dm] - \int_{\mathbb{M}} x_i^2 dm \quad \text{and}$$

$$\theta^m_{i+1,i+1} + \theta^m_{i+2,i+2} = \operatorname{trace}(\theta^m) - \theta^m_{ii} =$$

$$= \frac{1}{\kappa}\cdot[m(\mathbb{M}) - \int_{\mathbb{M}} x_0^2 dm] + \int_{\mathbb{M}} x_i^2 dm .$$

Now we get the inequalities using (cf. § 2.i.ã) $\int_{\mathbb{M}} x_i^2 dm > 0$ .

**(iv) Lemma.**

$$\mathcal{T}_m(\xi,\eta) = \langle\omega(\xi), \theta^m\cdot\omega(\eta)\rangle + \langle v(\xi), [m(\mathbb{M})\cdot I - \kappa\cdot\theta^m]\cdot v(\eta)\rangle +$$
$$+ \langle\omega(\xi)\times\int_{\mathbb{M}} x_0(q)r(q)\,dm(q), v(\eta)\rangle +$$
$$+ \langle v(\xi), \omega(\eta)\times\int_{\mathbb{M}} x_0(q)r(q)\,dm(q)\rangle \quad \textit{for all } \xi,\eta\in\mathfrak{g},$$

where I denotes the (3×3)-unit matrix.

**Proof.** According to § 2.iv,viii both sides of the proposed equation are bilinear and symmetric in $\xi,\eta\in\mathfrak{g}$. Hence it is sufficient, to prove it for $\xi=\eta\in\mathfrak{g}$. We compute

$$\mathcal{T}_m(\xi,\xi) \underset{\S\,2.\text{iv}}{=} \int_{\mathbb{M}} (\xi\cdot q\,|\,\underbrace{\xi\cdot q}_{\underset{\S\,1.\text{vi}}{=}\,(-\kappa\cdot\langle v(\xi), r(q)\rangle,\ (\omega(\xi)\times r(q)) + x_0(q)v(\xi))})_\kappa\, dm(q)$$

$$\underset{[2],(2.1)}{=} \int_{\mathbb{M}} [\kappa\cdot\langle v(\xi), r(q)\rangle^2 + \langle(\omega(\xi)\times r(q)) + x_0(q)v(\xi), (\omega(\xi)\times r(q)) + x_0(q)v(\xi)\rangle]\,dm(q)$$

$$= \int_{\mathbb{M}} \langle\omega(\xi)\times r(q), \omega(\xi)\times r(q)\rangle dm(q) +$$
$$+ \int_{\mathbb{M}} [x_0^2(q)\cdot\langle v(\xi), v(\xi)\rangle + \kappa\cdot\langle v(\xi), r(q)\rangle^2]\,dm(q) +$$
$$+ 2\cdot\langle v(\xi), \omega(\xi)\times\int_{\mathbb{M}} x_0(q)r(q)\,dm(q)\rangle \quad ,$$

where $\int_{\mathbb{M}} \langle\omega(\xi)\times r(q), \omega(\xi)\times r(q)\rangle dm(q) \underset{(\text{ii})}{=} \langle\omega(\xi), \theta^m\cdot\omega(\xi)\rangle$

and $\int_{\mathbb{M}} [x_0^2(q)\cdot\langle v(\xi),v(\xi)\rangle + \kappa\cdot\langle v(\xi),n(q)\rangle^2]dm(q) =$

$$\underset{[2],(2.1),(2.5)}{\overset{\uparrow}{=}} \langle v(\xi),v(\xi)\rangle\cdot m(\mathbb{M}) -$$

$$- \kappa\cdot\int_{\mathbb{M}} [\langle v(\xi),v(\xi)\rangle\cdot\langle n(q),n(q)\rangle - \langle v(\xi),n(q)\rangle^2]dm(q) =$$

$$\underset{(ii)}{=} \langle v(\xi),[m(\mathbb{M})\cdot I - \kappa\cdot\theta^m]\cdot v(\xi)\rangle .$$

**(v) Corollary** (use § 4.iii). If the center of mass of m is the point **e**, then for all $\xi,\eta\in\mathfrak{g}$

$$\mathcal{T}_m(\xi,\eta) = \langle\omega(\xi),\theta^m\cdot\omega(\eta)\rangle + \langle v(\xi),[m(\mathbb{M})\cdot I - \kappa\cdot\theta^m]\cdot v(\eta)\rangle .$$

**(vi) Lemma.** $\mathcal{T}_m$ *is positive definite.*

**Proof.** After choosing another initial position of the body we may assume, that the center of mass of m is the point **e**. According to (v) and § 1.iii we have to show:

$$\theta^m \textit{ and } m(\mathbb{M})\cdot I - \kappa\cdot\theta^m \textit{ are positive definite.}$$

The positive definiteness of $\theta^m$ follows from (i) and condition ã) in § 2.i, since for $w\in\mathbb{R}^3\setminus\{o\}$ the function $\sin_\kappa^2(\mathrm{dist}(..,c_w(\mathbb{R})))$ is non-negative, continuous and its zeros are exactly the points of the geodesic line $c_w(\mathbb{R})$ (which is always contained in a 2-dim. totally geodesic submanifold of $\mathbb{M}$).

From the proof of (iv) we know for $w\in\mathbb{R}^3$

$$\langle w,[m(\mathbb{M})\cdot I - \kappa\cdot\theta^m]\cdot w\rangle = \int_{\mathbb{M}} [x_0^2(q)\cdot\langle w,w\rangle + \kappa\cdot\langle w,n(q)\rangle^2]dm(q) .$$

If $w\neq o$ and $\kappa>0$, then the integrand is non-negative, continuous and its zeros are contained in $\mathbb{M}\cap x_0^{-1}(\{0\})$, which is a 2-dim. totally geodesic submanifold of $\mathbb{M}$. Hence the positive definiteness of $m(\mathbb{M})\cdot I - \kappa\cdot\theta^m$ follows for $\kappa>0$ from condition ã) in § 2.i, and for $\kappa\le 0$ it is a consequence of the same property of $\theta^m$ and $m(\mathbb{M})>0$.

