DEMONSTRATIO MATHEMATICA
Vol. XXIV No 3-4 1991

Peter Dombrowski, Jürgen Zitterbarth

ON THE PLANETARY MOTION IN THE 3-DIM. STANDARD SPACES $\mathbb{M}^3_\kappa$ OF CONSTANT CURVATURE $\kappa\in\mathbb{R}$

0. Introduction

(i) On the history of the problem:

Between 1848 and 1851 Janos BOLYAI made already the proposal (see [Bo],p.156-157), to study the planetary motion around the sun in the noneuclidean hyperbolic 3-space with a *"radial" field of attraction* by the sun, *which – at a distance* r *from the sun – is reciprocally proportional to the area of the 2-dim. distance sphere of radius* r *in that geometry*. [With this proposal he intended to find results which could – possibly – allow to decide, whether the Euclidean resp. the hyperbolic geometry provides the "better" model for describing, what physicists can measure in reality.] – About the same time (~1851/52?) P. G. DIRICHLET mentioned (orally) to R. LIPSCHITZ (see [Lp_2],p.117, footnote), that he had investigated the theory of gravitational attraction according to NEWTON's law in the hyperbolic 3-space. – [After having treated the noneuclidean geometry of LOBATSCHEWSKI and BOLYAI in the spirit of Riemannian geometry (in particular see [Be]), E. BELTRAMI proved 1869 (see [St],p.472), that the constant speed *geodesics are the only path's of point-like particles moving in a Riemannian manifold* M *in the absence of external forces* (i.e. the corresponding competent Lagrangian is just the kinetic energy), thereby implicitly suggesting, that the LAGRANGE equations for the path $c:I \longrightarrow M$ of a particle in the Riemannian manifold M with an acceleration induced by some "force" F should be like $\nabla_\partial \dot{c} = F(\dot{c})$ with a vector field $F:TM \longrightarrow TM$ along $\pi:TM \longrightarrow M$ as "force".] — E. Schering described 1870 (see [Sch_1],p.318) the *analytic expression* $-k\cdot\coth(r)$ ($k\in\mathbb{R}_+$) *for* Newton's *gravitational potential* of a central star *in hyperbolic* 3-*space*, r being the function which measures the distance from that star, and R. Lipschitz (see [Lp_1],p.55 and [Lp_2],p.117, footnote) considered 1870 all functions V(r) depending only on the distance r from a fixed point in an n-dim. space form, which are harmonic (i.e. $\Delta V(r)=0$), thereby determining implicitly Newton's *gravitational potential for the* n-*dim. space forms*, and these *were explicitly listed* 1873 in an article by E. Schering (see [Sch_2],p.153, resp. [Ki_2],p.27).

W. Killing published in 1885 a fundamental article with a great variety of (*impressive*!) *results on* the dynamics in 1-connected n-dim. space forms of constant curvature κ, in particular about *the planetary motion in* $\mathbb{M}^3_\kappa$ (see [Ki_2],p.7-9), *verifying* (*resp. detecting "the correct" version of*) Kepler's *laws for* $\kappa\neq 0$. — H. Liebmann described 1905 the latter results on the planetary motion in more detail, especially he discussed the *geometric nature of the "conics", which occur as the* (*"plane"*) Kepler's *orbits of planets* in the noneuclidean space forms $\mathbb{M}^2_\kappa$ ($\kappa\neq 0$) (see [Li],p.219-236 and p.182-196).

(ii) Plan of this article:

We want to give in this article a rather complete account of the planetary motion in the simply connected space forms $\mathbb{M}^n_\kappa$ of constant curvature κ (in dimension n=3) and this in a *uniform* treatment *with respect to the curvature constant* κ. We try to formulate the corresponding results in explicit dependence on κ ($\in\mathbb{R}$) in order to gain perfect control about the changes of the "laws of nature" when κ changes, in particular to see how stably these laws behave when perturbing the value $\kappa=0$, i.e. when passing from Euclidean to noneuclidean geometry.

Of course, most of the important results are stated resp. mentioned already somewhere in the literature (in particular see [Ki_2] and [Li], loc.cit.). But in many cases these statements are (in the available literature) rather incomplete, sometimes not presented in an optimal form, or their proofs are not quite sufficient resp. rather "out of date", since several appropriate tools (e.g. technics of covariant derivation or BUSEMANN functions in the hyperbolic spaces) were not available, when these articles were written. So we thought it to be worthwile to make these (wonderful classical) results and proofs of them accessible in a rather rigid and complete version, including several improvements (e.g. concerning the completeness of the list of all possible "KEPLER orbits" in $\mathbb{M}^2_\kappa$).

(iii) Organization of the article:

§ 1 summarizes the definitions and basic properties of (what we call) the "*κ-geometric*" functions $\sin_\kappa$, $\cos_\kappa$,... [which essentially are the trigonometric ones for $\kappa>0$ resp. the hyperbolic ones for $\kappa<0$] in a uniform way w.r.t. $\kappa\in\mathbb{R}$ (avoiding the complex domain) and pointing out their real analytic dependence on κ.

§ 2 describes some basic geometry of the *space forms* $\mathbb{M}^n_\kappa$ *represented as* (what we call) *the* WEIERSTRASS-*models* (i.e. as hypersurfaces in $\mathbb{R}^{n+1}$, where $\mathbb{R}^{n+1}$ is endowed with a Riemannian metric if $\kappa\geq0$ and with a Lorentzian metric if $\kappa<0$). KILLING used these models in his publications successfully, but he attributed the invention of these models to his teacher WEIERSTRASS in 1872 (see [Ki_1],p.74, footnote). [These models, well-known for $\kappa\in\{0,+1,-1\}$, seem to be not so familiar for $\kappa\neq0,\pm1$, at least they differ from the corresponding models one usually finds today.] The WEIERSTRASS models for $\mathbb{M}^n_\kappa$ prove however to be optimally apt for getting the LIE group G^n_κ of all orientation preserving isometries of $\mathbb{M}^n_\kappa$ (resp. its LIE algebra $\mathfrak{g}^n_\kappa$), considered as a subgroup of $GL_+(n+1,\mathbb{R})$ (resp. of $\mathfrak{gl}(n+1,\mathbb{R})$), *analytically dependent on* κ, which was essential for the subsequent article (see [Zi_2]) in this journal, studying motions of rigid bodies in $\mathbb{M}^3_\kappa$.

§ 3 contains, aside from a rather subtle version of the "*inverse of the law of energy*" (see § 3.ii,iv), a list (without proofs) of "*a priori information*" about maximal solutions of the differential equation describing a mechanical process of one degree of freedom. These results are essentially known, but a direct reference (for all needed information) seemed to be not available in the literature.

§ 4 gives an analytic description (and a metric classification) of *conics in* $\mathbb{M}_\kappa^2$ which is important for verifying KEPLER's first law, later.

§ 5 summarizes the basic (differential) "*equations of motion*" for a path $c:I \longrightarrow \mathbb{M}_\kappa^n$ moving *in a "central force field"*, it is shown that such *motions are "plane"* and satisfy KEPLER's *second law*, moreover the equations are specialized, when this "central field" is induced by NEWTON's *gravitational potential* (of a central "star").

§ 6 gives, finally, a complete list of all possible motions of point-like particles in $\mathbb{M}_\kappa^3$ with an acceleration induced by NEWTON's gravitational potential.

The results and proofs of this article are in part taken from informal lecture notes on differential geometry and mechanics by the first resp. from the doctoral thesis of the second author.

1. The κ-geometric functions $\sin_\kappa$ and $\cos_\kappa$ for $\kappa\in\mathbb{R}$

(i) The geometry (resp. the mechanics) in Riemannian manifolds of constant sectional curvature κ (∈ℝ) is governed by the two *κ-geometric functions on* ℝ defined by

$$(1.1)\quad \begin{cases} \sin_\kappa(x) := \sum_{\nu=0}^{\infty} \frac{(-\kappa)^\nu}{(2\nu+1)!}\cdot x^{2\nu+1} = \begin{cases} \frac{1}{\sqrt{-\kappa}}\cdot\sinh(\sqrt{-\kappa}x & \textit{for } \kappa<0 \\ x & \textit{for } \kappa=0 \\ \frac{1}{\sqrt{\kappa}}\cdot\sin(\sqrt{\kappa}x) & \textit{for } \kappa>0 \end{cases} \\ \cos_\kappa(x) := \sum_{\nu=0}^{\infty} \frac{(-\kappa)^\nu}{(2\nu)!}\cdot x^{2\nu} = \begin{cases} \cosh(\sqrt{-\kappa}x) & \textit{for } \kappa<0 \\ 1 & \textit{for } \kappa=0 \\ \cos(\sqrt{\kappa}x) & \textit{for } \kappa>0 \end{cases} \end{cases}$$

hence $\sin_\kappa(x)$ and $\cos_\kappa(x)$ are real analytic functions of $(\kappa,x)\in\mathbb{R}^2$.

$\sin_\kappa$ and $\cos_\kappa$ are as well characterized as solutions on $\mathbb{R}$ of the first order ODE system

$$(1.1')\quad \begin{cases} \sin_\kappa' = \cos_\kappa \quad \textit{and} \quad \cos_\kappa' = -\kappa\sin_\kappa \quad \textit{with} \\ \sin_\kappa(0) = 0 \quad \textit{and} \quad \cos_\kappa(0) = 1 , \end{cases}$$

resp. as a (maximally defined) fundamental system of solutions of the second order ODE system

$$(1.1'')\quad \begin{cases} y'' + \kappa\cdot y = 0 \quad \textit{with} \\ \sin_\kappa(0) = \cos_\kappa'(0) = 0 \quad \textit{and} \quad \sin_\kappa'(0) = \cos_\kappa(0) = 1 . \end{cases}$$

(ii) Moreover we shall use the functions

$$(1.2)\quad \begin{cases} \tan_\kappa(x) := \dfrac{\sin_\kappa(x)}{\cos_\kappa(x)} = \begin{cases} (-\kappa)^{-1/2}\tanh(\sqrt{-\kappa}x) & \textit{for } \kappa<0 \\ x & \textit{for } \kappa=0 \\ \kappa^{-1/2}\tan(\sqrt{\kappa}x) & \textit{for } \kappa>0 \end{cases} \\[2ex] \cot_\kappa(x) := \dfrac{\cos_\kappa(x)}{\sin_\kappa(x)} = \begin{cases} \sqrt{-\kappa}\coth(\sqrt{-\kappa}x) & \textit{for } \kappa<0 \\ \dfrac{1}{x} & \textit{for } \kappa=0 \\ \sqrt{\kappa}\cot(\sqrt{\kappa}x) & \textit{for } \kappa>0 \end{cases} \end{cases}$$

with

$$(1.2')\quad \tan_\kappa' = \cos_\kappa^{-2} , \qquad \cot_\kappa' = - \sin_\kappa^{-2}$$

and

$$(1.3)\quad \begin{cases} \psi_\kappa(x) := \displaystyle\int_0^x \sin_\kappa(t)\,dt = 2\cdot\sin_\kappa^2(\tfrac{x}{2}) = \\ \qquad = \begin{cases} \dfrac{x^2}{2} & \textit{for } \kappa=0 \\ \dfrac{1}{\kappa}\cdot(1-\cos_\kappa(x)) & \textit{for } \kappa\neq 0 \end{cases} \end{cases}$$

hence

$$(1.3'')\quad \psi_\kappa''(x)+\kappa\cdot\psi_\kappa(x) = 1 \quad \textit{with} \quad \psi_\kappa(0) = \psi_\kappa'(0) = 0 .$$

(iii) These functions fulfill the following relations:

$$(1.4)\quad \sin_\kappa(-x) = -\sin_\kappa(x) , \quad \cos_\kappa(-x) = \cos_\kappa(x) ,$$

$$(1.5)\quad \begin{cases} \sin_\kappa(x+y) = \sin_\kappa(x)\cos_\kappa(y) + \cos_\kappa(x)\sin_\kappa(y) , \\ \cos_\kappa(x+y) = \cos_\kappa(x)\cos_\kappa(y) - \kappa\cdot\sin_\kappa(x)\sin_\kappa(y) , \end{cases}$$

$$(1.6)\qquad \cos_\kappa^2(x) + \kappa\cdot\sin_\kappa^2(x) = 1\ ,\quad \cot_\kappa^2(x) = \sin_\kappa^{-2}(x) - \kappa\ ,$$

$$(1.7)\quad \begin{cases} \sin_\kappa(2x) = 2\cdot\sin_\kappa(x)\cos_\kappa(x) = \dfrac{2\cdot\tan_\kappa(x)}{1+\kappa\cdot\tan_\kappa^2(x)}\ , \\ \cos_\kappa(2x) = \cos_\kappa^2(x) - \kappa\cdot\sin_\kappa^2(x) = 2\cdot\cos_\kappa^2(x) - 1 = \\ \qquad = 1 - 2\kappa\cdot\sin_\kappa^2(x) = \dfrac{1-\kappa\cdot\tan_\kappa^2(x)}{1+\kappa\cdot\tan_\kappa^2(x)}\ , \end{cases}$$

$$(1.8)\quad \begin{cases} \tan_\kappa(x+y)\cdot(1-\kappa\cdot\tan_\kappa(x)\tan_\kappa(y)) = \tan_\kappa(x) + \tan_\kappa(y), \\ \cot_\kappa(x+y)\cdot(\cot_\kappa(x)+\cot_\kappa(y)) = \cot_\kappa(x)\cdot\cot_\kappa(y) - \kappa\ . \end{cases}$$

(iv) If we define

$$(1.9)\quad \begin{cases} \pi_\kappa := \begin{cases} +\infty & \textit{for } \kappa\le 0 \\ \pi/\sqrt{\kappa} & \textit{for } \kappa>0 \end{cases}, \\ \textit{then} \begin{cases} \sin_\kappa(t) > 0 & \textit{for all } t\in]0,\pi_\kappa[\\ \cos_\kappa(t) > 0 & \textit{for all } t\in]-\frac{1}{2}\cdot\pi_\kappa,\frac{1}{2}\cdot\pi_\kappa[\ , \end{cases} \end{cases}$$

and

$$(1.10)\quad \begin{cases} \textit{For all } \alpha,\beta\in\mathbb{R} \textit{ with } \alpha^2+\kappa\cdot\beta^2 = 1 \textit{ and } (\alpha\ge 0 \textit{ if } \kappa\le 0) \\ \textit{there exists exactly one } t\in]-\pi_\kappa,\pi_\kappa] \textit{ such that} \\ \cos_\kappa(t) = \alpha \textit{ and } \sin_\kappa(t) = \beta\ . \end{cases}$$

2. The WEIERSTRASS model $\mathbb{M}_\kappa^n$ for "the" n-dim. 1-connected complete Riemannian C^ω-manifold of constant curvature $\kappa\in\mathbb{R}$

(i) Notations for $\mathbb{R}^{n+1}$. The bilinear forms $<..,..>_\kappa$ and $(..|..)_\kappa$ on $\mathbb{R}^{n+1}$.

