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# ON JETS OF INFINITE ORDER AND ON SMOOTH DIFFERENTIAL OPERATORS

## 0. Introduction

Let $E := \mathbb{R}^n$, $F := \mathbb{R}^m$ and $\mathbb{N} = \{0,1,2,...\}$. For $f \in C^\infty(E,F)$ the partial derivatives are denoted $\partial^\alpha f := \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$, where $\alpha := (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$. We shall consider the map $B\colon E\pi C^\infty(E,F) \longrightarrow E\pi F^{\mathbb{N}^n}$ defined by

$$B(x,f) = (x,(\partial^\alpha f(x))_{\alpha\in\mathbb{N}^n}).$$

A well known theorem of E. Borel states the surjectivity of the map B which will be called the Borel map. In Sec. 1 it will be shown that B is even a $C^\infty$-quotient-map in the sense of [1]; see below. Using this result one obtains in Sec. 2 an elegant description of the smooth structure of the space $J^\infty(X,Y)$ of jets of infinite order between smooth manifolds X and Y, and in Sec. 3 an intrinsic characterization of smooth differential operators. For the notion of smooth structure we refer to [1].

The intrinsic characterization of $J^\infty(X,Y)$ can be generalized to the case of arbitrary smooth spaces X, Y; that of smooth differential operators to the case of convenient vector bundles (cf. [1]). Whether this is useful mainly depends on possible generalizations of the properties of the Borel map B. Surjectivity of B is known to hold for $E = \mathbb{R}^n$ and F any Fréchet space and to fail even for $E = \mathbb{R}$ if F is an arbitrary convenient vector space.

1. The universal property of the Borel map

A $C^\infty$-quotient-map is a map $q: X \longrightarrow Y$ between smooth spaces which is surjective and has the universal property that a map $h: Y \longrightarrow Z$ into any smooth space $Z$ is smooth iff $h \circ q$ is smooth.

If with respect to a smooth map $q: X \longrightarrow Y$ smooth curves lift (i. e. for each $c: \mathbb{R} \longrightarrow Y$ smooth there exist $\bar{c}: \mathbb{R} \longrightarrow X$ smooth with $c = q \circ \bar{c}$), then $q$ is certainly a $C^\infty$-quotient-map. In fact, surjectivity of $q$ follows trivially. And if $h \circ q$ is smooth then for any smooth curve $c: \mathbb{R} \longrightarrow Y$ also $h \circ c: \mathbb{R} \longrightarrow Z$ is smooth (since $h \circ c = h \circ (q \circ \bar{c}) = (h \circ q) \circ \bar{c}$), and this implies smoothness of $h$. Remark: there exist $C^\infty$-quotient-maps for which smooth curves do not even lift locally, cf. (7.1.8) in [1].

**Lemma.** Smooth curves lift with respect to the (smooth) map $B_o: C^\infty(E,F) \longrightarrow F^{\mathbb{N}^n}$ defined by $B_o(f) = (\partial^\alpha f(0))_{\alpha \in \mathbb{N}^n}$.

**Proof.** Let $c: \mathbb{R} \longrightarrow F^{\mathbb{N}^n}$ be a smooth curve. We want to obtain $\bar{c}: \mathbb{R} \longrightarrow C^\infty(E,F)$ smooth with $B_o \circ \bar{c} = c$.
By cartesian closedness (cf. (1.4.3) in [1]) smooth curves $\bar{c}: \mathbb{R} \longrightarrow C^\infty(E,F)$ correspond to smooth functions $\Gamma: \mathbb{R}\pi E \longrightarrow F$ via $\bar{c}(t)(x) = \Gamma(t,x)$. Putting $c(t) = (c_\alpha(t))_{\alpha \in \mathbb{N}^n}$ the condition $B_o \circ \bar{c} = c$ gives for $\Gamma$ the condition

$$\partial^{(0,\alpha)}\Gamma(t,0) = c_\alpha(t)$$

for $t \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$.
Taking the derivative of order $\alpha_o$ with respect to $t$ we get from this

$$\partial^{(\alpha_0,\alpha_1,\ldots,\alpha_n)}\Gamma(t,0) = (c_{\alpha_1,\ldots,\alpha_n})^{(\alpha_0)}(t)$$

for $t \in \mathbb{R}$, $(\alpha_0,\ldots,\alpha_n) \in \mathbb{N}^{n+1}$.
Thus we are looking for a smooth function $\Gamma: \mathbb{R}\pi\mathbb{R}^n \longrightarrow \mathbb{R}^m$ for which its jet of infinite order at all points of the closed set $A := \mathbb{R}\pi\{o\} \subset \mathbb{R}\pi\mathbb{R}^n$ is given. The Whitney extension theorem [6] gives a necessary and sufficient condition for the existence of such a smooth function $\Gamma$. This condition is simplified in our situation since on A all except the first

variable are constant. In the formulation of [4] the condition requires that for each $M > 0$ and each $k \in \mathbb{N}$ one has

$$(c_{\alpha 1,\dots,\alpha n})^{(\alpha_0)}(t) = \sum_{i=0}^{k} \frac{1}{i!}(c_{\alpha 1,\dots,\alpha n})^{(\alpha_0+i)}(s)\cdot(t-s)^i + $$
$$+ o(|t-s|^k)$$

uniformly as $|t-s| \longrightarrow 0$, $t,s \in [-M,M]$.
That this is satisfied follows from Taylor's formula.