**(vii) Lemma.** If $\gamma:J\longrightarrow G$ is a smooth curve and $\xi\in\mathfrak{g}$ , then $$(\mathrm{Ad}_\gamma^{-1}\xi)' = [\mathrm{Ad}_\gamma^{-1}\xi, \dot\gamma_L]\ ,^{3}$$

$$\omega((\mathrm{Ad}_\gamma^{-1}\xi)') = \omega(\mathrm{Ad}_\gamma^{-1}\xi)\times\omega(\dot\gamma) + \kappa\cdot v(\mathrm{Ad}_\gamma^{-1}\xi)\times v(\dot\gamma)\ ,$$

$$v((\mathrm{Ad}_\gamma^{-1}\xi)') = \omega(\mathrm{Ad}_\gamma^{-1}\xi)\times v(\dot\gamma) + v(\mathrm{Ad}_\gamma^{-1}\xi)\times\omega(\dot\gamma)\ .$$

**Proof.** The differentiability of $\mathrm{Ad}_\gamma^{-1}\xi : J \longrightarrow \mathfrak{g}$ follows from that of $\gamma$. Then, by differentiating $\gamma\cdot\gamma^{-1} = \mathrm{id}$, we get $(\gamma^{-1})' = -\dot\gamma_L\cdot\gamma^{-1}$ and use this to compute (cf. [2],(2.28))

$$\begin{aligned}(\mathrm{Ad}_\gamma^{-1}\xi)' &= (\gamma^{-1}\cdot\xi\cdot\gamma)' = \gamma^{-1}\cdot\xi\cdot\gamma' + (\gamma^{-1})'\cdot\xi\cdot\gamma = \\ &= \gamma^{-1}\cdot\xi\cdot\gamma\cdot\dot\gamma_L - \dot\gamma_L\cdot\gamma^{-1}\cdot\xi\cdot\gamma = \mathrm{Ad}_\gamma^{-1}\xi\cdot\dot\gamma_L - \dot\gamma_L\cdot\mathrm{Ad}_\gamma^{-1}\xi = \\ &= [\mathrm{Ad}_\gamma^{-1}\xi, \dot\gamma_L]\ .\end{aligned}$$

The other assertions follow by using § 1.iii.

**(viii) Lemma.** If the center of mass of m is the point e, then for a smooth curve $\gamma : J \longrightarrow G$ and $\xi\in\mathfrak{g}$ one obtains

$$\begin{aligned}(\mathcal{I}_{\dot\gamma}^{m}(\xi))' = &\langle\omega(\mathrm{Ad}_\gamma^{-1}\xi), \theta^m\cdot(\omega(\dot\gamma))' - \\ &- (\theta^m\cdot\omega(\dot\gamma))\times\omega(\dot\gamma) + \kappa\cdot(\theta^m\cdot v(\dot\gamma))\times v(\dot\gamma)\rangle + \\ &+ \langle v(\mathrm{Ad}_\gamma^{-1}\xi), [m(M)\cdot I - \kappa\cdot\theta^m]\cdot[(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma))] + \\ &+ \kappa\cdot[2\cdot\theta^m - \mathrm{trace}(\theta^m)\cdot I]\cdot(\omega(\dot\gamma)\times v(\dot\gamma))\rangle\end{aligned}$$

(cf. § 2.vii).

**Proof.** First § 2.vii implies, that $\mathcal{I}^m : TG \longrightarrow \mathfrak{g}^*$ is a $C^\infty$-map, and so the differentiability of $\mathcal{I}_{\dot\gamma}^m(\xi) : J \longrightarrow \mathbb{R}$ follows from that of $\dot\gamma : J \longrightarrow TG$ . We compute

$$\begin{aligned}(\mathcal{I}_{\dot\gamma}^{m}(\xi))' &\underset{\S\ 2.\mathrm{vii}}{=} (\mathcal{I}_m(\dot\gamma_L, \mathrm{Ad}_\gamma^{-1}\xi))' = \\ &\underset{\S\ 2.\mathrm{iv}}{=} \mathcal{I}_m((\dot\gamma_L)', \mathrm{Ad}_\gamma^{-1}\xi) + \mathcal{I}_m(\dot\gamma_L, (\mathrm{Ad}_\gamma^{-1}\xi)') = \\ &\underset{(\mathrm{v})\text{ and } S=e}{=} \langle\omega((\dot\gamma_L)'), \theta^m\cdot\omega(\mathrm{Ad}_\gamma^{-1}\xi)\rangle + \\ &\qquad + \langle v((\dot\gamma_L)'), [m(M)\cdot I - \kappa\cdot\theta^m]\cdot v(\mathrm{Ad}_\gamma^{-1}\xi)\rangle +\end{aligned}$$

---

[3] In [1],4.1.25.c is proved $(\mathrm{Ad}_\gamma\xi)' = [\dot\gamma_R, \mathrm{Ad}_\gamma\xi]$ (cf. § 2.iii).

$$+ \langle\omega(\dot\gamma_L),\theta^m\cdot\omega((Ad_\gamma^{-1}\xi)')\rangle +$$
$$+ \langle v(\dot\gamma_L),[m(M)\cdot I - \kappa\cdot\theta^m]\cdot v((Ad_\gamma^{-1}\xi)')\rangle \quad ,$$

where, because $\theta^m$ is symmetric (w.r.t. $\langle..,..\rangle$) and $\omega_i, v_i$ are linear on $\mathfrak{g}$ and $\omega(\dot\gamma_L)=\omega(\dot\gamma)$, $v(\dot\gamma_L)=v(\dot\gamma)$,

$$\langle\omega((\dot\gamma_L)'),\theta^m\cdot\omega(Ad_\gamma^{-1}\xi)\rangle + \langle v((\dot\gamma_L)'),[m(M)\cdot I - \kappa\cdot\theta^m]\cdot v(Ad_\gamma^{-1}\xi)\rangle =$$
$$= \langle\omega(Ad_\gamma^{-1}\xi),\theta^m\cdot(\omega(\dot\gamma))'\rangle + \langle v(Ad_\gamma^{-1}\xi),[m(M)\cdot I - \kappa\cdot\theta^m]\cdot(v(\dot\gamma))'\rangle$$