We look at $\mathbb{R}^{n+1}$ (with its standard coordinate functions $x_0,\ldots,x_n$) as $\mathbb{R}\times\mathbb{R}^n$, and correspondingly the elements

$$(2.0)\quad \begin{cases} \mathbf{a}\in\mathbb{R}^{n+1} \textit{ will be denoted by } \mathbf{a} = (a_0,a) \textit{ with } a_0\in\mathbb{R} \textit{ and} \\ a\in\mathbb{R}^n, \textit{ let } <..,..> \textit{ denote the canonical Euclidean inner} \\ \textit{product of } \mathbb{R}^n, \mathbf{o} \textit{ the origin, } \mathbf{e} := \mathbf{e}_0 := (1,0,\ldots,0)\ , \\ \mathbf{e}_1,\ldots,\mathbf{e}_n \textit{ the canonical basis of } \mathbb{R}^{n+1}. \end{cases}$$

For every $\kappa\in\mathbb{R}$ we define two symmetric bilinear forms $<..,..>_\kappa$ and $(..|..)_\kappa$ on $\mathbb{R}^{n+1}$: For all $\mathbf{a},\mathbf{b}\in\mathbb{R}^{n+1}$ (see (2.0))

$$(2.1)\quad \begin{cases} \langle a,b\rangle_\kappa := a_0b_0 + \kappa\cdot\langle a,b\rangle = a_0b_0 + \kappa\cdot\sum_{i=1}^{n} a_ib_i \\ (a|b)_\kappa := \begin{cases} a_0b_0 + \langle\ ,b\rangle & , \text{ if } \kappa=0 \\ \frac{1}{\kappa}\cdot a_{0\ 0} + \langle a,b\rangle , & \text{if } \kappa\neq 0 , \end{cases} \end{cases}$$

consequently

$$(2.2)\qquad \kappa\neq 0 \Rightarrow \langle ..,..\rangle_\kappa = \kappa\cdot(..|..)_\kappa .$$

Let e_κ denote the C^ω tensor field of type (0,2) on $\mathbb{R}^{n+1}$, such that for all $p\in\mathbb{R}^{n+1}$ and $v,w\in T_p\mathbb{R}^{n+1}$ (see (2.1)):

$$(2.3)\quad \begin{cases} e_\kappa(v,w) := (\vec{v}|\vec{w})_\kappa , \\ \qquad \textit{where } \vec{v} := (d_px_0(v),\dots,d_px_n(v)) \\ \qquad \textit{resp. } v = (t \mapsto p + t\cdot\vec{v})^\cdot(0) . \end{cases}$$

Therefore e_κ is a Riemannian resp. Lorentzian C^ω metric on $\mathbb{R}^{n+1}$ if $\kappa\geq 0$ resp. $\kappa<0$, and it is invariant under translations of $\mathbb{R}^{n+1}$, its Levi-Civita covariant derivative $\overset{o}{\nabla}$ therefore being the canonical torsion- and curvature-free covariant derivative of the abelian Lie (vector-) group $\mathbb{R}^{n+1}$. We denote by

$$(2.4)\quad \begin{cases} \mathbb{E}_\kappa^{n+1} := (\mathbb{R}^{n+1},e_\kappa) \textit{ this Riemannian resp. Lorentzian } C^\omega \\ \textit{manifold and } \overset{o}{\nabla} \textit{ its Levi-Civita covariant derivative} \\ \textit{with } \overset{o}{\nabla}_X Y := (\vec{Y})_* X , \quad \textit{i.e.} \quad (\overset{o}{\nabla}_X Y)^{\rightarrow} = X\cdot(\vec{Y}) \quad \textit{for} \\ \textit{all } X,Y\in\mathfrak{X}(\mathbb{R}^{n+1}). \end{cases}$$

(ii) The definition of $\mathbb{M}_\kappa^n$ as a certain submanifold of $\mathbb{R}^{n+1}$. $(\mathbb{M}_\kappa^n)_{\kappa\in\mathbb{R}}$ is a C^ω family of n-dim. Riemannian manifolds. The inner metric d_κ and the exponential map $\exp_\kappa$ of $\mathbb{M}_\kappa^n$

For $n\in\mathbb{N}$, $n\geq 2$, and $\kappa\in\mathbb{R}$ the

$$(2.5)\quad \begin{cases} \textit{n-dim. Riemannian } C^\omega \textit{ manifold } \mathbb{M}_\kappa^n \textit{ is defined to have} \\ \textit{the n-dim. regular } C^\omega \textit{ submanifold of } \mathbb{R}^{n+1} \text{ (see (2.1))} \\ \quad i\colon \{p\in\mathbb{R}^{n+1} \mid \langle p,p\rangle_\kappa=1 \text{ and } (\kappa\leq 0 \Rightarrow p_0>0)\} \hookrightarrow \mathbb{R}^{n+1} \\ \textit{as underlying } C^\omega \textit{ manifold and } g := i^*e_\kappa \textit{ as its} \\ \textit{Riemannian metric } \text{(see (2.3))}. \textit{ Let } \nabla \textit{ denote its Levi-} \\ \textit{Civita covariant derivative.} \end{cases}$$

Remarks.

a) $\mathbb{M}^n_\kappa$, as a C^ω-manifold, is a connected component of a quadric hypersurface of $\mathbb{R}^{n+1}$, namely for

$\kappa<0$: the "*upper*" *sheet* ($x_0(\mathbb{M}^n_\kappa)\subseteq\mathbb{R}_+$!) *of the two sheeted hyperboloid of revolution* (around the x_0-axis) in $\mathbb{R}^{n+1}$ with center o and principal axes $\mathbf{e}$, $(-\kappa)^{-1/2}\cdot\mathbf{e}_1,\ldots,(-\kappa)^{-1/2}\cdot\mathbf{e}_n$ (see (2.0)),

$\kappa=0$: the *affine hyperplane* $\mathbf{e}+(\{0\}\times\mathbb{R}^n) = x_0^{-1}(\{1\})$ of $\mathbb{R}^{n+1}$,

$\kappa>0$: the *ellipsoid of revolution* (around the x_0-axis) in $\mathbb{R}^{n+1}$ with center o and principal axes $\mathbf{e}$, $\kappa^{-1/2}\cdot\mathbf{e}_1,\ldots,\kappa^{-1/2}\cdot\mathbf{e}_n$,

hence ($n\geq2$!):

(2.6) $\mathbf{e}\in\mathbb{M}^n_\kappa$, $\mathbb{M}^n_\kappa$ *is 1-connected and* ($\mathbb{M}^n_\kappa$ *compact* $\Leftrightarrow$ $\kappa>0$).

b) For $\kappa\geq0$ the positive definiteness of g follows from the same property of e_κ, for $\kappa<0$ it follows from the well known fact, that in Lorentzian vector spaces the orthogonal complement of a time-like vector is a space-like vector subspace.

c) The Riemannian manifold $\mathbb{M}^n_\kappa$, as a complete one of constant curvature κ (see (2.9),(2.6)), seems to have been first introduced (for n=3 and $\kappa\neq0$) by K. Weierstrass during Seminar talks he gave in 1872 (according to [Ki_1],p.74, footnote) and $\mathbb{M}^n_\kappa$ was subsequently used successfully by his doctorand W. Killing (see e.g. [Ki_2],p.4).

d) We have inserted in Weierstrass' definition the $\mathbb{M}^n_0$ for $\kappa=0$, to get a "C^ω *family*" $(\mathbb{M}^n_\kappa)_{\kappa\in\mathbb{R}}$ of 1-connected (see a)), complete Riemannian manifolds of constant curvature κ (see (2.6),(2.9)). This family seems to be optimally apt for *studying the dependence on the curvature constant* κ of geometric theorems resp. of "physical laws of nature" in spaces of constant curvature κ: The main reason for this being at first:

$(\mathbb{M}^n_\kappa)_{\kappa\in\mathbb{R}}$ *is a* C^ω *family of n-dim. Riemannian manifolds.*

[This concept means: If J is a k-dim. C^ω manifold, we call $(M_\iota)_{\iota\in J}$ a C^ω family of n-dim. Riemannian manifolds, if there is a (n+k)-dim. C^ω manifold N, a n-dim. C^ω foliation M of N, a Riemannian C^ω metric g for the vector subbundle TM of TN over N (not only over M !), thus turning each leaf M(q) of M through a point $q\in N$ into a Riemannian C^ω manifold, and there exists a C^ω map $f:J \longrightarrow N$ such that for all $\iota\in J$ the Riemannian manifold M_ι is C^ω isometric to the leaf $M(f(\iota))$. In our special case of the family $(\mathbb{M}^n_\kappa)_{\kappa\in\mathbb{R}}$ we have $J := \mathbb{R}$, we can take for N the (n+1)-dim. regular algebraic (hence C^ω) hypersurface $N := \varphi^{-1}(\{1\})$ of the open submanifold $\tilde{N}$ of $\mathbb{R}^{n+2}$ with $\tilde{N} := ((\mathbb{R}_+\times\mathbb{R}^n)\times\mathbb{R}) \cup (\mathbb{R}^{n+1}\times\mathbb{R}_+)$ and $\varphi := (y_0^2 + (y_1^2+\ldots+y_n^2)\cdot y_{n+1})|\tilde{N}$, where $y_0,\ldots,y_{n+1}:\mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ are the canonical coordinate functions, the foliation M of N is given by the hyperplane sections y_{n+1} = const of N, the leaf of M through $(e,\kappa)\in N$ being $\mathbb{M}^n_\kappa\times\{\kappa\}$. A Riemannian C^ω metric g for the vector subbundle TM of TN over N is given by: For all $q\in N(\subseteq\mathbb{R}^{n+2})$ and all $v,w\in T_qM$ we have (where T_qM is identified with a vector subspace of $T_q\mathbb{R}^{n+2}$):

$$(2.7_1)\quad g(v,w) := \frac{1}{q_{n+1}}\cdot v_0w_0 + \sum_{i=1}^n v_iw_i \qquad \textit{for } q_{n+1}\in\mathbb{R}^*$$

$$(2.7_2)\quad g(v,w) := \frac{q_{n+1}}{q_0^2}\cdot\Big(\sum_{i=1}^n q_iv_i\Big)\cdot\Big(\sum_{i=1}^n q_iw_i\Big) + \sum_{i=1}^n v_iw_i \qquad \textit{for } q_0\in\mathbb{R}^*,$$

where $q_i := y_i(q)$ and $v_i := dy_i(v)$ for $i=0,\ldots,n+1$. Observe, that for all $q\in N=\varphi^{-1}(\{1\})$ we have (by definition of φ) at least $q_0\neq0$ or $q_{n+1}\neq0$, therefore g is defined by (2.7_1) and (2.7_2) on all of N. Moreover, these two C^ω definitions (2.7_1) and (2.7_2) of g on open subsets of N coincide on $N \cap \{q\in\mathbb{R}^{n+2}\,|\,q_0,q_{n+1}\in\mathbb{R}^*\}$: Since for $v\in T_qM$ we have $d_q\varphi(v) = d_qy_{n+1}(v) = 0$ (the leaf of M through $q\in N$ is the connected component of $\varphi^{-1}(\{1\}) \cap y_{n+1}^{-1}(\{y_{n+1}(q)\})$!), it follows:

$$\frac{1}{2}\cdot d_q\varphi(v) \underset{\substack{\uparrow\\ d_qy_{n+1}(v)=0}}{=} q_0v_0 + q_{n+1}\cdot\sum_{i=1}^n q_iv_i = 0\ ,\ \textit{i.e.}\ v_0 = -\frac{q_{n+1}}{q_0}\cdot\sum_{i=1}^n q_iv_i .$$

Substituting this value of v_0 and the analogous value of w_0 in (2.7_1) gives (2.7_2). Finally $f:\mathbb{R} \longrightarrow N$ can be chosen as $\kappa \mapsto (1,0,\ldots,0,\kappa)$, the isometry of $\mathbb{M}^n_\kappa$ with the leaf

$M(f(\kappa)) = \mathbb{M}^n_\kappa \times \{\kappa\}$ is given by $p \mapsto (p,\kappa)$.]