The author thanks J. Mather for having suggested the use of Whitney's theorem for the above lemma. For our purpose it would be enough that smooth curves lift locally, a result proved in [2] (Lemma 2.5, p. 98) or [3] (Lemma, p. 5).

**Theorem.** Smooth curves lift with respect to the Borel map $B: E\pi C^\infty(E,F) \longrightarrow E\pi F^{\mathbb{N}^n}$ and thus B is a $C^\infty$- quotient-map.

**Proof.** Let $c: \mathbb{R} \longrightarrow E\pi F^{\mathbb{N}^n}$ be a smooth curve. We put $c(t) = (c_1(t),c_2(t))$ where $c_1: \mathbb{R} \longrightarrow E$ and $c_2: \mathbb{R} \longrightarrow F^{\mathbb{N}^n}$ are smooth. By the Lemma $c_2$ lifts with respect to $B_o$, i. e. there exists $\bar{c}_2: \mathbb{R} \longrightarrow C^\infty(E,F)$ smooth with $c_2 = B_o \circ \bar{c}_2$.
One then defines $\gamma: \mathbb{R} \longrightarrow C^\infty(E,F)$ by $\gamma(t)(x) = \bar{c}_2(t)(x - c_1(t))$. By cartesian closedness $\gamma$ is smooth since the associated map $\mathbb{R}\pi E \longrightarrow F$ is smooth. One easily verifies that $\bar{c}: \mathbb{R} \longrightarrow E\pi C^\infty(E,F)$ defined by $\bar{c}(t) = (c_1(t),\gamma(t))$ is smooth and lifts c with respect to B.

## 2. The smooth structure of the jet-space $J^\infty(X,Y)$

We consider smooth manifolds X and Y. The space $C^\infty(X,Y)$ of all smooth maps $X \longrightarrow Y$ has by cartesian closedness a natural smooth structure. On $X\pi C^\infty(X,Y)$, equipped with the product structure, we consider the equivalence relation defined by

$$(x_1,f_1) \sim (x_2,f_2)$$

iff $x_1 = x_2$ and $(\phi\circ f_1\circ c)^{(i)}(0) = (\phi\circ f_2\circ c)^{(i)}(0)$ for each $i \in \mathbb{N}$, each $c: \mathbb{R} \longrightarrow X$ smooth with $c(0) = x_1$ and each $\phi: Y \longrightarrow \mathbb{R}$ smooth. We put:

$$J^\infty(X,Y) := (X\pi C^\infty(X,Y))/\sim$$

this set being provided with the quotient smooth structure. The elements of $J^{\infty}(X,Y)$ are the jets of infinite order. Obviously the map $pr_1: X\pi C^{\infty}(X,Y) \longrightarrow X$ factors over the canonical projection $\pi: X\pi C^{\infty}(X,Y) \longrightarrow J^{\infty}(X,Y)$ inducing a map $\sigma: J^{\infty}(X,Y) \longrightarrow X$. From $\sigma\circ\pi = pr_1$ and since $\pi$ is a $C^{\infty}$-quotient-map we deduce that $\sigma$ is smooth.

We next consider the evaluation map $ev: X\pi C^{\infty}(X,Y) \longrightarrow Y$. Since $(x_1,f_1) \sim (x_2,f_2)$ implies $\phi(f_1(x_1)) = \phi(f_2(x_2))$ for all $\phi \in C^{\infty}(Y,\mathbb{R})$ we deduce $ev(x_1,f_1) = ev(x_2,f_2)$ and hence the smooth map ev also factors over $\pi$ inducing a map $\tau: J^{\infty}(X,Y) \longrightarrow Y$ which by the same arguments is a smooth map.

For $\psi \in J^{\infty}(X,Y)$, $\sigma(\psi)$ is called the source, $\tau(\psi)$ the target of $\psi$.

Let $u: U \longrightarrow E$ and $v: V \longrightarrow F$ be local coordinates for X resp. Y. For simplicity we assume u and v to be surjective. Then if $x_i \in U$ and $f_i \in C^{\infty}(X,Y)$ with $f_i(x_i) \in V$ for $i = 1,2$ one has $(x_1,f_1) \sim (x_2,f_2)$ iff $x_1 = x_2$ and for each $\alpha \in \mathbb{N}^n$

$$\partial^{\alpha}(v\circ f_1\circ u^{-1})(u(x_1)) = \partial^{\alpha}(v\circ f_2\circ u^{-1})(u(x_2)).$$

One therefore gets a bijection between $E\pi F^{\mathbb{N}^n}$ and

$$J^{\infty}_{U,V}(X,Y) := \{\psi \in J^{\infty}(X,Y);\ \sigma(\psi) \in U \text{ and } \tau(\psi) \in V\}$$

These bijections form an atlas making $J^{\infty}(X,Y)$ a smooth Fréchet manifold.