and, using (vii),

$$\langle\omega(\dot\gamma_L),\theta^m\cdot\omega((Ad_\gamma^{-1}\xi)')\rangle + \langle v(\dot\gamma_L),[m(M)\cdot I - \kappa\cdot\theta^m]\cdot v((Ad_\gamma^{-1}\xi)')\rangle =$$
$$= \langle\theta^m\cdot\omega(\dot\gamma),\omega(Ad_\gamma^{-1}\xi)\times\omega(\dot\gamma) - \kappa\cdot v(Ad_\gamma^{-1}\xi)\times v(\dot\gamma)\rangle +$$
$$+ \langle[m(M)\cdot I - \kappa\cdot\theta^m]\cdot v(\dot\gamma),\omega(Ad_\gamma^{-1}\xi)\times v(\dot\gamma) + v(Ad_\gamma^{-1}\xi)\times\omega(\dot\gamma)\rangle =$$
$$= - \langle\omega(Ad_\gamma^{-1}\xi),(\theta^m\cdot\omega(\dot\gamma))\times\omega(\dot\gamma) + ([m(M)\cdot I - \kappa\cdot\theta^m]\cdot v(\dot\gamma))\times v(\dot\gamma)\rangle +$$
$$- \langle v(Ad_\gamma^{-1}\xi),\kappa\cdot(\theta^m\cdot\omega(\dot\gamma))\times v(\dot\gamma) + ([m(M)\cdot I - \kappa\cdot\theta^m]\cdot v(\dot\gamma))\times\omega(\dot\gamma)\rangle =$$
$$= - \langle\omega(Ad_\gamma^{-1}\xi),(\theta^m\cdot\omega(\dot\gamma))\times\omega(\dot\gamma) - \kappa\cdot(\theta^m\cdot v(\dot\gamma))\times v(\dot\gamma)\rangle +$$
$$+ \langle v(Ad_\gamma^{-1}\xi),[m(M)\cdot I - \kappa\cdot\theta^m]\cdot(\omega(\dot\gamma)\times v(\dot\gamma)) +$$
$$+ \kappa\cdot[\theta^m\cdot(\omega(\dot\gamma)\times v(\dot\gamma)) - (\theta^m\cdot\omega(\dot\gamma))\times v(\dot\gamma) - \omega(\dot\gamma)\times(\theta^m\cdot v(\dot\gamma))]\rangle \; .$$

The proof is completed by the observation

$(\theta^m\cdot a)\times b + a\times(\theta^m\cdot b) = [\mathrm{trace}(\theta^m)\cdot I - \theta^m]\cdot(a\times b)$ *for all* $a,b\in\mathbb{R}^3$

because of the symmetry of $\theta^m$.

## 6. Equations of motion

We use the notations introduced in § 1 and § 2.

**(i) Theorem.** If the center of mass of the rigid body with mass distribution m is the point $e\in M$, then a $C^2$ curve $\gamma: J \longrightarrow G$ is a process of motion of that rigid body without external forces if and only if $\dot\gamma_L: J \longrightarrow \mathfrak{g}$ is a solution of the following ODE system:

$$\theta^m\cdot(\omega(\dot\gamma))' - (\theta^m\cdot\omega(\dot\gamma))\times\omega(\dot\gamma) + \kappa\cdot(\theta^m\cdot v(\dot\gamma))\times v(\dot\gamma) = o \; ,$$

$$[m(M)\cdot I - \kappa\cdot\theta^m]\cdot[(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma))] +$$
$$+ \kappa\cdot[2\cdot\theta^m - \mathrm{trace}(\theta^m)\cdot I]\cdot(\omega(\dot\gamma)\times v(\dot\gamma)) = o \; .$$

**(ii) Remarks.**

a) From § 1.iii we get $\omega(\dot\gamma) = \omega^o(\dot\gamma_L)$ and

$v(\dot\gamma) = v^o(\dot\gamma_L)$, where $(\omega^o, v^o): \mathfrak{g} \longrightarrow \mathbb{R}^6$ is a $\mathbb{R}$-vector space isomorphism and $\theta^m$ and $m(\mathbb{M})\cdot I - \kappa\cdot\theta^m$ are symmetric and positive definite, therefore invertible, according to § 2.viii and the proof of § 5.vi. Hence the ($\mathbb{R}^3$-valued) equations in (i) are a (non-linear!) first order ODE in $\mathfrak{g}$ for $\dot\gamma_L$ and can be written as $(\dot\gamma_L)' = Q_m\circ\dot\gamma_L$ with a polynomial map $Q_m: \mathfrak{g} \longrightarrow \mathfrak{g}$, which is homogeneous of degree two (and induced by the rigid body).

b) If the tensor of inertia $\theta^m$ is diagonal, then the equations in (i) take the form (use § 5.iii):

$$\theta^m_{11}\cdot(\omega_1(\dot\gamma))' + [\theta^m_{33}-\theta^m_{22}]\cdot[\omega_2(\dot\gamma)\omega_3(\dot\gamma) - \kappa\cdot v_2(\dot\gamma)v_3(\dot\gamma)] = 0,$$
$$\theta^m_{22}\cdot(\omega_2(\dot\gamma))' + [\theta^m_{11}-\theta^m_{33}]\cdot[\omega_3(\dot\gamma)\omega_1(\dot\gamma) - \kappa\cdot v_3(\dot\gamma)v_1(\dot\gamma)] = 0,$$
$$\theta^m_{33}\cdot(\omega_3(\dot\gamma))' + [\theta^m_{22}-\theta^m_{11}]\cdot[\omega_1(\dot\gamma)\omega_2(\dot\gamma) - \kappa\cdot v_1(\dot\gamma)v_2(\dot\gamma)] = 0,$$
$$\int_{\mathbb{M}}(x_0^2+\kappa x_1^2)\,dm\cdot(v_1(\dot\gamma))' + \int_{\mathbb{M}}(x_0^2-\kappa x_1^2)\,dm\cdot[\omega(\dot\gamma)\times v(\dot\gamma)]_1 = 0,$$
$$\int_{\mathbb{M}}(x_0^2+\kappa x_2^2)\,dm\cdot(v_2(\dot\gamma))' + \int_{\mathbb{M}}(x_0^2-\kappa x_2^2)\,dm\cdot[\omega(\dot\gamma)\times v(\dot\gamma)]_2 = 0,$$
$$\int_{\mathbb{M}}(x_0^2+\kappa x_3^2)\,dm\cdot(v_3(\dot\gamma))' + \int_{\mathbb{M}}(x_0^2-\kappa x_3^2)\,dm\cdot[\omega(\dot\gamma)\times v(\dot\gamma)]_3 = 0,$$

whereby the coefficients of the first 3 equations can be written as (use § 5.iii):