If we introduce now on $\mathbb{R}^{n+1}$ its EULER *vector field* E by (see (2.3))

(2.8_1) $E \in \mathfrak{X}(\mathbb{R}^{n+1})$ *with* $E^{\to} = \mathrm{Id}$, *then* $\overset{o}{\nabla}_X E = X$ *for* $X \in \mathfrak{X}(\mathbb{R}^{n+1})$

(see (2.4)) and if $i : \mathbb{M}^n_\kappa \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion map, then (see (2.2),...,(2.5)):

(2.8_2) $\langle i,i\rangle_\kappa = 1$, $(E\circ i)^{\to} = i$, $(\kappa\neq 0 \Rightarrow \kappa\cdot e_\kappa(E\circ i, E\circ i)=1)$,

and

(2.8_3) $i_* T_p \mathbb{M}^n_\kappa = \{v \in T_p\mathbb{R}^{n+1} \mid \langle p, v^{\to}\rangle_\kappa = 0\}$ *for all* $p \in \mathbb{M}^n_\kappa$,

in particular (see (2.2),(2.3),(2.8_2),(2.8_3)):

(2.8_4) $\kappa\cdot e_\kappa(E\circ i, i_* X) = 0$ *for all* $X \in \mathfrak{X}(\mathbb{M}^n_\kappa)$.

Therefore (see (2.8_2),(2.8_3),(2.8_4)) we obtain a *unit normal field* ξ_κ *along* $i : \mathbb{M}^n_\kappa \hookrightarrow \mathbb{E}^{n+1}_\kappa$ by setting

$$(2.8_5)\quad \begin{cases} \xi_\kappa := \begin{cases} (\partial/\partial x_0)\circ i & \text{for } \kappa=0 \\ -\sqrt{|\kappa|}(E\circ i) & \text{for } \kappa\neq 0 \end{cases}, \quad i.e. \\ \text{for all } p\in\mathbb{M}^n_\kappa \text{ one has } \xi_\kappa(p)^{\to} = \begin{cases} e & \text{if } \kappa=0 \\ -\sqrt{|\kappa|}\cdot p & \text{if } \kappa\neq 0 \end{cases}, \end{cases}$$

more precisely (see (2.1),(2.8_2),(2.8_4),(2.8_1))

$$(2.8_6)\quad \begin{cases} e_\kappa(\xi_\kappa,\xi_\kappa) = \operatorname{sgn}\kappa := \begin{cases} +1 & \text{for } \kappa\geq 0 \\ -1 & \text{for } \kappa<0 \end{cases}, \\ e_\kappa(\xi_\kappa, i_* X) = 0 \text{ and } \overset{o}{\nabla}_X \xi_\kappa = -\sqrt{|\kappa|}\, i_* X \text{ for } X\in\mathfrak{X}(\mathbb{M}^n_\kappa), \end{cases}$$

therefore $\mathbb{M}^n_\kappa$ *is an umbilical hypersurface in* $\mathbb{E}^{n+1}_\kappa$ *of mean curvature* $\sqrt{|\kappa|}$ (w.r.t. the normal field ξ_κ, in particular $\mathbb{M}^n_0$ is totally geodesic in $\mathbb{E}^{n+1}_0$).

Moreover we have the following relation between the LEVI-CIVITA covariant derivative ∇ of $\mathbb{M}^n_\kappa$ and the one $\overset{o}{\nabla}$ of $\mathbb{E}^{n+1}_\kappa$ (see (2.4)), depending in a real analytic way on $\kappa\in\mathbb{R}$:

(2.9) $i_*(\nabla_X Y) = \overset{o}{\nabla}_X i_* Y - \kappa\cdot g(X,Y)\cdot(E\circ i)$ *for all* $X,Y\in\mathfrak{X}(\mathbb{M}^n_\kappa)$,

in particular (see (2.5),(2.8_4),(2.9))

(2.10) $g(\nabla_X Y, Z) = e_\kappa(\overset{o}{\nabla}_X i_* Y, i_* Z)$ *for all* $X,Y,Z\in\mathfrak{X}(\mathbb{M}^n_\kappa)$,

and we get from (2.9) for every C^∞

(2.9_1) *path* $c:I \longrightarrow \mathbb{M}^n_\kappa$: $(i_*\nabla_\partial \dot{c})^{\rightarrow} = (i\circ c)'' - \kappa\cdot g(\dot{c},\dot{c})\cdot(i\circ c)$,

where I is an open interval of $\mathbb{R}$, ∂ the canonical vector field d/dx on $\mathbb{R}$ and where we define for any C^∞ path $\gamma:I \longrightarrow V$ in a real vector space V the C^∞ path $\gamma':I \longrightarrow V$ in the elementary way by

(2.11) $(\dot{\gamma}(t))^{\rightarrow} := \gamma'(t) := \lim_{h\to 0}(\frac{1}{h}\cdot(\gamma(t+h)-\gamma(t))$ *for all* $t\in I$

and then $\gamma'' := (\gamma')'$ etc.

Finally one gets from (2.10),(2.9) and (2.8_4) immediately

$\forall_{X,Y,Z,W\in\mathfrak{X}(\mathbb{M}^n_\kappa)}$ $g(\nabla_X\nabla_Y Z,W) = e_\kappa(\overset{o}{\nabla}_X\overset{o}{\nabla}_Y i_*Z, i_*W) + \kappa\cdot g(X,W)g(Y,Z)$.

Since the curvature tensor of $\overset{o}{\nabla}$ vanishes, it follows from the last statement and (2.10) for the curvature tensor R of $\mathbb{M}^n_\kappa$

(2.12) $\begin{cases} \mathbb{M}^n_\kappa \textit{ is of constant curvature } \kappa, \\ \textit{and moreover } \mathbb{M}^n_\kappa \textit{ is complete.} \end{cases}$

The completeness of $\mathbb{M}^n_\kappa$ follows by the fact, that (see (2.9_1)) a C^∞ path $c:I \longrightarrow \mathbb{M}^n_\kappa$ is a maximally defined unit speed geodesic iff $i\circ c:I \longrightarrow \mathbb{R}^{n+1}$ is a maximal solution of $(i\circ c)'' + \kappa\cdot(i\circ c) = 0$, which implies by (1.1″): $I=\mathbb{R}$ and (see (2.11))

(2.13_0) $c(t) = \cos_\kappa(t)c(0) + \sin_\kappa(t)\dot{c}(0)^{\rightarrow}$ *for all* $t\in\mathbb{R}$,

i.e. the *exponential map* $\exp_\kappa$ of $\mathbb{M}^n_\kappa$ satisfies

(2.13) $\begin{cases} \textit{for all } p\in\mathbb{M}^n_\kappa,\ u\in T^1_p\mathbb{M}^n_\kappa \textit{ and } t\in\mathbb{R}: \\ i\circ\exp_\kappa(tu) = \cos_\kappa(t)p + \sin_\kappa(t)(i_*u)^{\rightarrow}, \end{cases}$

from where one checks easily

(2.13_1) $\exp_\kappa$ *has at each point of* $\mathbb{M}^n_\kappa$ *the injectiv. radius* π_κ.

From (2.13),(2.13_1) one derives immediately (see (1.1),(1.3),(2.1)):

(2.14) $\begin{cases} \textit{The inner metric } d_\kappa \textit{ of } \mathbb{M}^n_\kappa \textit{ satisfies for all } p,q\in\mathbb{M}^n_\kappa: \\ \cos_\kappa(d_\kappa(p,q)) = \langle p,q\rangle_\kappa \textit{ resp. } 2\psi_\kappa(d_\kappa(p,q)) = (q-p|q-p)_\kappa. \end{cases}$

From (1.9),(2.14) follows

(2.14$_1$) $\operatorname{diam}(\mathbb{M}^n_\kappa)=\pi_\kappa$, *and for all* $p,q\in\mathbb{M}^n_\kappa$: $d_\kappa(p,q)=\pi_\kappa \Leftrightarrow q=-p$.

Therefore, according to (2.13$_1$),(2.14$_1$), we have

$$(2.15)\quad \left\{\begin{array}{l} \textit{for every } p\in\mathbb{M}^n_\kappa \textit{ is } \exp_\kappa|\{v\in T_p\mathbb{M}^n_\kappa \mid \|v\|_g<\pi_\kappa\} \\ \textit{a } C^\omega \textit{ diffeomorphism onto } \mathbb{M}^n_\kappa\setminus\{-p\} \end{array}\right.$$

and consequently there exists (see (2.15)) a well-defined C^ω unit vector field U_p along the "collapsing" map $\mathbb{M}^n_\kappa(p) \longrightarrow \{p\}$, i.e. a

$$(2.16)\quad \left\{\begin{array}{l} C^\omega \text{ map } U_p:\mathbb{M}^n_\kappa(p):=\mathbb{M}^n_\kappa\setminus\{p,-p\} \longrightarrow T^1_p\mathbb{M}^n_\kappa \textit{ characterized by} \\ \exp_\kappa(d_\kappa(p,q)\cdot U_p(q)) = q \textit{ for all } q\in\mathbb{M}^n_\kappa(p):=\mathbb{M}^n_\kappa\setminus\{p,-p\}, \end{array}\right.$$

i.e. $U_p(q) = \dot c(0)$, where $c:[0,d_\kappa(p,q)] \longrightarrow \mathbb{M}^n_\kappa$ is the unique unit speed geodesic with $c(0) = p$ and $c(d_\kappa(p,q)) = q$.

Moreover this last geodesic $c:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ is explicitly given by (see (2.14$_1$)):

$$(2.16_1)\quad \left\{\begin{array}{l} c(t) = \dfrac{1}{\sin_\kappa(\ell)}\cdot(\sin_\kappa(\ell-t)\cdot p + \sin_\kappa(t)\cdot q) \quad \textit{for } t\in\mathbb{R} \\ \textit{with } \ell := d_\kappa(p,q) . \end{array}\right.$$

[For proving $\langle c,c\rangle_\kappa = 1$ use (1.5) and $\cos_\kappa(\ell) = \langle p,q\rangle_\kappa$ (see (2.14)) and then verify $c'' + \kappa\cdot c = o$, see the remark before (2.13$_0$).]

From (2.16),(1.1) follows:

$$(2.16_2)\quad \left\{\begin{array}{l} \textit{The } C^\omega \textit{ map } d_\kappa(p,..)\cdot U_p \textit{ resp. } (\sin_\kappa\circ d_\kappa(p,..))\cdot U_p \\ \textit{from } \mathbb{M}^n_\kappa\setminus\{p,-p\} \textit{ to } T_p\mathbb{M}^n_\kappa \textit{ has a } C^\omega \textit{ extension onto} \\ \mathbb{M}^n_\kappa\setminus\{-p\} \textit{ resp. onto all of } \mathbb{M}^n_\kappa \textit{ with value } o \textit{ at } p, \end{array}\right.$$

the first extension being the inverse of the C^ω diffeomorphism (2.15), the second extension $\ell:\mathbb{M}^n_\kappa \longrightarrow T_p\mathbb{M}^n_\kappa$ characterized by: $i(p)+(i_*\ell)^{\rightarrow}:\mathbb{M}^n_\kappa \longrightarrow \mathbb{E}^{n+1}_\kappa$ is the composition of the inclusion map $i:\mathbb{M}^n_\kappa \longrightarrow \mathbb{E}^{n+1}_\kappa$ with the $(..|..)_\kappa$-orthogonal projection of $i(\mathbb{M}^n_\kappa)$ onto $i(p)+(i_*T_p\mathbb{M}^n_\kappa)^{\rightarrow}$, and e.g. in case $p=e$ this amounts to $(i_*\ell(q))^{\rightarrow} = (0,q)$ for $q=(q_0,q)\in\mathbb{M}^n_\kappa$ (see (2.0)).

Finally it follows from (2.1),(2.3),(2.5),(2.13),(2.13$_1$), that for $k\in\{2,\ldots,n\}$ the canonical "inclusion" map

$\mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{n+1}$ $((a_0,\ldots,a_k) \mapsto (a_0,\ldots,a_k,0,\ldots,0))$ induces

(2.17_0) $\left\{\begin{array}{l}\text{"the canonical" isometric, distance preserving,} \\ \text{totally geodesic } C^\omega \text{ immersion } \mathbb{M}^k_\kappa \hookrightarrow \mathbb{M}^n_\kappa\end{array}\right.$,

and moreover, if $k\in\{1,\ldots,n\}$ and $\mathcal{V}_k:=\{V|V$ is a $(k+1)$-dim. vector subspace of $\mathbb{R}^{n+1}$ with $V\cap\mathbb{M}^n_\kappa\neq\emptyset\}$, then (2.13) and the transversality of E to $\mathbb{M}^n_\kappa$ (see (2.8_2), (2.8_3)) imply:

(2.17) $\left\{\begin{array}{l} V \mapsto V\cap\mathbb{M}^n_\kappa \textit{ is a bijection of } \mathcal{V}_k \textit{ onto the set } \mathcal{A}_k \textit{ of} \\ \textit{all complete, connected k-dim. totally geodesic} \\ \textit{submanifolds } A \textit{ of } \mathbb{M}^n_\kappa\textit{, each such } A \textit{ being isometric to} \\ \mathbb{M}^k_\kappa\textit{, with a distance preserving inclusion } A \hookrightarrow \mathbb{M} \end{array}\right.$.

(iii) The orientation of $\mathbb{M}^n_\kappa$.