**Theorem.** The smooth structure considered on $J^{\infty}(X,Y)$ is the same as the one coming from its structure of Fréchet manifold.

**Proof.** It is enough to prove the result for $X = E$, $Y = F$. For $(x_i,f_i) \in E\pi C^{\infty}(E,F)$ we have, using Borel map B:

$$(x_1,f_1) \sim (x_2,f_2) \iff B(x_1,f_1) = B(x_2,f_2).$$

Thus the surjective map B induces a bijective map $\bar{B}$ making commutative the diagram

$$\begin{array}{ccc} E\pi C^{\infty}(E,F) & \longrightarrow & E\pi F^{\mathbb{N}^n} \\ \downarrow & \nearrow_{\bar{B}} & \\ J^{\infty}(E,F) & & \end{array}$$

Since B is smooth and $\pi$ is a $C^{\infty}$-quotient-map, $\bar{B}$ is smooth. Since $\pi$ is smooth and B is a $C^{\infty}$-quotient-map, $(\bar{B})^{-1}$ is smooth.

## 3. Smooth differential operators

Let X, Y be smooth manifolds with Lindelöf property, $\omega$: X $\longrightarrow$ Y a $C^{\infty}$- quotient - map; V $\longrightarrow$ X and W $\longrightarrow$ Y smooth vector bundles with finite dimensional fibers. The space $\Gamma(X,V)$ of smooth sections X $\longrightarrow$ V is a Fréchet space; cf. (4.6.24) in [1].

A map $\Phi$: $\Gamma(X,V) \longrightarrow \Gamma(Y,W)$ is called $\omega$-local if $(\Phi f)(y)$ depends only on the germ of f at $\omega(y)$. The most important special case is the case X = Y and $\omega$ = Id.

The notion of $\omega$-local operator was used by Slovàk in [5] where he proved, using Whitney's extension theorem, that $\omega$-localness implies the following stronger property:

$(\Phi f)(y)$ only depends on the $\infty$-jet of f at $\omega(y)$.

**Theorem.** Let $\Phi$: $\Gamma(X,V) \longrightarrow \Gamma(Y,W)$ be an $\omega$-local operator. Then $\Phi$ is smooth iff expressed in local coordinates $(\Phi f)(y)$ depends smoothly on the $\infty$-jet of f at $\omega(y)$.

**Proof.** it is enough to consider the case where X = E and V = E$\pi$F is a product bundle and similarly Y = G and W = G$\pi$H, there $G = \mathbb{R}^p$, $H = \mathbb{R}^q$. In this case we obtain $\Phi$: $C^{\infty}(E,F) \longrightarrow C^{\infty}(G,H)$. By cartesian closedness $\Phi$ is smooth iff the associated map $\tilde{\Phi}$: $G\pi C^{\infty}(E,F) \longrightarrow H$ defined by $\tilde{\Phi}(y,f) = (\Phi f)(y)$ is smooth.

We further consider the map $\Psi$ which is the composite of maps

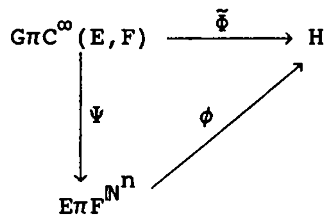
$$G\pi C^{\infty}(E,F) \xrightarrow{\omega\ \pi\ \mathrm{Id}} E\pi C^{\infty}(E,\ F) \xrightarrow{B} E\pi F^{\mathbb{N}^n}$$

$$(y,f) \longmapsto (\omega(y),f) \longmapsto (\omega(y),(\partial^{\alpha}f(\omega(y)))_{\alpha}).$$

The fact that $(\Phi f)(y)$ only depends on $\omega(y)$ and the $\infty$-jet of f at $\omega(y)$ is expressed by the property

$$\Psi(y_1,f_1) = \Psi(y_2,f_2) \Longrightarrow \tilde{\Psi}(y_1,f_1) = \tilde{\Psi}(y_2,f_2).$$

We therefore get a unique map $\phi$ making commutative the diagram




which means that $(\Psi f)(y) = \phi(\omega(y),(\partial^\alpha f(\omega(y)))_\alpha)$.

If we suppose $\Phi$ to be smooth, then $\tilde{\Phi}$ is smooth and using that $\Psi = B\circ(\omega\pi \mathrm{Id})$ is a $C^\infty$-quotient-map the smoothness of $\phi$ follows. Conversely $\phi$ smooth implies $\tilde{\Phi}$ and thus also $\Phi$ to be smooth.

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