$$\theta^m_{11} = \int_{\mathbb{M}}(x_2^2+x_3^2)\,dm, \quad \theta^m_{33}-\theta^m_{22} = \int_{\mathbb{M}}(x_2^2-x_3^2)\,dm \quad \text{etc.}$$

and the coefficients of the last 3 equations can be written as

$$\int_{\mathbb{M}}(x_0^2+\kappa x_1^2)\,dm = m(\mathbb{M}) - \kappa\cdot\theta^m_{11},$$
$$\int_{\mathbb{M}}(x_0^2-\kappa x_1^2)\,dm = m(\mathbb{M}) - \kappa\cdot\theta^m_{22} - \kappa\cdot\theta^m_{33} \quad \text{etc.}$$

c) If the tensor of inertia $\theta^m$ is diagonal and $(\xi_1,\ldots,\xi_6)$ denotes the basis of $\mathfrak{g}$ dual to $(\omega_1^o,\omega_2^o,\omega_3^o,v_1^o,v_2^o,v_3^o)$, then § 5.viii implies

$$\begin{pmatrix}(\mathcal{J}^m_\gamma(\xi_1))' \\ \vdots \\ (\mathcal{J}^m_\gamma(\xi_6))'\end{pmatrix} = \begin{pmatrix}\omega_1(\mathrm{Ad}_\gamma^{-1}\xi_1) & \cdots\cdots & v_3(\mathrm{Ad}_\gamma^{-1}\xi_1) \\ \vdots & & \vdots \\ \omega_1(\mathrm{Ad}_\gamma^{-1}\xi_6) & \cdots\cdots & v_3(\mathrm{Ad}_\gamma^{-1}\xi_6)\end{pmatrix}\cdot\begin{pmatrix}\text{l.h.} \\ \text{side} \\ \text{of} \\ \text{ODE} \\ \text{in b)}\end{pmatrix},$$

the matrix in this equation being the transposed of the representation matrix of the automorphism $\mathrm{Ad}_\gamma^{-1}$ with respect to the basis $(\xi_1,\ldots,\xi_6)$.

**d)** The additional assumptions on m can always be satisfied by choosing another initial position of the (unchanged) body. More precisely, for a given m there always exists an isometry $f\in G$ of $\mathbb{M}$, such that the Borel measure $f*m$ (with $(f*m)(B):=m(f(B))$ for all Borel subsets $B\subseteq\mathbb{M}$) has center of mass **e** and a tensor of inertia in diagonal form.

**e)** In case $\kappa=0$ the first 3 equations of the ODE system in b) become the classical Euler equations and the last 3 equations become equivalent to the classical condition that the center of mass moves with constant speed along a geodesic of $\mathbb{M}$ (use § 3.vi).

**f)** In case $\kappa\neq0$ the center of mass rather rarely moves along a geodesic of $\mathbb{M}$ (which drastically contrasts the case $\kappa=0$), see below § 7.iv and § 8.iv, and the example of a "symmetric ball" studied in § 7 will show, that this fact does not depend on our choice of the concept "center of mass", which might have been suspected to be not appropriate in this respect. We are rather inclined to see the reason for this phenomenon of a nongeodesic motion of the center of mass in case $\kappa\neq0$ in the different structure of the Lie group G (in contrast to the case $\kappa=0$), more precisely in the lack of a subgroup of translations in G for $\kappa\neq0$.

**Proof of (i).** Let $\gamma:J\longrightarrow G$ be a process of motion of the rigid body with mass distribution m without external forces. Then $\vartheta^{m}_{\gamma}:J\longrightarrow\mathfrak{g}^*$ is constant according to § 2.vii and from § 5.viii we get for all $\xi\in\mathfrak{g}$

$$0=(\vartheta^{m}_{\gamma}(\xi))' = <\omega(\mathrm{Ad}_{\gamma}^{-1}\xi),\theta^{m}\cdot(\omega(\dot\gamma))' - \\ -(\theta^{m}\cdot\omega(\dot\gamma))\times\omega(\dot\gamma)+\kappa\cdot(\theta^{m}\cdot v(\dot\gamma))\times v(\dot\gamma)> + \\ + <v(\mathrm{Ad}_{\gamma}^{-1}\xi),[m(\mathbb{M})\cdot I-\kappa\cdot\theta^{m}]\cdot[(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma))] + \\ +\kappa\cdot[2\cdot\theta^{m}-\mathrm{trace}(\theta^{m})\cdot I]\cdot(\omega(\dot\gamma)\times v(\dot\gamma))> .$$

For every $t\in J$, however, $\mathrm{Ad}_{\gamma(t)}^{-1}:\mathfrak{g}\longrightarrow\mathfrak{g}$ is bijective as well as $(\omega,v)|\mathfrak{g}:\mathfrak{g}\longrightarrow\mathbb{R}^3\times\mathbb{R}^3$. Since the last equation holds for all $\xi\in\mathfrak{g}$, it implies that $\dot\gamma_L:J\longrightarrow\mathfrak{g}$ is a solution of the ODE in (i).

Apply uniqueness for the above ODE in $\mathfrak{g}$ and for the ODE $Y' = Y\cdot\dot{\gamma}_L$ in $\mathfrak{M}(4,\mathbb{R})$ to complete the proof.

7. Motion of a ball with rotational symmetric distribution of mass in the absence of external forces

We use the notations introduced in § 1 and § 2 and specify m as follows:

(i) Let the point $p_0$ from § 2.i be equal to **e** and let m be invariant under the isotropy subgroup of G for the point $\mathbf{e}\in\mathbb{M}$, i.e. $f^*m=m$ for all $f\in G_{\mathbf{e}}$ (cf. § 6.ii.d), in other words the rigid body is a *geodesic ball* of radius $< \frac{1}{8}\cdot\pi_\kappa$ with metric center **e** and a *full rotational symmetric distribution of mass* (e.g. including the case of a geodesic ball of constant density of mass).