$\mathbb{M}^n_\kappa$ admits a canonical orientation induced by the one of $\mathbb{R}^{n+1}$:

(2.18) $\left\{\begin{array}{l} \textit{For every } p\in\mathbb{M}^n_\kappa \textit{ a basis } (v_1,\ldots,v_n) \textit{ of } T_p\mathbb{M}^n_\kappa \textit{ is} \\ \textit{positively oriented iff (see (2.3),(2.5)) the basis} \\ (p,(i_*v_1)^\rightarrow,\ldots,(i_*v_n)^\rightarrow) \textit{ of } \mathbb{R}^{n+1} \textit{ is pos. oriented.} \end{array}\right.$

(iv) The Lie group G^n_κ of all orientation preserving isometries of $\mathbb{M}^n_\kappa$ and the Lie algebra $\mathfrak{g}^n_\kappa$ of this group.

Let $GL(n+1,\mathbb{R})\times\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ $((\mathbb{A},p) \mapsto \mathbb{A}\cdot p)$ denote the canonical group action of $GL(n+1,\mathbb{R})$ on $\mathbb{R}^{n+1}$ from the left.

Then for every

(2.19) $\left\{\begin{array}{l} \mathbb{A}\in GL(n+1,\mathbb{R}) \textit{ with } \mathbb{A}\cdot\mathbb{M}^n_\kappa\subseteq\mathbb{M}^n_\kappa\textit{, i.e. (see (2.5),(2.6)):} \\ \langle\mathbb{A}\cdot p,\mathbb{A}\cdot p\rangle_\kappa=\langle p,p\rangle_\kappa \textit{ for } p\in\mathbb{M}^n_\kappa \textit{ and, if } \kappa\leq 0\textit{: } x_0(\mathbb{A}\cdot p)>0, \end{array}\right.$

there is a

(2.20) *unique* C^ω *diffeomorph.* $f^{\mathbb{A}}:\mathbb{M}^n_\kappa \longrightarrow \mathbb{M}^n_\kappa$ *with* $i\circ f^{\mathbb{A}} = \mathbb{A}\cdot i$,

and because of

(2.21) $(\mathbb{A}_*v)^\rightarrow = \mathbb{A}\cdot(v^\rightarrow)$ *for* $v\in T\mathbb{R}^{n+1}$

it follows from (2.18),(2.20),(2.21):

(2.22) $f^{\mathbb{A}}$ *is orientation preserving* $\Leftrightarrow$ $\mathbb{A}\in GL_+(n+1,\mathbb{R})$
$\Leftrightarrow$ $\det(\mathbb{A})>0$.

Moreover (2.19),(2.2),(2.3),(2.5) imply

(2.23) for $\kappa\neq 0$: $f^{\mathbb{A}}:\mathbb{M}^n_\kappa \longrightarrow \mathbb{M}^n_\kappa$ *is a* C^ω *isometry,*

(2.24) for $\kappa=0$: $\mathbb{A} = \left(\begin{array}{c|c} 1 & o^* \\ \hline a & A \end{array}\right)$ *with* $a\in\mathbb{R}^n$ *and* $A\in GL(n,\mathbb{R})$

(where $o, a\in\mathbb{R}^n$ are considered as $(n\times 1)$-matrices and B^* is the transposed of a matrix B).

If we introduce therefore

$$(2.25)\quad \begin{cases} G^n_\kappa := \{\mathbb{A}\in GL_+(n+1,\mathbb{R})\mid \mathbb{A}\cdot\mathbb{M}^n_\kappa\subseteq\mathbb{M}^n_\kappa\} & \textit{for } \kappa\neq 0, \\ G^n_0 := \{\mathbb{A}\in GL_+(n+1,\mathbb{R})\mid \mathbb{A}\cdot\mathbb{M}^n_0\subseteq\mathbb{M}^n_0 \textit{ and } A\in SO(n)\} , \end{cases}$$

then one proves (in a standard way) for all $\kappa\in\mathbb{R}$:

(2.26) G^n_κ *is a connected* $\binom{n+1}{2}$*-dim.* LIE *subgroup of* $GL_+(n+1,\mathbb{R})$, *which is compact iff* $\kappa>0$,

and (see (2.20))

(2.27) $G^n_\kappa \longrightarrow \mathrm{Isom}_0(\mathbb{M}^n_\kappa)$ $(\mathbb{A}\mapsto f^{\mathbb{A}})$ *is a* C^ω *isomorphism onto the* LIE *group of all orientation preserving isometries of* $\mathbb{M}^n_\kappa$, *which acts transitively on* $\mathbb{M}^n_\kappa$ *(even transitively on the bundle of positive orthonormal n-frames of* $\mathbb{M}^n_\kappa$*). Moreover* $\mathrm{Isom}_0(\mathbb{M}^n_\kappa)$ *is the connected component of* Id *in* $\mathrm{Isom}(\mathbb{M}^n_\kappa)$.

In the following we identify $\mathrm{Isom}_0(\mathbb{M}^n_\kappa)$ with G^n_κ under the isomorphism (2.27), see also (2.20).

The LIE *algebra* $\mathfrak{g}^n_\kappa$ *of* G^n_κ $(\cong\mathrm{Isom}_0(\mathbb{M}^n_\kappa))$ will be as usual identified with a certain vector subspace of the vector space $\mathfrak{M}(n+1,\mathbb{R})$ of all real $(n+1)\times(n+1)$-matrices, namely with the image under the composition of the canonical maps

$$T_{\mathbb{I}}G^n_\kappa \xrightarrow{\ j_*\ } T_{\mathbb{I}}\mathfrak{M}(n+1,\mathbb{R}) \xrightarrow{\ (..)^{\rightarrow}\ } \mathfrak{M}(n+1,\mathbb{R}) ,$$

where $\mathbb{I}$ is the unit $(n+1)\times(n+1)$-matrix, $j:G^n_\kappa \hookrightarrow \mathfrak{M}(n+1,\mathbb{R})$ the inclusion map, $(..)^{\rightarrow}$ being defined for the vector space $\mathfrak{M}(n+1,\mathbb{R})$ analogously to (2.3). Under this identification we get:

(2.28) *The* LIE *algebra* $\mathfrak{g}^n_\kappa$ *of* G^n_κ ($\cong \mathrm{Isom}_0(\mathbb{M}^n_\kappa)$, see (2.27)) *is the* LIE *subalgebra of* $\mathfrak{gl}(n+1,\mathbb{R})$ (:= LIE algebra of real $(n+1)\times(n+1)$-matrices with the commutator as bracket operation) *formed by all matrices*

$$(v;V)_\kappa := \left(\begin{array}{c|c} 0 & -\kappa v^* \\ \hline v & V \end{array}\right) \quad \textit{with } v\in\mathbb{R}^n \textit{ and } V\in\mathfrak{M}_{ss}(n,\mathbb{R}),$$

with the (commutator) bracket

$$(2.28_0) \qquad [(v;V)_\kappa,(w;W)_\kappa] = (Vw-Wv;(VW-WV)-\kappa(vw^*-wv^*))_\kappa \ ,$$

where the elements of $\mathbb{R}^n$ are considered as $(n\times1)$-matrices and A^* is the transposed of a matrix A and $\mathfrak{M}_{ss}(n,\mathbb{R})$ denotes the $\binom{n}{2}$-dim. vector space of all real skew-symmetric $(n\times n)$-matrices.

Using (2.28_0) one can compute the KILLING *form* B^n_κ *of* $\mathfrak{g}^n_\kappa$ as:

$$(2.28_1) \qquad B^n_\kappa((v;V)_\kappa,(w;W)_\kappa) = -(n-1)(2\kappa\cdot\langle v,w\rangle + \langle V,W\rangle) =$$

$$(2.28_2) \qquad = (n-1)(\text{trace of the matrix } (v;V)_\kappa\cdot(w;W)_\kappa) \ ,$$

where $\langle V,W\rangle := \mathrm{trace}(V^*\cdot W)$ is the canonical Euclidean inner product on $\mathfrak{M}(n,\mathbb{R})$. Therefore, if $n\geq2$, B^n_κ is negative definite for $\kappa>0$, degenerate for $\kappa=0$ and nondegenerate and indefinite for $\kappa<0$.

Remark. *The description* (2.28),(2.28_0) *of the* LIE *algebras* $\mathfrak{g}^n_\kappa$ *of the isometry groups of* $\mathbb{M}^n_\kappa$ *(for all* $\kappa\in\mathbb{R}$) *as a certain* C^ω *family of* LIE *subalgebras of* $\mathfrak{gl}(n+1,\mathbb{R})$ [which should be understood in an analogous sense as in § 2.ii.d], *depending analytically on* $\kappa\in\mathbb{R}$, *is the main feature of the* WEIERSTRASS *models* $\mathbb{M}^n_\kappa$ ($\kappa\in\mathbb{R}$, see (2.5)), when studying the mechanics of rigid bodies in spaces of constant curvature with the aim of controlling how the results do depend on κ, when κ varies in the real numbers.

Moreover the *isotropy group of the point* e *in* $\mathrm{Isom}_0(\mathbb{M}^n_\kappa)$ (see (2.0),(2.6)) corresponds under the isomorphism (2.27) to the

$$(2.29)\qquad \textit{subgroup}\quad G^n_\kappa|_e := \{(1\times A) := \left(\begin{array}{c|c} 1 & o^* \\ \hline o & A \end{array}\right) | A\in SO(n)\}\quad \textit{of } G^n_\kappa$$

and we call the corresponding Lie subalgebra of $\mathfrak{g}^n_\kappa$ (see (2.28),(2.29)):

$$(2.29_0)\quad \begin{cases} \mathfrak{s}^n_\kappa := \{(o;V)_\kappa | V\in\mathfrak{M}_{ss}(n,\mathbb{R})\} \\ \textit{the "isotropy subalgebra" of } \mathfrak{g}^n_\kappa. \end{cases}$$

On the other hand we call the following vector subspace of $\mathfrak{g}^n_\kappa$ (which is a Lie subalgebra of $\mathfrak{g}^n_\kappa$ only for $\kappa=0$, see (2.28),(2.28$_0$))

$(2.29_1)\quad \mathfrak{t}^n_\kappa := \{(v;O)_\kappa | v\in\mathbb{R}^n\}$ *the "translation subspace" of* $\mathfrak{g}^n_\kappa$,

thereby giving rise to the

$$(2.29_2)\quad \begin{cases} \textit{vectorspace splitting}\quad \mathfrak{g}^n_\kappa = \mathfrak{s}^n_\kappa \oplus \mathfrak{t}^n_\kappa\ , \\ (v;V)_\kappa \mapsto ((o;V)_\kappa,(v;O)_\kappa)\ , \end{cases}$$

which is orthogonal with respect to the Killing form of $\mathfrak{g}^n_\kappa$ (see (2.28$_1$)).

Then the exponential function for matrices (defined by the classical power series;

$$(2.30)\quad \begin{cases} \mathrm{Exp} := \mathrm{Exp}^{(n+1)}: \mathfrak{M}(n+1,\mathbb{R}) \longrightarrow GL_+(n+1,\mathbb{R}) \\ \textit{maps } \mathfrak{g}^n_\kappa \textit{ into } G^n_\kappa \textit{ and } \mathfrak{s}^n_\kappa \textit{ onto } G^n_\kappa|_e\ , \end{cases}$$

more precisely (see (2.29),(2.29$_0$)): for all $V\in\mathfrak{M}_{ss}(n,\mathbb{R})$:

$$(2.30_0)\qquad \mathrm{Exp}((o;V)_\kappa) = (1\times A)\quad \textit{with}\quad A := \mathrm{Exp}^{(n)}(V) \in SO(n)\ .$$

Finally, if $a_1,\ldots,a_n$ denote the canonical basis vectors of $\mathbb{R}^n$, then one checks easily (see(2.28)) for all $t\in\mathbb{R}$:

$$(2.31)\qquad \mathrm{Exp}((ta_1;O)_\kappa) = \begin{pmatrix} \cos_\kappa(t) & -\kappa\sin_\kappa(t) & 0 & \cdots & 0 \\ \sin_\kappa(t) & \cos_\kappa(t) & 0 & \cdots & 0 \\ 0 & 0 & 1 & & O \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & O & & 1 \end{pmatrix} \in G^n_\kappa$$

and similar for $a_2,\ldots,a_n$.

More generally, if $u \in S^{n-1} = \{a\in\mathbb{R}^n | \langle a,a\rangle=1\}$, then (see (2.0),(2.8$_3$)) there is a unique $u\in T^1_e M^n_\kappa$ with $(i_* u)^\to = (0,u) \in \mathbb{R}^{n+1}$ and the orbit of the point $e\in M^n_\kappa$ under the 1-parameter subgroup $(\mathrm{Exp}((tu;O)_\kappa))_{t\in\mathbb{R}}$ in G^n_κ is (see (2.13),(2.30))

$$(2.32)\quad \begin{cases} \textit{the geodesic } c_u:\mathbb{R}\longrightarrow \mathbb{M}^n_\kappa \textit{ of } \mathbb{M}^n_\kappa \textit{ with } c_u(0)=e \textit{ and} \\ \dot c_u(0)=u\ ,\ \textit{i.e. for all } t\in\mathbb{R}: \\ \mathrm{Exp}((tu;O)_\kappa)\cdot e = c_u(t) = \cos_\kappa(t)e+\sin_\kappa(t)\cdot(0,u)\ . \end{cases}$$

(v) The distance function r and the radial unit vector field R in $\mathbb{M}^n_\kappa$ with respect to the point $e\in\mathbb{M}^n_\kappa$. The (n-1)-dim. volume of geodesic spheres in $\mathbb{M}^n_\kappa$.