(ii) For this m, because of the symmetry obviously holds: The center of mass of m is the point **e** and the tensor of inertia $\theta^m$ is scalar, i.e. $\theta^m = \vartheta\cdot I$, where I is the identity and (cf. § 5.iii) $\frac{\vartheta}{2} = \int_{\mathbb{M}} x_1^2 dm = \int_{\mathbb{M}} x_2^2 dm = \int_{\mathbb{M}} x_3^2 dm > 0$ . So § 6.i,ii.b imply:

A smooth curve $\gamma:J \longrightarrow G$ is a process of motion of the ball with mass distribution m without external forces iff

$$(\omega(\dot{\gamma}))' = o \quad \textit{and} \quad (v(\dot{\gamma}))' + K\cdot(\omega(\dot{\gamma})\times v(\dot{\gamma})) = o \;,$$

where
$$K := \frac{\int_{\mathbb{M}}(x_0^2-\kappa x_1^2)\,dm}{\int_{\mathbb{M}}(x_0^2+\kappa x_1^2)\,dm} = \frac{m(\mathbb{M})-2\kappa\vartheta}{m(\mathbb{M})-\kappa\vartheta}\begin{cases}\in\ ]1,\infty[ & \text{for } \kappa<0\\ =1 & \text{for } \kappa=0\\ \in\ ]0,1[ & \text{for } \kappa>0\end{cases} .$$

(iii) This ODE system is de facto linear and we get:

A smooth curve $\gamma:J \longrightarrow G$ is a process of motion of the ball with mass distribution m without external forces iff there exist a positively oriented orthonormal basis $(e_1,e_2,e_3)$ of $\mathbb{R}^3$ and real numbers $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$ , such that

$$\omega(\dot{\gamma}) = \lambda_1 e_1 \quad \textit{and} \quad v(\dot{\gamma}) = \lambda_2 e_1 + \lambda_3\cdot[\sin(K\lambda_1 x)e_2 + \cos(K\lambda_1 x)e_3]$$

with K from (ii).

**Proof.** Verify first by simple computation, that the $\dot\gamma_L$ given by the last equations (cf. § 6.ii.a) is a solution of the ODE in (ii), and show then, that all initial conditions in $\mathfrak{g}$ can be realized by such curves.

**(iv)** Let m,K be as in (i),(ii) and let $\gamma:J \longrightarrow G$ be a process of motion of the ball with mass distribution m without external forces. Then one obtains for the *orbit* $\gamma\cdot e:J \longrightarrow \mathbb{M}$ *of the metric center* e of the ball under $\gamma$ and any fixed $t\in J$:

$\gamma\cdot e$ is a curve in $\mathbb{M}$ of *constant speed*

$$\|v(\dot\gamma(t))\|$$

and – if this is $\neq 0$ – of *constant* (geodesic) *curvature*

$$|1-K|\cdot\|\omega(\dot\gamma(t))\times v(\dot\gamma(t))\|\cdot\|v(\dot\gamma(t))\|^{-2}$$

and – if this is $\neq 0$ too – of *constant torsion*

$$(1-K)\cdot\langle\omega(\dot\gamma(t)),v(\dot\gamma(t))\rangle\cdot\|v(\dot\gamma(t))\|^{-2} ,$$

where $\quad 1-K = \kappa\cdot\frac{\vartheta}{m(\mathbb{M})-\kappa\vartheta} = O(\kappa) \quad$ *for* $\kappa \longrightarrow 0$ , *but* $\vartheta > 0$

(cf. (ii)), in particular

$\gamma\cdot e:J \longrightarrow \mathbb{M}$ is a geodesic $\Leftrightarrow$

$\Leftrightarrow \Big[\kappa=0$ *or* [$\omega(\dot\gamma(t))$ and $v(\dot\gamma(t))$ are linearly dependent]$\Big]$ ,

$\gamma\cdot e:J \longrightarrow \mathbb{M}$ is a plane curve[4] $\Leftrightarrow$

$\Leftrightarrow \Big[\kappa=0$ *or* [$\omega(\dot\gamma(t))$ and $v(\dot\gamma(t))$ are linearly dependent]
*or* [$\omega(\dot\gamma(t))$ and $v(\dot\gamma(t))$ are orthogonal]$\Big]$ .

**Proof.** With $(e_1,e_2,e_3)$ and $\lambda_1,\lambda_2,\lambda_3$ from (iii) we get first (using § 3.vii)

$$\|(\gamma\cdot e)^{\cdot}\| = \|v(\dot\gamma)\| = \sqrt{\lambda_2^2+\lambda_3^2} \quad \text{constant,}$$

$$(v(\dot\gamma))' = K\lambda_1\lambda_3\cdot[\cos(K\lambda_1 x)e_2 - \sin(K\lambda_1 x)e_3] ,$$

$$(v(\dot\gamma))' \perp v(\dot\gamma) ,$$

$$\omega(\dot\gamma)\times v(\dot\gamma) = \lambda_1\lambda_3\cdot[\sin(K\lambda_1 x)e_3 - \cos(K\lambda_1 x)e_2] ,$$

$$(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma)) = -(1-K)\lambda_1\lambda_3\cdot[\cos(K\lambda_1 x)e_2-\sin(K\lambda_1 x)e_3] .$$

So the curvature of $\gamma\cdot e$ is equal to $|1-K|\cdot|\lambda_1\lambda_3|\cdot\|v(\dot\gamma)\|^{-2}$ , hence constant, and $|\lambda_1\lambda_3| = \|\omega(\dot\gamma)\times v(\dot\gamma)\|$ .

Next we obtain

$$u := [(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma))]' + \omega(\dot\gamma)\times[(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma))] =$$

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[4] i.e. contained in a 2-dim. totally geodesic submanifold of $\mathbb{M}$

$= - (1-K)^2\lambda_1^2\lambda_3 \cdot [\sin(K\lambda_1 x)e_2 + \cos(K\lambda_1 x)e_3]$ ,
$\det(v(\dot\gamma),(v(\dot\gamma))' + (\omega(\dot\gamma)\times v(\dot\gamma)),u) =$
$= (1-K)\cdot\lambda_1\lambda_2\cdot(|1-K|\cdot|\lambda_1\lambda_3|)^2$ constant,
where $\lambda_1\lambda_2 = \langle\omega(\dot\gamma),v(\dot\gamma)\rangle$ and $|1-K|\cdot|\lambda_1\lambda_3| = \kappa_{\gamma\cdot e}\cdot\|v(\dot\gamma)\|^2$ . So § 3.vii yields the assertion on the torsion of $\gamma\cdot$**e**.