If we denote (see (2.0),(2.6))

$$(2.33_0)\qquad e := (1,0,\dots,0)\in\mathbb{M}^n_\kappa \quad \textit{and} \quad \mathbb{M}^n_\kappa(e) := \mathbb{M}^n_\kappa\setminus\{e,-e\}\ ,$$

then the function r, which measures the distance of the points of $\mathbb{M}^n_\kappa$ from e satisfies (see (2.14),(2.14$_1$),(1.9))

$$(2.33)\quad \begin{cases} r := d_\kappa(e,..):\mathbb{M}^n_\kappa\longrightarrow\mathbb{R} \textit{ is } C^0,\ \ r|\mathbb{M}^n_\kappa(e) \textit{ is } C^\omega \textit{ and } >0, \\ \textit{and for } p=(p_0,p)\in\mathbb{M}^n_\kappa:\ p_0=\cos_\kappa(r(p)),\ \|p\|=\sin_\kappa(r(p)) \\ \textit{and if } p\in\mathbb{M}^n_\kappa(e):\quad (0,p) = \sin_\kappa(r(p))\cdot i_*U_e(p)^{\rightarrow}\ . \end{cases}$$

Then we define on $\mathbb{M}^n_\kappa(e)$ *the e-radial unit vector field* R by

$$(2.34_0)\qquad R(p) := (\exp_\kappa(x\cdot U_e(p)))^{\cdot}(r(p)) \quad \textit{for } p\in\mathbb{M}^n_\kappa$$

(see (2.15),(2.16)), R(p) being the velocity vector at time r(p) of the unique unit speed geodesic $c:\mathbb{R}\longrightarrow\mathbb{M}^n_\kappa$ joining $e = c(0)$ and $p = c(r(p))$, thus explaining the terminology "e-radial" for the vector field R. Then

$$(2.34)\quad \begin{cases} g(R,R) = 1\ ,\quad \nabla_R R = O \quad \textit{and moreover} \quad R = \mathrm{grad}(r)\ , \\ \textit{hence R is } C^\omega, \end{cases}$$

as one verifies easily, and (2.34$_0$),(2.13) imply:

$$(2.34_1)\qquad (i_*R)^{\rightarrow} = -\kappa\cdot\sin_\kappa(r)\cdot e + \cos_\kappa(r)\cdot(i_*U_e)^{\rightarrow}\ .$$

Furthermore, if $p\in\mathbb{M}^n_\kappa(e)$ and $a\in T^1_p\mathbb{M}^n_\kappa$ is a unit vector orthogonal to $U_e(p)$ (see (2.16)), then we define

$$(2.35_0)\quad S^a_p := [\exp_\kappa(r(p)\cdot(\cos(x)\cdot U_e(p)+\sin(x)\cdot a))]^{\cdot}(0)\in T_p\mathbb{M}^n_\kappa$$

and we obtain from (2.35$_0$),(2.13) immediately

$$(2.35_1)\qquad (i_*S^a_p)^{\rightarrow} = \sin_\kappa(r(p))\cdot(i_*a)^{\rightarrow}\ ,$$

which together with (2.34_1), (2.5), (2.3) gives

(2.35_2) $\quad g(R(p),S_p^a) = 0$ *and* $\|S_p^a\| = \sin_\kappa(r(p))$.

Moreover, if we consider $U_e : \mathbb{M}_\kappa^n(e) \longrightarrow T\mathbb{M}_\kappa^n$ as a C^ω vector field in $\mathbb{M}_\kappa^n$ along the constant map $\mathbb{M}_\kappa^n(e) \longrightarrow \mathbb{M}_\kappa^n$ $(q \mapsto p)$, we obtain from (2.34_0) resp. (2.35_0) immediately (by the standard methods for computing covariant derivatives for vector fields along maps):

(2.35_3) $\quad \nabla_{R(p)}U_e = 0$ *resp.* $\nabla_{(S_p^a)}U_e = a$.

From the last equation, from (2.34_1), (2.35_1) and (2.9) follows but $\nabla_{(S_p^a)}R = \cot_\kappa(r(p))\cdot S_p^a$ which implies [using $\nabla_R R=0$ (see (2.34)) and that the set $\{S_p^a \mid a\in T_p^1\mathbb{M}_\kappa^n$ and $g(a,U_e(p))=0\}$ spans the orthogonal complement of $R(p)$ in $T_p\mathbb{M}_\kappa^n$ (see (2.35_1), (2.35_2))]:

(2.35) $\quad \nabla_X R = \cot_\kappa(r)\cdot[X - g(X,R)\cdot R]$ *for all* $X\in\mathfrak{X}(\mathbb{M}_\kappa^n(e))$.

Remarks.

a) (2.34_0), (2.34), (2.35_0), (2.35_2) amount essentially to the so-called "Gauss Lemma" for $\exp_\kappa$ in a very explicit form for the special Riemannian manifold $\mathbb{M}_\kappa^n$, which however here follows from elementary computations (without using Jacobi fields !) only using the explicit description (2.13) of the exponential map $\exp_\kappa$ of $\mathbb{M}_\kappa^n$ in $\mathbb{R}^{n+1}$.

b) We have the following geometric interpretation of S_p^a: The point $p\in\mathbb{M}_\kappa^n(e)$ lies on the *geodesic* (distance) *sphere* $S_e^{n-1}(\rho)$ of radius $\rho := r(p) \in\ ]0,\pi_\kappa[$ (see (2.33_0), (2.14_1)), which is (see (2.15)) the image under $\exp_\kappa(\rho\cdot\ldots)$ of the Euclidean unit sphere $T_e^1\mathbb{M}_\kappa^n$ in $T_e\mathbb{M}_\kappa^n$ of radius 1 around the origin of $T_e\mathbb{M}_\kappa^n$, $T_e^1\mathbb{M}_\kappa^n$ being isometric to the standard sphere S^{n-1} of $\mathbb{E}^n$. (2.35_0) says then, that S_p^a is tangent to $S_e^{n-1}(\rho)$ at p, namely the image under $(\exp_\kappa(\rho\cdot\ldots))_*$ of the unit vector $(\cos(x)\cdot U_e(p) + \sin(x)\cdot a)^\cdot(0)$ tangent to $T_e^1\mathbb{M}_\kappa^n$ at $U_e(p)$. Therefore, due to (2.35_1) the (n-1)-dim. volume element of $S_e^{n-1}(\rho)$ (pulled back to $T_e^1\mathbb{M}_\kappa^n$ via $\exp_\kappa(\rho\cdot\ldots)$) is $\sin_\kappa(\rho)^{n-1}$

times the volume element of $T^1_e M^n_\kappa$ (isometric to S^{n-1}), therefore for

$$(2.36)\quad \begin{cases} \textit{every } \rho\in]0,\pi_\kappa[\textit{ the geodesic sphere } S^{n-1}_e(\rho):=r^{-1}(\{\rho\}) \\ \textit{is a regular } (n-1)\textit{-dim. } C^\omega \textit{ submanifold of } \mathbb{M}^n_\kappa \textit{ with} \\ (n-1)\textit{-dim. volume } \operatorname{vol}(S^{n-1}_e(\rho)) = \sin_\kappa(\rho)^{n-1}\cdot\sigma_{n-1}\ , \end{cases}$$

where

$$(2.36_1)\quad \sigma_{n-1} := \textit{Euclidean } (n-1)\textit{-dim. volume of } S^{n-1} \textit{ in } \mathbb{E}^n = 2\cdot\pi^{n/2}/\Gamma(\tfrac{n}{2})$$

(e.g. $\sigma_0=2$, $\sigma_1=2\pi$, $\sigma_2=4\pi$, $\sigma_3=2\pi^3$, ...).

(vi) The Busemann function $\beta_c:\mathbb{M}^n_\kappa \longrightarrow \mathbb{R}$ of a unit speed geodesic $c:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ for $\kappa<0$.

a) Suppose now $\kappa<0$. Then in the Lorentzian vector space $(\mathbb{R}^{n+1},\langle..,..\rangle_\kappa)$ (see (2.1),(2.2)) we have the *future time* resp. *future light cone*:

$$(2.37)\quad \begin{cases} C := \{p\in\mathbb{R}^{n+1} \mid \langle p,p\rangle_\kappa>0 \textit{ and } p_0>0\} \supseteq \mathbb{M}^n_\kappa \quad \text{(see (2.5))} \\ \partial C := \{b\in\mathbb{R}^{n+1} \mid \langle b,b\rangle_\kappa=0 \textit{ and } b_0>0\} = \\ \qquad = \textit{asymptotic cone of } \mathbb{M}^n_\kappa \\ \textit{and (Lorentzian geometry !) for all } v,w\in C\cup\partial C \textit{ one has} \\ \langle v,w\rangle_\kappa \geq 0 \textit{ and } \text{“=”} \textit{ iff } \mathbb{R}_+v = \mathbb{R}_+w \subseteq \partial C. \end{cases}$$

(Remember, that (because of $\kappa<0$) $\mathbb{M}^n_\kappa$ is a hyperboloid of revolution in $\mathbb{R}^{n+1}$, § 2.ii.a). Now, for any unit speed geodesic $c:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ we have (using (2.13),(2.2),(2.3),(2.5), (2.8_3) and $\kappa<0$):

$$(2.38)\quad \begin{cases} c(t) = \cos_\kappa(t)c(0) + \sin_\kappa(t)c'(0) \quad \textit{for all } t\in\mathbb{R} \textit{ and} \\ \langle c,c\rangle_\kappa=1\ ,\quad c_0 = \langle c,e\rangle_\kappa\geq 1\ ,\quad \langle c,c'\rangle_\kappa=0\ ,\quad \langle c',c'\rangle_\kappa=\kappa. \end{cases}$$

Then we define the Busemann *vector* b_c *of* c by

$$(2.39)\quad \begin{cases} b_c := c(0) + (1/\sqrt{-\kappa})\cdot c'(0) = \\ \qquad = e^{-\sqrt{-\kappa}t}\cdot(c(t) + (1/\sqrt{-\kappa})\cdot c'(t)) \in \mathbb{R}^{n+1} \quad \textit{for } t\in\mathbb{R}, \end{cases}$$

where the second equation follows from (2.38) and one has the identities (which follow from (1.1)) in case $\kappa<0$:

$$(2.40)\quad \begin{cases} 2\cdot\cos_\kappa(x) = e^{\sqrt{-\kappa}x} + e^{-\sqrt{-\kappa}x}, \\ 2\sqrt{-\kappa}\cdot\sin_\kappa(x) = e^{\sqrt{-\kappa}x} - e^{-\sqrt{-\kappa}x}. \end{cases}$$

If we introduce the *"reverse" geodesic* $c^\vee:\mathbb{R}\longrightarrow \mathbb{M}^n_\kappa$ *of* c by

$$(2.41)\quad \begin{cases} c^\vee(x) := c(-x)\ ,\quad \textit{then}\quad b_{c^\vee} = c(0) - (1/\sqrt{-\kappa})\cdot c'(0) \\ \textit{and}\quad \langle b_c,b_c\rangle_\kappa = \langle b_{c^\vee},b_{c^\vee}\rangle_\kappa = 0 \quad \textit{and}\quad \langle b_c,b_{c^\vee}\rangle_\kappa = 2 \end{cases}$$

(see (2.38)). From (2.39),(2.41) and (2.17) one obtains immediately

$$(2.42)\quad \begin{cases} 2\cdot c(x) = e^{\sqrt{-\kappa}t}\cdot b_c + e^{-\sqrt{-\kappa}t}\cdot b_{c^\vee}\ , \\ \textit{hence}\quad c(\mathbb{R}) = \mathbb{M}^n_\kappa \cap \operatorname{Span}\{b_c,b_{c^\vee}\}\ . \end{cases}$$

From (2.38),(2.42) we obtain

$$2 \le e^{\sqrt{-\kappa}t}\cdot\langle b_c,e\rangle_\kappa + e^{-\sqrt{-\kappa}t}\cdot\langle b_{c^\vee},e\rangle_\kappa \quad \textit{for all } t\in\mathbb{R},$$

which implies $\langle b_c,e\rangle_\kappa > 0$, therefore (see (2.37),(2.41)) $b_c\in\partial C$ and (together with (2.42) and its differentiated version) we get

$$(2.43)\quad b_c = \lim_{t\to\infty}[2e^{-\sqrt{-\kappa}t}c(t)] = \lim_{t\to\infty}[(2/\sqrt{-\kappa})e^{-\sqrt{-\kappa}t}c'(t)] \in \partial C\ ,$$

i.e.: b_c *is the "renormalized" direction* of c(t) *resp.* c'(t) *in* $\mathbb{R}^{n+1}$ *if* t *tends to* $+\infty$ and this direction b_c lies on the cone ∂C of asymptotic directions for the hyperboloid $\mathbb{M}^n_\kappa$ in $\mathbb{R}^{n+1}$. Oppositely:

$$(2.44)\quad \begin{cases} \textit{For every } b\in\partial C \textit{ and } p\in\mathbb{M}^n_\kappa \textit{ there exists a unique unit} \\ \textit{speed geodesic}\quad c:\mathbb{R}\longrightarrow\mathbb{M}^n_\kappa \quad \textit{with } c(0)=p \textit{ and } b_c\in\mathbb{R}_+b, \end{cases}$$

namely $c(t) := \cos_\kappa(t)p + \sqrt{-\kappa}\sin_\kappa(t)(\langle b,p\rangle_\kappa^{-1}\cdot b - p)$ for all $t\in\mathbb{R}$ (see (2.37)).

b) Remark. For a complete 1-connected n-dim. Riemannian manifold M of strictly negative sectional curvature one is used to form a "closure" $\overline{M}$ of M by adding to M an *ideal boundary* "∂M" *of* M, consisting of *"points at infinity"*, which are equivalence classes of unit speed geodesics $c,\tilde{c},\ldots:\mathbb{R}\longrightarrow M$ with respect to the relation:

c *asymptotic to* $\tilde{c}$ $:\Leftrightarrow$

$:\Leftrightarrow$ *There exists* $\rho\in\mathbb{R}_+$ *with* $d(c(t),\tilde{c}(t)) < \rho$ *for all* $t\in\mathbb{R}_+$, *the equivalence class of* c *with respect to this relation being denoted by* $c(\infty)$,

and $c(\infty)$ is called the point of c at $+\infty$ .. A topology on $\overline{M} := M \cup \partial M$ is then introduced, called the "cone toplogy", a terminology which seems to be motivated by our example in question $M:=\mathbb{M}^n_\kappa$. Because, since $\kappa<0$, the function $\cos_\kappa|[0,+\infty[$ is strictly monotonic increasing, therefore in case $M:=\mathbb{M}^n_\kappa$ the last definition amounts to

$$(2.45_0)\quad \begin{cases} c(\infty)=\tilde{c}(\infty) & \textit{iff there exists } \rho\in\mathbb{R}_+ \\ \textit{with } \langle c(t),\tilde{c}(t)\rangle_\kappa < \rho & \textit{for all } t\in\mathbb{R}_+ . \end{cases}$$

This and the first equation of (2.43) (applied to c and $\tilde{c}$) implies therefore

$$(2.45)\qquad c(\infty) = \tilde{c}(\infty) \iff (\langle b_c,b_{\tilde{c}}\rangle_\kappa=0 \underset{(2.37)}{\iff})\ \mathbb{R}_+\cdot b_c = \mathbb{R}_+\cdot b_{\tilde{c}} .$$

The *cone topology* on $\overline{\mathbb{M}^n_\kappa}:=\mathbb{M}^n_\kappa\dot{\cup}\partial\mathbb{M}^n_\kappa$ is then by definition the one, which makes the bijective map (see (2.5),(2.44),(2.45))

$$(2.46)\quad h:\overline{\mathbb{M}^n_\kappa} \longrightarrow (C\dot{\cup}\partial C)/\mathbb{R}_+ \quad \textit{with} \quad h(p) := \begin{cases} \mathbb{R}_+\cdot p & \textit{if } p\in\mathbb{M}^n_\kappa \\ \mathbb{R}_+\cdot b_c & \textit{if } p=c(\infty)\in\partial\mathbb{M}^n_\kappa \end{cases}$$

into a homeomorphism. In this way $(\overline{\mathbb{M}^n_\kappa},\partial\mathbb{M}^n_\kappa)$ becomes canonically homeomorphic to (D^n,S^{n-1}), since the map $C\dot{\cup}\partial C \longrightarrow \mathbb{R}^n$ $(v=(v_0,v) \mapsto (\sqrt{-\kappa}v_0)^{-1}\cdot v)$ induces a homeomorphism of $\partial C/\mathbb{R}_+$ resp. $(C\dot{\cup}\partial C)/\mathbb{R}_+$ of onto the unit sphere S^{n-1} resp. the unit disc D^n in $\mathbb{R}^n$.

Since the metric space $(\mathbb{M}^n_\kappa,d_\kappa)$ is complete, one can not extend the metric d_κ onto all of $\overline{\mathbb{M}^n_\kappa}$, however the BUSEMANN *functions* which we will describe now will measure in a certain sense distances of points of $\mathbb{M}^n_\kappa$ from the points $c(\infty)\in\partial\mathbb{M}^n_\kappa$ at infinity:

c) For every unit speed geodesic $c:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ we define *the* BUSEMANN *function* $\beta_c:\mathbb{M}^n_\kappa \longrightarrow \mathbb{R}$ *of* c by (see (2.37)):

$$(2.47)\quad \begin{cases} \beta_c(p) := (1/\sqrt{-\kappa})\cdot\ln(\langle p,b_c\rangle_\kappa) & \textit{for all } p\in\mathbb{M}^n_\kappa, \\ \textit{therefore } \beta_c \textit{ is } C^\omega. \end{cases}$$

This description of β_c is extrinsic since $b_c\in\partial C$ does not belong to $\mathbb{M}^n_\kappa$, however from (2.47) we get via (2.43),(2.14), (2.40) and using $\lim_{t\to\infty} d_\kappa(p,c(t)) = +\infty$ the following intrinsic one:

$$(2.47_1) \qquad \beta_c(p) = \lim_{t\to\infty}[d_\kappa(p,c(t))-t] \quad \textit{for all } p\in\mathbb{M}^n_\kappa.$$

This means, that $\beta_c(p)$ *is some "renormalized distance of* p *from* c(∞)" (see b)) and which can be symbolized (since $t=d_\kappa(c(0),c(t))$) by "$\beta_c(p) = d_\kappa(p,c(\infty)) - d_\kappa(c(0),c(\infty))$". The last interpretation of β_c however deserves a warning: This "distance from c(∞)" is determined by c(∞) only up to an additive constant, more precisely (see (2.45),(2.47)):

If $c,\tilde{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ are unit speed geodesics, then

$$(2.47_2) \qquad \beta_c - \beta_{\tilde{c}} : \mathbb{M}^n_\kappa \longrightarrow \mathbb{R} \quad \textit{is constant iff} \quad c(\infty) = \tilde{c}(\infty) .$$

Finally we mention another *intrinsic description of* $\beta_c(p)$ *for* $p\in\mathbb{M}^n_\kappa$ (without any limiting process !) *in terms of "polar coordinates with respect to* $(c(0),\dot{c}(0))$", more precisely a description which only involves the distance from p to c(0) and the angle between $\dot{c}(0)$ and the initial velocity vector $u_0(p)\in T^1_{c(0)}\mathbb{M}^n_\kappa$ of the unique unit speed geodesic joining c(0) and p, namely: If we set (see (2.16))

$$(2.47_3) \quad \begin{cases} r_0:=d_\kappa(c(0),..):\mathbb{M}^n_\kappa \longrightarrow \mathbb{R} \quad \textit{and} \\ U_0:=U_{c(0)}:\mathbb{M}^n_\kappa\setminus\{c(0)\} \longrightarrow T^1_{c(0)}\mathbb{M}^n_\kappa , \\ \textit{i.e. for all } p\in\mathbb{M}^n_\kappa \text{ (see (2.16),(2.13),(2.16}_1\text{)):} \\ p = \exp_\kappa(r_0(p)\cdot U_0(p)) = \\ \quad = \cos_\kappa(r_0(p))\cdot c(0) + \sin_\kappa(r_0(p))\cdot (i_*U_0(p))^{\rightarrow} , \end{cases}$$

then one deduces (using (2.39),(2.47_3),(2.14),(2.38),(2.5), (2.3)) easily for all $p\in\mathbb{M}^n_\kappa$

$$(2.47_4) \quad \langle p,b_c\rangle_\kappa = \cos_\kappa(r_0(p)) - \sqrt{-\kappa}\cdot\sin_\kappa(r_0(p))\cdot g(U_0(p),\dot{c}(0))$$

(see (2.16_1)), which together with (2.47) provides the announced intrinsic description of $\beta_c(p)$.

Moreover one sees from (2.47_1) that

$$(2.47_5) \qquad (\beta_{f\circ c})\circ f = \beta_c \quad \textit{for every isometry} \quad f:\mathbb{M}^n_\kappa \longrightarrow \mathbb{M}^n_\kappa .$$

Going back again to (2.47_2) we observe, that according to (2.47_2) the differential resp. the gradient vector field of β_c

does only depend on $c(\infty)$, more precisely: If we introduce for convenience of the computation the constant vector field $B_c : \mathbb{M}^n_\kappa \longrightarrow T\mathbb{R}^{n+1}$ along $i : \mathbb{M}^n_\kappa \hookrightarrow \mathbb{R}^{n+1}$ with

$$(2.48_0)\quad \begin{cases} \vec{B_c} = b_c \ , \\ \textit{therefore}\quad e_\kappa(B_c, B_c) = 0 \ \textit{and}\ \overset{o}{\nabla}_X B_c = 0 \ \textit{for}\ X \in \mathfrak{X}(\mathbb{M}^n_\kappa) \end{cases}$$

(as follows from $b_c \in \partial C$, see (2.43),(2.37), resp. from (2.4)), then, using the unit normal field ξ_κ of $\mathbb{M}^n_\kappa$ in $\mathbb{E}^{n+1}_\kappa$, the definition (2.47) of the BUSEMANN function can be written as (see (2.2),(2.3),(2.8_5)):

$$(2.48_1)\qquad \beta_c = (1/\sqrt{-\kappa}) \cdot \ln(\sqrt{-\kappa} \cdot e_\kappa(\xi_\kappa, B_c)) \ .$$

Therefore for all $X \in \mathfrak{X}(\mathbb{M}^n_\kappa)$ we obtain from (2.48_1) (using (2.8_6),(2.48_0)):

$$(2.48_2)\qquad X \cdot \beta_c = - \frac{e_\kappa(i_* X, B_c)}{e_\kappa(\xi_\kappa, B_c)} = - e_\kappa\Big(\xi_\kappa + \frac{1}{e_\kappa(\xi_\kappa, B_c)} \cdot B_c, i_* X\Big) \ .$$

Since $\xi_\kappa + e_\kappa(\xi_\kappa, B_c)^{-1} \cdot B_c$ is orthogonal to ξ_κ (see (2.8_6)), it is tangential to $\mathbb{M}^n_\kappa$, and therefore we get from (2.48_2)

$$(2.48_3)\quad \begin{cases} i_* \operatorname{grad}\beta_c = - \Big(\xi_\kappa + \dfrac{1}{e_\kappa(\xi_\kappa, B_c)} \cdot B_c\Big) \ , \\ \textit{hence}\quad \|\operatorname{grad}\beta_c\| = 1 \ . \end{cases}$$

Then the Hessian of β_c can be computed for all $X \in \mathfrak{X}(\mathbb{M}^n_\kappa)$ as

$$(2.48_4)\quad \begin{cases} (\operatorname{Hess}\beta_c)(X) := \nabla_X(\operatorname{grad}\beta_c) = \\ \qquad = \sqrt{-\kappa} \cdot [X - g(X, \operatorname{grad}\beta_c) \cdot \operatorname{grad}\beta_c] \ , \end{cases}$$

in particular the Laplacian of β_c in $\mathbb{M}^n_\kappa$ is constant:

$$(2.48_5)\qquad \Delta\beta_c := \operatorname{trace}(\operatorname{Hess}\beta_c) = (n-1) \cdot \sqrt{-\kappa} \ .$$

We are now able to prove for every C^∞ path $\tilde{c} : I \longrightarrow \mathbb{M}^n_\kappa$:

$$(2.49)\quad \begin{cases} \tilde{c} : I \longrightarrow \mathbb{M}^n_\kappa \ \textit{is a maximal integral curve of}\ -\operatorname{grad}\beta_c \ \textit{iff} \\ I = \mathbb{R} \ \textit{and}\ \tilde{c}\ \textit{is a unit speed geodesic with}\ \tilde{c}(\infty) = c(\infty) . \end{cases}$$

[Proof: First one obtains immediately from (2.47_1) and (2.13_1):

$$(2.50)\quad \beta_c(t) = -t \quad \textit{for}\ t \in \mathbb{R},\ \textit{therefore}\quad g(\operatorname{grad}\beta_c \circ c, \dot{c}) = -1 \ .$$

But $\operatorname{grad}\beta_c \circ c$ (see (2.48_3)) and $\dot{c}$ are both *unit* vector fields, so it follows from (2.50) and CAUCHY-SCHWARZ (in-)equality:

(2.50$_1$) $\quad \dot{c} = -(\text{grad}\beta_c)\circ c$.

If $\tilde{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ is now an arbitrary unit speed geodesic with $\tilde{c}(\infty)=c(\infty)$ then we get from (2.47$_2$), that $\text{grad}\beta_{\tilde{c}} = \text{grad}\beta_c$ and therefore (apply (2.50$_1$) for $\tilde{c}$ instead of c): $\dot{\tilde{c}} = -(\text{grad}\beta_{\tilde{c}})\circ\tilde{c} = -(\text{grad}\beta_c)\circ\tilde{c} =$, whereby (2.49)"⇐" is proved. Suppose oppositely, that $\tilde{c}:I \longrightarrow \mathbb{M}^n_\kappa$ is a maximal integral curve of $-\text{grad}\beta_c$ and without loss of generality $0\in I$. Choose then the unique unit speed geodesic $\hat{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ with

$\hat{c}(0) = \tilde{c}(0)$ *and* $\hat{c}(\infty) = \tilde{c}(\infty)$, (see (2.44),(2.45)).

Then $\hat{c}$ is according to the (already proved !) statement (2.49)"⇐" a maximal integral curve of $-\text{grad}\beta_c$. Therefore by the uniqueness property of maximal integral curves of a vector field with the same initial values it follows $I=\mathbb{R}$ and $\hat{c}=\tilde{c}$, in particular $\tilde{c}$ is (as $\hat{c}$) a unit speed geodesic with $\tilde{c}(\infty)=c(\infty)$.]