**(v)** Let m be as above and let $\gamma:J\longrightarrow G$ be a process of motion of the ball with mass distribution m without external forces. Define $\omega_e(\dot\gamma):J\longrightarrow T_e\mathbb{M}$ by $\omega_e(\dot\gamma)^{\rightarrow} = (0,\omega(\dot\gamma))$ and let $t_0\in J$. We can interpret $\mathbb{R}\cdot\omega_e(\dot\gamma(t_0))$ as *rotation axis of* $\gamma$ *at the moment* $t_0$ *in time in body coordinates* and state:

$\gamma$ *transports the vector* $\omega_e(\dot\gamma(t_0))$ *parallelly,* i.e. the vector field defined by $t\mapsto\gamma(t)_*\omega_e(\dot\gamma(t_0))$ is a parallel field (in the sense of Levi-Civita in $\mathbb{M}$) along the orbit $\gamma\cdot$**e** of the ball's center.

This vector field is equal to the one defined by $t\mapsto\gamma(t)_*\omega_e(\dot\gamma(t))$ . We can interpret $\mathbb{R}\cdot\gamma(t)_*\omega_e(\dot\gamma(t))$ as a *rotation axis* of $\gamma$ at the moment t in time *in space coordinates*.

**Proof.** $\omega(\dot\gamma)$ is constant according to (iii). Apply § 3.viii. ·

## 8. Motion of a symmetric gyroscope without external forces

We use the notations introduced in § 1 and § 2 and specify m as follows:

**(i)** Let the center of mass of m be the point **e**, let m be invariant under the group of rotations around the axis $\mathrm{Span}(\mathbf{e},\mathbf{e}_1)\cap\mathbb{M}$ (the image of a geodesic in $\mathbb{M}$, cf. [2],(2.13)) and let $\theta^m_{11}\neq\theta^m_{22}$ . Hence the tensor of inertia $\theta^m$ is a diagonal matrix with $\theta^m_{22}=\theta^m_{33}$ , but m (and $\theta^m$) possesses no symmetry with respect to any other axis. We call the body a (free movable) *symmetric gyroscope*.

**(ii)** For this m the results § 6.i,ii.b imply: A smooth curve $\gamma:J\longrightarrow G$ is a process of motion of the gyroscope with mass distribution m without external forces iff $\dot\gamma_L:J\longrightarrow\mathfrak{g}$ is a solution of the following ODE system (cf. § 6.ii.a):

$$(\omega_1(\dot\gamma))' = 0 ,$$

$$(\omega_2(\dot\gamma))' + K\cdot[\omega_3(\dot\gamma)\omega_1(\dot\gamma)-\kappa v_3(\dot\gamma)v_1(\dot\gamma)] = 0 ,$$
$$(\omega_3(\dot\gamma))' - K\cdot[\omega_1(\dot\gamma)\omega_2(\dot\gamma)-\kappa v_1(\dot\gamma)v_2(\dot\gamma)] = 0 ,$$
$$(v_1(\dot\gamma))' + C_1\cdot[\omega(\dot\gamma)\times v(\dot\gamma)]_1 = 0 ,$$
$$(v_2(\dot\gamma))' + C_2\cdot[\omega(\dot\gamma)\times v(\dot\gamma)]_2 = 0 ,$$
$$(v_3(\dot\gamma))' + C_2\cdot[\omega(\dot\gamma)\times v(\dot\gamma)]_3 = 0 ,$$

where

$$K := \frac{\theta^m_{11}-\theta^m_{22}}{\theta^m_{22}} , \quad C_1 := \frac{\int_{\mathbb{M}}(x_0^2-\kappa x_1^2)\,dm}{\int_{\mathbb{M}}(x_0^2+\kappa x_1^2)\,dm} , \quad C_2 := \frac{\int_{\mathbb{M}}(x_0^2-\kappa x_2^2)\,dm}{\int_{\mathbb{M}}(x_0^2+\kappa x_2^2)\,dm}$$

with $K \in \mathbb{R}^*$ and $\begin{cases} \kappa<0 \Rightarrow (C_1,C_2\in]1,\infty[ \textit{ and } C_1\neq C_2) \\ \kappa=0 \Rightarrow C_1=C_2=1 \\ \kappa>0 \Rightarrow (C_1,C_2\in]0,1[ \textit{ and } C_2\neq C_3) \end{cases}$ .

In case $\kappa=0$ cf. § 6.ii.e.

(iii) Let $m,C_2$ be as above and let $\kappa\neq 0$. Then for a smooth curve $\gamma:J \longrightarrow G$ the following two statements a) and b) are equivalent (where $\nabla$ denotes the Levi-Civita covariant derivative in $\mathbb{M}$):

a) $\gamma$ is a process of motion without external forces of the gyroscope with mass distribution m, such that the symmetry axis of the gyroscope, i.e. the vector $u\in T_e\mathbb{M}$ with $\vec{u}=e_1$ , is $\nabla$-parallelly transported by $\gamma$, i.e. the vector field along the orbit $\gamma\cdot e$ of the center of mass, defined by $t \mapsto \gamma(t)_* u$ , is $\nabla$-parallel.

b) One of the following two conditions $\alpha$) and $\beta$) is fulfilled:

$\alpha$) There exist numbers $\lambda_1,\lambda_2\in\mathbb{R}$, such that
$$\omega(\dot\gamma) = (\lambda_1,0,0) \quad\text{and}\quad v(\dot\gamma) = (\lambda_2,0,0) .$$
Hence $\gamma$ is a left coset of an one-parameter subgroup of G, which can be interpreted as a screw motion in the direction of the gyroscope's axis of symmetry.

$\beta$) There exist numbers $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$, such that $\omega(\dot\gamma) = (\lambda_1,0,0)$ and $v(\dot\gamma) = (0,\lambda_2\sin(C_2\lambda_1 x+\lambda_3),\lambda_2\cos(C_2\lambda_1 x+\lambda_3))$ .
Here the center of mass e is either at rest (if $\lambda_2=0$) or moves with constant speed (of value $|\lambda_2|$) along a plane curve with constant curvature (of value $|C_2-1|\cdot|\lambda_1|\cdot|\lambda_2|^{-1}$) and the gyroscope's axis of symmetry

stays orthogonal to the plane in which the center of mass e moves.

**Proof.** According to § 3.viii we have to look for those solutions of the ODE in (ii), which satisfy $\omega_2(\dot\gamma)=0$ *and* $\omega_3(\dot\gamma)=0$. Since $\kappa\neq 0$ by our assumption, the second and third equation of the ODE imply $v_1(\dot\gamma)v_3(\dot\gamma)=0$ *and* $v_1(\dot\gamma)v_2(\dot\gamma)=0$. But § 6.ii.a shows, that $\dot\gamma_L$ is an integral curve of a $C^\omega$ vector field on $\mathfrak{g}$. Hence $\dot\gamma_L$ and (cf. § 1.iii) $v(\dot\gamma)$ are $C^\omega$ maps and we can conclude:

$$v_1(\dot\gamma)=0 \quad or \quad v_2(\dot\gamma)=v_3(\dot\gamma)=0 .$$

1st case: $v_2(\dot\gamma)=v_3(\dot\gamma)=0$.
Together with $\omega_2(\dot\gamma)=\omega_3(\dot\gamma)=0$ we get $\omega(\dot\gamma)\times v(\dot\gamma)=0$. This leads to the situation $\alpha$) of b).

2nd case: $v_1(\dot\gamma)=0$.
The fourth equation of the ODE is satisfied because of $\omega_2(\dot\gamma)=\omega_3(\dot\gamma)=0$ and the last two equations yield

$$(v_2(\dot\gamma))' - C_2\omega_1(\dot\gamma)\cdot v_3(\dot\gamma) = (v_3(\dot\gamma))' + C_2\omega_1(\dot\gamma)\cdot v_2(\dot\gamma) = 0 .$$

Together with the first equation of the ODE this gives the situation $\beta$) of b). Concerning the geometry of $\gamma\cdot e$ one finds by computation using § 3.vi that

$\|(\gamma\cdot e)^\cdot\| = \|v(\dot\gamma)\| = |\lambda_2|$ is constant, furthermore

$(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma)) =$

$$= (C_2-1)\cdot\lambda_1\lambda_2\cdot(0,\cos(C_2\lambda_1 x+\lambda_3),-\sin(C_2\lambda_1 x+\lambda_3)) \in v(\dot\gamma_L)^\perp ,$$

hence (curvature of $\gamma\cdot e$) $= |C_2-1|\cdot|\lambda_1|\cdot|\lambda_2|^{-1}$ is constant,

finally $[(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma))]' + \omega(\dot\gamma)\times[(v(\dot\gamma))'+(\omega(\dot\gamma)\times v(\dot\gamma))] =$

$$= -(C_2-1)^2\lambda_1^2\cdot v(\dot\gamma) , \quad \text{hence} \quad \nabla_\partial\nabla_\partial(\gamma\cdot e)^\cdot = -(C_2-1)^2\lambda_1^2\cdot(\gamma\cdot e)^\cdot$$

and the curve $\gamma\cdot e$ is plane.
With $E_1,E_2,E_3$ from § 3.i we have $[t\mapsto\gamma(t)_*u] = E_1$ and $\mathrm{Span}((\gamma\cdot e)^\cdot,\nabla_\partial(\gamma\cdot e)^\cdot) \subseteq \mathrm{Span}(E_2,E_3) = E_1^\perp$.

**(iv)** Let m,K be as above and let $\kappa\neq 0$. Then for a smooth curve $\gamma:J\longrightarrow G$ the following two statements a) and b) are equivalent:

a) $\gamma$ is a process of motion without external forces of the gyroscope with mass distribution m, such that the orbit $\gamma \cdot e$ of the center of mass e under $\gamma$ is a geodesic in $\mathbb{M}$.

b) One of the following four conditions $\alpha$),..,$\delta$) is fulfilled:

$\alpha$) There exist numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, such that $v(\dot{\gamma}) = o$ and $\omega(\dot{\gamma}) = (\lambda_1, \lambda_2 \cos(K\lambda_1 x+\lambda_3), \lambda_2 \sin(K\lambda_1 x+\lambda_3))$ . Here e is at rest, i.e. $\gamma \cdot e$ is constant.

$\beta$) The condition $\alpha$) of (iii)b) is satisfied.

$\gamma$) There exists a vector $a \in \mathbb{R}^3$ and numbers $\lambda_1, \lambda_2 \in \mathbb{R}$, such that
$a_1 = 0$ and $\omega(\dot{\gamma}) = \lambda_1 a$ , $v(\dot{\gamma}) = \lambda_2 a$ .
Hence $\gamma$ is a left coset of an one-parameter subgroup of G, which can be interpreted as a screw motion in the direction orthogonal to the gyroscope's axis of symmetry.

$\delta$) $\kappa > 0$ and there exist a vector $a \in \mathbb{R}^3$ and $\sigma \in \{+1,-1\}$, such that $a_1 \neq 0$ and $\omega(\dot{\gamma}) = \sigma \cdot \sqrt{\kappa} \cdot a$ , $v(\dot{\gamma}) = a$ .
Hence $\gamma$ is a left coset of an one-parameter subgroup of G, which can be interpreted as a screw motion in the direction diagonal to the gyroscope's axis of symmetry. [Screw motions in such directions can occur as process of motion without external forces only in case $\kappa>0$ and if in addition this special proportionality between the translational and the angular velocity holds.]

**Proof.** According to § 3.vi we have to look for those solutions of the ODE in (ii), which satisfy $(v(\dot{\gamma}))' + (\omega(\dot{\gamma}) \times v(\dot{\gamma})) = o$ . Because of our assumption $\kappa \neq 0$ (hence $C_1, C_2 \neq 1$) this is compatible with the last 3 equations of the ODE only in case $(v(\dot{\gamma}))' = o$ *and* $\omega(\dot{\gamma}) \times v(\dot{\gamma}) = o$ . So we have $v(\dot{\gamma})$ constant and $\omega(\dot{\gamma})$ and $v(\dot{\gamma})$ linearly dependent. The further discussion is left to the reader.

**(v)** Let m be as in (i) and suppose $\kappa \neq 0$. If $\gamma: J \longrightarrow G$ is a process of motion of the gyroscope with mass distribution m without external forces, then the orbit $\gamma \cdot e$ of the center of mass e under $\gamma$ is a totally geodesic immersion in $\mathbb{M}$, i.e. its geodesic curvature equals zero, only if the parametrization $\gamma \cdot e: J \longrightarrow \mathbb{M}$ is already a constant speed geodesic in $\mathbb{M}$ (use (ii) and § 3.ii,v).