From (2.49) one gets immediately

(2.51) $\left\{\begin{array}{l} \textit{If } \tilde{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa \textit{ is any unit speed geodesic and there} \\ \textit{exists } t_0\in\mathbb{R} \textit{ with } \dot{\tilde{c}}(t_0) = -\text{grad}\beta_c(c(t_0)) \textit{, then} \\ \tilde{c}(\infty) = c(\infty) . \end{array}\right.$

[Proof: After re-scaling $\tilde{c}$ we may assume $t_0=0$. Let then (see (2.49)) $\hat{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ denote the maximal integral curve of $-\text{grad}\beta_c$ with $\hat{c}(0)=\tilde{c}(0)$, therefore

$\dot{\hat{c}}(0) = -\text{grad}\beta_c(\hat{c}(0)) = -\text{grad}\beta_c(\tilde{c}(0)) = \dot{\tilde{c}}(0)$.

But according to (2.49) $\hat{c}$ is a unit speed geodesic with $\hat{c}(\infty)=c(\infty)$. Because of the uniqueness theorem for maximal geodesics with the same initial velocity vectors it follows therefore $\tilde{c}=\hat{c}$ and thus $\tilde{c}(\infty)=\hat{c}(\infty)=c(\infty)$.]

d) The *horosphere* $H := H(p,c(\infty))$ *through* $p\in\mathbb{M}^n_\kappa$ *with limit point* $c(\infty)\in\partial\mathbb{M}^n_\kappa$ (see b)) is defined as usual to be the regular (n-1)-dim. level submanifold of the BUSEMANN function β_c through p in $\mathbb{M}^n_\kappa$, i.e. (see (2.47))

(2.52) $\quad H = \beta_c^{-1}(\{\beta_c(p)\}) = \{q\in\mathbb{M}^n_\kappa \mid \langle q-p,b_c\rangle_\kappa=0\}$.

[We observe: Because of (2.47$_2$) for any unit speed geodesic $\tilde{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ with $\tilde{c}(\infty)=c(\infty)$ holds $\beta_{\tilde{c}}^{-1}(\{\beta_{\tilde{c}}(p)\}) = \beta_c^{-1}(\{\beta_c(p)\})$, i.e. H in fact depends only on $c(\infty)$.]

Moreover: Because of (2.44),(2.47$_5$) and the fact, that the group G_κ^n acts transitively on the set of all unit speed geodesics in $\mathbb{M}_\kappa^n$, it follows from (2.52):

(2.52$_1$) *Each two horospheres of* $\mathbb{M}_\kappa^n$ *are congruent in* $\mathbb{M}_\kappa^n$.

Remark. Since $V := \{v\in\mathbb{R}^{n+1} \mid \langle v,b_c\rangle_\kappa=0\}$ is a n-dim. vector subspace of $\mathbb{R}^{n+1}$, which is *tangent to the asymptotic cone* ∂C *for the hyperboloid* $\mathbb{M}_\kappa^n$ *along the generator* $\mathbb{R}_+\cdot b_c$ (see (2.37),(2.43)), we get from (2.52) the following extrinsic description of the horosphere H as the "*hyperplane section*" of $\mathbb{M}_\kappa^n$ with the n-dim. affine subspace of $\mathbb{R}^{n+1}$ *through* p, which is parallel to V in $\mathbb{R}^{n+1}$.

From the last remark and (2.52) follows, that if e.g. $c(0)=e$ and $c'(0)=e_1$, then (see (2.39)) $b_c = e + (1/\sqrt{-\kappa})\cdot e_1$ resp. (see (2.47)) $\beta_c(e)=0$ and:

$$(2.52_2)\quad \begin{cases} \{a\in\mathbb{R}^n \mid a_1=0\} \longrightarrow \mathbb{R}^{n+1} \\ (a \longmapsto (-\frac{\kappa}{2}\cdot\langle a,a\rangle\ ,\ \frac{\sqrt{-\kappa}}{2}\cdot\langle a,a\rangle\ ,\ a_2,\ldots,a_n)) \\ \textit{induces a } C^\omega \textit{ diffeomorphism of } \mathbb{R}^{n-1} \textit{ onto the} \\ \textit{horosphere } H = \beta_c^{-1}(\{0\}) \textit{ of } \mathbb{M}_\kappa^n. \end{cases}$$

From (2.52) follows, that $\zeta := -(\mathrm{grad}\beta_c)\circ j$ is a unit normal field along $j:H \hookrightarrow \mathbb{M}_\kappa^n$ and therefore it follows directly from (2.48$_2$) and [Do],4.5.ii,iii, that the shape operator of H with respect to the normal field ζ at $p\in H$ equals $\sqrt{-\kappa}\cdot \mathrm{Id}_{T_pH}$, i.e. H is an *umbilical hypersurface in* $\mathbb{M}_\kappa^n$ *of constant mean curvature* $\sqrt{-\kappa}$ (w.r.t. ζ) and if $n\geq 3$, the (n-1)-dim. riemannian submanifold H of $\mathbb{M}_\kappa^n$ is of *vanishing sectional curvature.*

From the description (2.52) of the horosphere H it is clear ($\mathbb{M}_\kappa^n$ is the boundary of an unbounded convex body in $\mathbb{R}^{n+1}$, hence $\mathbb{M}_\kappa^n$ has the "two-piece property") that $\mathbb{M}_\kappa^n\setminus H$ consists of two components H^+, H^- where

(2.53) $H^+ = \{q \in \mathbb{M}^n_\kappa \mid \beta_c(q) > \beta_c(p)\} = \mathbb{M}^n_\kappa \cap \{q \in \mathbb{R}^{n+1} \mid \langle q-p, b_c \rangle_\kappa > 0\}$

and H^- is defined correspondingly.

Because of (2.50) the sets H^+, H^- can be characterized (without using b_c or β_c) in the following way:

(2.53_1) $\left\{\right.$ *H^+ resp. H^- is the connected component of $\mathbb{M}^n_\kappa \setminus H$, which contains the points $c(t)$ for $t \longrightarrow -\infty$ resp. $t \longrightarrow +\infty$.*

Moreover the normal field property of $-(\mathrm{grad}\beta_c)\circ j$ for $j: H \hookrightarrow \mathbb{M}^n_\kappa$ allows (using (2.47_2), (2.49), (2.50), (2.51), (2.53_1)) to characterize the asymptotic behaviour of a unit speed geodesic $\tilde{c}: \mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ with respect to the given c (see b), (2.45_0)) as follows:

(2.54) $\left\{\right.$ $\tilde{c}(\infty) = c(\infty) \iff$ *There exists $t_0 \in J$ with $\tilde{c}(t_0) \in H$ and $\tilde{c}$ intersects H at the time t_0 orthogonally, passing from H^+ to H^-.*

(If one side of (2.54) is true, then $\tilde{c}(]-\infty, t_0[) \subseteq H^+$ and $\tilde{c}(]t_0, \infty[) \subseteq H^-$.)

If we choose now for the given unit speed geodesic $c: \mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ the special horosphere (see (2.50) and the remark after (2.54)):

$$(2.55_0) \quad \left\{ \begin{array}{l} H_c := H(c(0), c(\infty)) := \beta_c^{-1}(\{0\}) , \\ \text{then} \quad H_c^+ = \beta_c^{-1}(\mathbb{R}_+) \quad \text{and} \quad H_c^- = \beta_c^{-1}(\mathbb{R}_-) , \end{array} \right.$$

and we get, that the BUSEMANN *function* β_c *measures the oriented distance* of the points in $\mathbb{M}^n_\kappa$ *from the horosphere* H_c

$$(2.55) \qquad \beta_c(p) = \left\{ \begin{array}{ll} + \operatorname{dist}(p, H_c) & \text{for } p \in H_c^+ \cup H_c \\ - \operatorname{dist}(p, H_c) & \text{for } p \in H_c^- \cup H_c , \end{array} \right.$$

where $\operatorname{dist}(p, H_c) = \inf\{d_\kappa(p,q) \mid q \in H_c\}$. Moreover, as one can show [using (2.47_2), (2.47_3), the CAUCHY-SCHWARZ inequality and the strict monotony of (see (1.1))

$$(2.56)\quad \begin{cases} \cos_\kappa(x) + \sqrt{-\kappa}\cdot\sin_\kappa(x) = e^{\sqrt{-\kappa}\cdot x} & \textit{and} \\ \cos_\kappa(x) - \sqrt{-\kappa}\cdot\sin_\kappa(x) = e^{-\sqrt{-\kappa}\cdot x} & \textit{for } \kappa<0 \;] \end{cases}$$

that the latter infimum is attained exactly at the one point $(q=)\ \tilde{c}(\beta_C(p)) \in H_C$, where $\tilde{c}:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ is the unique unit speed geodesic (see (2.44),(2.45)) with $\tilde{c}(0) = p$ and $\tilde{c}(\infty) = c(\infty)$.

(vii) Basic geometry and polar coordinates in $\mathbb{M}^2_\kappa$.

In all of this section (vii) we assume n=2.

a) Then due to the canonical orientation of $\mathbb{M}^2_\kappa$ (see (2.18)) we have a uniquely determined *complex* C^ω *structure* J for $\mathbb{M}^2_\kappa$, i.e. a C^ω tensor field J on $\mathbb{M}^2_\kappa$ of type (1,1), such that for all $p\in\mathbb{M}^2_\kappa$ and $u\in T^1_p\mathbb{M}^2_\kappa$ one has

(2.57) (u,Ju) *is a pos. oriented orthonormal 2-frame of* $T_p\mathbb{M}^2_\kappa$,

i.e. J_p is the rotation of $T_p\mathbb{M}^2_\kappa$ about the angle $\pi/2$ in the positive sense w.r.t the given orientation, in particular

$$(2.58)\quad \textit{for all } v,w\in T_p\mathbb{M}^2_\kappa: \begin{cases} (J\circ J)(v) = -v\ ,\ g(Jv,Jw) = g(v,w) \\ \textit{and}\ \ g(Jv,w) = -g(v,Jw)\ , \end{cases}$$

and from (2.58) follows easily:

(2.59) *J is* ∇*-parallel, i.e.* $\nabla_X JY = J\nabla_X Y$ *for all* $X,Y\in\mathfrak{X}(\mathbb{M}^2_\kappa)$.

b) The *oriented area form* σ of the oriented Riemannian manifold $\mathbb{M}^2_\kappa$ can be described as the differential 2-form with (see a)):

$$(2.60)\quad \begin{cases} \sigma(X,Y) := g(JX,Y) & \textit{for all } X,Y\in\mathfrak{X}(\mathbb{M}^2_\kappa), \\ \textit{therefore } \sigma \textit{ is } \nabla\textit{-parallel} \end{cases}$$

(together with g and J, see (2.59)), moreover one has for all $X,Y,Z\in\mathfrak{X}(\mathbb{M}^2_\kappa)$ the identity:

$$(2.60_1)\quad \sigma(X,Y)\sigma(Z,W) = g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\ .$$

c) Because of (ii), Remark a) we have, if $e=(1,0,0)\in\mathbb{R}^3$:

(2.61_0) $\mathbb{M}^2_\kappa(e) := \mathbb{M}^2_\kappa\backslash\{e,-e\}$ *is* C^ω *diffeomorphic to* $\mathbb{R}^2\backslash\{o\}$

and if $r := d_\kappa(e,..):\mathbb{M}^2_\kappa \longrightarrow \mathbb{R}$ and $R := \mathrm{grad}(r)$ on $\mathbb{M}^2_\kappa(e)$

(see (v)a)), then it follows from (2.34),(2.58):

$$(2.61)\quad \begin{cases} (R,JR)\ \textit{is a positively oriented orthonormal}\ C^{\omega}\ \textit{frame} \\ \textit{field on}\ \mathbb{M}^2_\kappa(e)\ \textit{and therefore}\ (\text{see } (2.35),(2.60)) \\ \nabla_X R = \cot_\kappa(r)\cdot\sigma(R,X)\cdot JR \quad \textit{for all}\ X\in\mathfrak{X}(\mathbb{M}^2_\kappa(e)), \end{cases}$$

and (see (2.60)):

$$(2.61_1)\quad dr(R) = g(R,R) = \sigma(R,JR) = 1\ ,\quad dr(JR) = g(R,JR) = 0.$$

If we define therefore the "*polar angle form* θ *on* $\mathbb{M}^2_\kappa(e)$ *w.r.t.* e" as usual as the Pfaffian form on $\mathbb{M}^2_\kappa(e)$ with (see (2.60),(1.9)):

$$(2.62_0)\quad \begin{cases} \theta(X) := \dfrac{\sigma(R,X)}{\sin_\kappa(r)} \quad \textit{for all}\ X\in\mathfrak{X}(\mathbb{M}^2_\kappa), \\ \textit{i.e.}\quad \theta(R) = 0 \quad \textit{and} \quad \theta(JR) = \dfrac{1}{\sin_\kappa(r)}\ , \end{cases}$$

then evidently (use (2.61),(2.62_0),(2.59) and ∇ torsionfree!)

$$(2.62)\quad \begin{cases} (dr,\sin_\kappa(r)\theta)\ \textit{is the dual frame field of}\ (R,JR), \\ \nabla_X R = \cos_\kappa(r)\cdot\theta(X)\cdot JR \quad \textit{for}\ X\in\mathfrak{X}(\mathbb{M}^2_\kappa(e)) \quad \textit{and} \quad d\theta = o\ , \end{cases}$$

consequently

$$(2.62_1)\qquad g = dr\odot dr + \sin^2_\kappa(r)\cdot(\theta\odot\theta) \quad \textit{and} \quad \sigma = \sin_\kappa(r)\cdot(dr\wedge\theta)\ .$$

Moreover, if $U_e:\mathbb{M}^2_\kappa(e) \longrightarrow T\mathbb{M}^2_\kappa$ is the vector field along the map $\mathbb{M}^2_\kappa(e) \longrightarrow \mathbb{M}^2_\kappa$ $(p \mapsto e)$, which was defined in (2.16), then one has

$$(2.63)\qquad \nabla_X U_e = \theta(X)\cdot JU_e \quad \textit{for all}\ X\in\mathfrak{X}(\mathbb{M}^2_\kappa).$$

[Because of (2.61) it suffices to verify this for $X\in\{R,JR\}$ in which cases the assertion (2.63) follows from (2.35_3), observing that for $p\in\mathbb{M}^2_\kappa(e)$ one finds easily (with use of (2.35_0),(2.35_1)) $\sin_\kappa(r(p))\cdot JR(p) = S^a_p$ with $a := JU_e(p)$.]

d) Suppose now, that we have chosen

(2.64_0) *a fixed unit tangent vector* $u\in T^1_e\mathbb{M}^2_\kappa$ *of* $\mathbb{M}^2_\kappa$ *at* e

(which we can interpret as a constant vector field $\mathbb{M}^2_\kappa(e) \longrightarrow T\mathbb{M}^2_\kappa$ along the map $\mathbb{M}^2_\kappa(e) \longrightarrow \mathbb{M}^2_\kappa$ $(p \mapsto e)$).