9. A review of some further results of [6]

The machinery for describing the motion of the rigid body in the present paper was – for simplicity – developed for special initial data. Many objects introduced here with respect to $e \in \mathbb{M}$ could have been introduced as well with respect to an arbitrary reference point $p \in \mathbb{M}$, e.g. the angular and translational velocity with respect to p as $T_p\mathbb{M}$-valued mappings using the ideas of § 1.iv,v. The just mentioned mappings give rise to a vector space isomorphism $\mathfrak{g} \longrightarrow T_p\mathbb{M} \times T_p\mathbb{M}$. Using this isomorphism and the Riemannian metric g of $\mathbb{M}$, the momentum mapping $\mathcal{J}^m: TG \longrightarrow \mathfrak{g}^*$ induces the maps $L_p^m, P_p^m: TG \longrightarrow T_p\mathbb{M}$, where, via E. Noehter's theorem and the just mentioned identifications, $L_p^m$ corresponds to the operation of the isotropy subgroup of G for the point $p \in \mathbb{M}$ and $P_p^m$ corresponds, in case $\kappa=0$, to the operation of the translation subgroup ($\leq G$), but allows in case $\kappa \neq 0$ no independent interpretation through Noether's theorem. Given a process of motion $\gamma: J \longrightarrow G$, we call $L_p^m \circ \dot{\gamma}$ the *angular momentum* with respect to p of the body (with mass distribution m) moving under $\gamma$ and $P_p^m \circ \dot{\gamma}$ the *translational momentum* with respect to p of the body (with mass distribution m) moving under $\gamma$. In [6] the following integral formulas for these momenta are proved: If $t \in J$ with $-p \notin \gamma(t) \cdot U_\varepsilon(p_0)$, where $\varepsilon$ and $p_0$ are chosen as in § 2.i and $-p$ denotes the antipodal point to p (which lies in $\mathbb{M}$ only for $\kappa>0$), then

$$(L_p^m \circ \dot{\gamma})(t) = \int_{\mathbb{M}} [r_p(\gamma(t) \cdot q) \times \|{}^p_{\gamma(t) \cdot q}((\gamma \cdot q)^{\cdot}(t))] dm(q)$$

and $$(P_p^m \circ \dot{\gamma})(t) = \int_{\mathbb{M}} Y^\gamma_{t,q}(1) dm(q) ,$$

where $r_p := (\frac{\sin_\kappa}{x} \circ d(p,..)) \cdot \exp_p^{-1} : \mathbb{M} \setminus \{-p\} \longrightarrow T_p\mathbb{M}$ (interpret $\frac{\sin_\kappa}{x}$ and $\exp_p^{-1}$ analogously to § 1.ii)[5],

$c^\gamma_{t,q}(x) := \exp_{\gamma(t) \cdot q}(x \cdot \exp^{-1}_{\gamma(t) \cdot q}(p))$ is the shortest geodesic

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[5] The extension of $r_p$, in case $\kappa>0$, to $\mathbb{M}$ by requiring $r_p(-p) := o$ is $C^\omega$ on the whole of $\mathbb{M}$.

from $(-p\neq)\ \gamma(t)\cdot q = c^{\gamma}_{t,q}(0)$ to $p = c^{\gamma}_{t,q}(1)$ ,
$\|^{p}_{\gamma(t)\cdot q} : T_{\gamma(t)\cdot q}\mathbb{M} \longrightarrow T_p\mathbb{M}$ denotes the Levi-Civita parallel transport along $c^{\gamma}_{t,q}|[0,1]$ and $Y^{\gamma}_{t,q}$ denotes the Jacobi field along $c^{\gamma}_{t,q}$ with $Y^{\gamma}_{t,q}(0) = (\gamma\cdot q)^{\cdot}(t)$ and $(\nabla_{\partial}Y^{\gamma}_{t,q})(0) = o$ .

It is further proved, that the derivatives $(L^{m}_{p}\circ\dot{\gamma})'$, $(P^{m}_{p}\circ\dot{\gamma})'$ possess similar integral formulas [only replace $(\gamma\cdot q)^{\cdot}$ by $\nabla_{\partial}(\gamma\cdot q)^{\cdot}$]. If we recall § 6.ii.c and use $\vartheta^{m}_{\dot{\gamma}}(\xi_i) = x_i\circ(L^{m}_{e}\circ\dot{\gamma})^{\rightarrow}$ and $\vartheta^{m}_{\dot{\gamma}}(\xi_{3+i}) = x_i\circ(P^{m}_{e}\circ\dot{\gamma})^{\rightarrow}$ for $i\in\{1,2,3\}$, where $\xi_1,\ldots,\xi_6$ are chosen as in § 6.ii.c, then the last result enables us to formulate the equations of motion for the rigid body *with external forces*, if these external forces are given as a law of acceleration for the points of the body.

Now consider that the body is *fixed in one point* $p\in(U_{\varepsilon}(p_0)\subseteq)\mathbb{M}$. Then the tangential space $T_p\mathbb{M}$ at $\mathbb{M}$ in p is a distinguished Euclidean space for this problem and in [6] a Euclidean mechanical model in $T_p\mathbb{M}$ is given, which describes modulo the exponential map $\exp_p$ exactly the corresponding (non Euclidean) motion in $\mathbb{M}$. This model consists of the *induced mass distribution* $\tau_p m$ in $T_p\mathbb{M}$, defined by

$$(\tau_p m)(B) := \int_{\exp_p(B\cap U_{\pi_\kappa}(o))} \left(\frac{\sin_\kappa}{x}\right)^2 (d(p,q))\,dm(q)$$ [6]

for all Borel subsets $B\subseteq T_p\mathbb{M}$, and of the induced external forces. If the external forces in $\mathbb{M}$ are given as a law of acceleration for the points of the body, then, the *induced law of acceleration* in $T_p\mathbb{M}$ is obtained by "conjugation" with $\exp_p$.

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[6] If we denote by V resp. $V_p$ the *volume* measure of $(\mathbb{M},g)$ resp. of $(T_p\mathbb{M},g_p)$, then V induces via the map $(\exp_p|U_{\pi}(o))^{-1}$ a measure on $T_p\mathbb{M}$, whose density with respect to $V_p$ is equal to the function $\left(\left(\frac{\sin_\kappa}{x}\right)^2\circ\|\,.\,.\,\|\right)\cdot\chi_{U_{\pi_\kappa}(o)} : T_p\mathbb{M}\longrightarrow[0,\infty[$ .

Other concepts for describing the external forces require additional modifications in defining the appropriate "induced forces" in $T_p\mathbb{M}$.

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