Then we can introduce the two C^{ω} *functions* $\cos\varphi_u$, $\sin\varphi_u:\mathbb{M}^2_\kappa(e) \longrightarrow \mathbb{R}$ by (see (2.16),(2.60),(2.64_0)):

$$(2.64)\quad \begin{cases} \cos\varphi_u := g(u,U_e)\ ,\ \sin\varphi_u := \sigma(u,U_e)\ :\ \mathbb{M}^2_\kappa(e) \longrightarrow \mathbb{R}\ , \\ \textit{i.e.}\ \ U_e = (\cos\varphi_u)\cdot u + (\sin\varphi_u)\cdot Ju \ \ \textit{and therefore} \\ \forall_{p\in\mathbb{M}^2_\kappa(e)}\ \ p = \exp_\kappa(r(p)\cdot(\cos\varphi_u(p)\cdot u + \sin\varphi_u(p)\cdot Ju))\ , \end{cases}$$

where $U_e(p)\in T^1_e\mathbb{M}^2_\kappa$ is the initial velocity vector of the unit speed geodesic $c:\mathbb{R}\longrightarrow\mathbb{M}^2_\kappa$ joining $e = c(0)$ with $p = c(r(p)) \in \mathbb{M}^2_\kappa(e)$, i.e. $U_e(p)$ is the direction under which an observer, situated at e, "sees" the point p, and we call therefore the three functions r, $\cos\varphi_u$, $\sin\varphi_u$ the *"polar coordinate functions for $\mathbb{M}^2_\kappa$ w.r.t.* (e,u)".

[Warning: We have *not* introduced a continuous angle function $\varphi_u:\mathbb{M}^2_\kappa(e)\longrightarrow\mathbb{R}$ (which is impossible because of (2.16_0)), but only $\cos\varphi_u,\sin\varphi_u$. See however Remark 2) below.]

Then one obtains from (2.60_1),(2.64))

$$(2.64_1)\quad \begin{cases} \cos\varphi_u^{\ 2} + \sin\varphi_u^{\ 2} = 1\ , \\ \textit{i.e.}\ \ (\cos\varphi_u,\sin\varphi_u):\mathbb{M}^2_\kappa(e)\longrightarrow S^1\ \ \textit{is a}\ C^\omega\ \textit{map.} \end{cases}$$

and from (2.64),(2.63) one gets (because of $X\cdot\cos\varphi_u = g(u,\nabla_X U_e)$, ...)

$$(2.65)\quad \begin{cases} d(\cos\varphi_u) = -(\sin\varphi_u)\cdot\theta \ \ \textit{and}\ \ d(\sin\varphi_u) = (\cos\varphi_u)\cdot\theta\ , \\ \textit{in particular for every}\ C^1\ \textit{path}\ \ c:I\longrightarrow\mathbb{M}^2_\kappa(e)\ : \\ (\cos\varphi_u\circ c)' = -(\sin\varphi_u\circ c)\theta(\dot c),\ (\sin\varphi_u\circ c)' = (\cos\varphi_u\circ c)\theta(\dot c). \end{cases}$$

Remark.

1) The first equation of (2.62) says, that the *connection form* $\omega:=\omega_{12}:=-g(E_2,\nabla E_1)$ with respect to the orthonormal frame field$(E_1,E_2):=(R,JR)$in the sense of E. Cartan satisfies (see(2.62_0))(2.65_1) $\omega = -\cos_\kappa(r)\cdot\theta$, *therefore* $d\omega = \kappa\cdot\sigma$ (see (2.65),(2.62),(2.62_1)).

2) The *oriented angle* function

$\angle_o(..,..):T^1_e\mathbb{M}^2_\kappa\times T^1_e\mathbb{M}^2_\kappa\longrightarrow]-\pi,\pi]$ for the oriented 2-dim. Euclidean vector space $T^1_e\mathbb{M}^2_\kappa$ (where (u,Ju) is a positively oriented orthonormal 2 frame, see (2.64_0)) is as usual defined by (see (2.60))

$$(2.66)\quad \begin{cases} \textit{For all } v,w\in T^1_e\mathbb{M}^2_\kappa: \quad \measuredangle_o(v,w)\in\,]-\pi,\pi] \textit{ and} \\ \cos(\measuredangle_o(v,w)) = g(v,w)\ ,\quad \sin(\measuredangle_o(v,w)) = \sigma(v,w)\ , \end{cases}$$

using $g(v,w)^2+\sigma(v,w)^2=1$ (cf. (2.60_1)), i.e. the point $(g(v,w),\sigma(v,w))$ lies on the unit circle S^1. This function $\measuredangle_o$ satisfies for all $v,w,\tilde{w}\in T^1_e\mathbb{M}^2_\kappa$:

$$(2.66_1)\quad \begin{cases} w\neq v \Rightarrow \measuredangle_o(v,w) = -\measuredangle_o(w,v)\ , \\ \measuredangle_o(v,\tilde{w}) - \measuredangle_o(v,w) \equiv \measuredangle_o(w,\tilde{w}) \mod 2\pi\ . \end{cases}$$

(verify, that cos and sin have equal value for the left and the right hand side of the congruence (2.66_1)). Moreover: The comparison of (2.64),(2.66) implies for all $p\in\mathbb{M}^2_\kappa$:

$$(2.66_2)\quad \begin{cases} (\cos\varphi_u)(p) = \cos(\measuredangle_o(u,U_e(p)))\ , \\ (\sin\varphi_u)(p) = \sin(\measuredangle_o(u,U_e(p)))\ , \end{cases}$$

which provides the following *geometric interpretation of the functions* $\cos\varphi_u$, $\sin\varphi_u:\mathbb{M}^2_\kappa(e)\longrightarrow\mathbb{R}$: For every $p\in\mathbb{M}^2_\kappa(e)$ the number $(\cos\varphi_u)(p)$ resp. $(\sin\varphi_u)(p)$ is the cosine resp. the sine of the oriented angle between the fixed direction u in $T_e\mathbb{M}^2_\kappa$ (see (2.64_0)) and the direction $U_e(p)$, under which an observer at **e** “sees” the point **p**. This interpretation becomes more satisfactory, using a covering model for $\mathbb{M}^2_\kappa(e)$:

3) Consider (see (1.9))

$$(2.67_0)\quad \begin{cases} \tilde{M}:=]0,\pi_\kappa[\times\mathbb{R} \textit{ as an open } C^\omega \textit{ submanifold of } \mathbb{R}^2 \textit{ and let} \\ \rho,\varphi:]0,\pi_\kappa[\times\mathbb{R}\longrightarrow\mathbb{R} \textit{ denote the first resp. second} \\ \textit{projection.} \end{cases}$$

Then the C^ω map

$$(2.67)\quad \begin{cases} f^u := \exp_\kappa(\rho\cdot(\cos(\varphi)\cdot u + \sin(\varphi)\cdot Ju)):\tilde{M}\longrightarrow\mathbb{M}^2_\kappa(e) \textit{ is} \\ \textit{a universal covering for } \mathbb{M}^2_\kappa(e) \text{ (see (2.15),}(2.61_0)) \end{cases}$$

which evidently satisfies (compare (2.16) with (2.67)):

$$(2.67_1)\qquad r\circ f^u = \rho \quad \textit{and} \quad U_e\circ f^u = \cos(\varphi)\cdot u + \sin(\varphi)\cdot Ju\ ,$$

i.e. (see (2.64),(2.67_1)):

$$(2.67_2)\qquad (\cos\varphi_u)\circ f^u = \cos(\varphi) \quad \textit{and} \quad (\sin\varphi_u)\circ f^u = \sin(\varphi)\ ,$$

whence one deduces

(2.67_3) $\quad (f^u)^*dr = d\rho$ *and* (using (2.65)) $(f^u)^*\theta = d\varphi$,

and therefore (see (2.62),(2.67_3))

(2.67_4) $\quad (f^u)_*(\frac{\partial}{\partial\rho}) = R\circ f^u$ *and* $(f^u)_*(\frac{\partial}{\partial\varphi}) = (\sin_\kappa(r)\cdot JR)\circ f^u$.

Finally the comparison of (2.66_2) and (2.67_2) yields

(2.67_5) $\quad \varphi \equiv \measuredangle_0(u,U_e\circ f^u)$ mod 2π .

Now suppose, that $r\in\mathbb{N}\cup\{\infty,\omega\}$ and we are given a

(2.68_0) $\quad C^r$ *map* $h:N\longrightarrow \mathbb{M}^2_\kappa(e)$ *of a 1-connected* C^r *manifold* N

into $\mathbb{M}^2_\kappa(e)$. Then because of (2.67) and the simple connectedness of N there exists by the monodromy principle a

$$(2.68)\left\{\begin{array}{l} C^r \textit{ lifting } \hbar:N\longrightarrow \tilde{M} \textit{ of } h \textit{ w.r.t. } f^u:\tilde{M}\longrightarrow \mathbb{M}^2_\kappa(e) ,\\ \textit{i.e. } f^u\circ\hbar = h \end{array}\right.$$

and we obtain for this map $\hbar:N\longrightarrow\tilde{M}$ (see (2.68),(2.67_1), (2.67_3))

$$(2.68_1)\left\{\begin{array}{l} r\circ h = \rho\circ\hbar \ , \quad h^*\theta = d(\varphi\circ\hbar) \quad \textit{and} \\ U_e\circ h = \cos(\varphi\circ\hbar)\cdot u + \sin(\varphi\circ\hbar)\cdot Ju \ , \quad \textit{i.e.} \\ (\cos\varphi_u)\circ h = \cos(\varphi\circ\hbar) \quad \textit{and} \quad (\sin\varphi_u)\circ h = \sin(\varphi\circ\hbar) \ , \end{array}\right.$$

and we get from (2.68),(2.67_5) and (2.66_1)

(2.68_2) $\forall_{p,q\in N}\ \varphi(\hbar(q))-\varphi(\hbar(p)) \equiv \measuredangle_0(U_e(h(p)),U_e(h(q)))$ mod 2π.

Moreover, if there exists

$$(2.69_0)\left\{\begin{array}{l} q_0\in N, \textit{ such that } h(q_0)\in \exp_\kappa(]0,\pi_\kappa[\cdot u) \ ,\\ \textit{i.e. } U_e(h(q_0))=u \ , \end{array}\right.$$

then $(r(h(q_0)),0)\in\tilde{M}$ and $f^u((r(h(q_0)),0)) = h(q_0)$ due to (2.16),(2.67),(2.69_0) and consequently according to the monodromy principle there exist a

$$(2.69)\left\{\begin{array}{l} \textit{unique lift of } h:(N,q_0)\longrightarrow(\mathbb{M}^2_\kappa(e),\exp_\kappa(]0,\pi_\kappa[\cdot u))\\ \textit{with respect to } f^u:\tilde{M}\longrightarrow\mathbb{M}^2_\kappa(e) \ , \textit{ we call it } \hbar^{(u,q_0)},\\ \textit{s. t. } f^u\circ\hbar^{(u,q_0)}=h \textit{ and } \hbar^{(u,q_0)}(q_0)=(r(h(q_0)),0),\\ \textit{i.e. } \rho\circ\hbar^{(u,q_0)}(q_0)=r\circ h(q_0) \textit{ and } \varphi\circ\hbar^{(u,q_0)}(q_0)=0. \end{array}\right.$$

For this $\hbar^{(u,q_0)}$ the statements (2.68_1),(2.68_2) hold m.m., moreover

(2.69_1) *for all* $q\in N$: $\varphi\circ\tilde{h}^{(u,q_0)}(q) \equiv \measuredangle_o(u,U_e(h(q)))$ mod 2π .

Via (2.67_3) the polar angle form θ gets now an interpretation appealing to its name:

Suppose that $c:[\alpha,\beta] \longrightarrow \mathbb{M}^2_\kappa(e)$ is an arbitrary C^1 path and $\tilde{c}:[\alpha,\beta] \longrightarrow \tilde{M}$ any lift of c with respect to f^u (see (2.67)), then (see (2.67_3))

$$(2.70)\quad \begin{cases} \int_\alpha^\beta \theta(\dot{c}(t))dt = \int_c \theta = \int_{f^u\circ\tilde{c}} \theta = \int_{\tilde{c}} (f^u)^*\theta = \int_{\tilde{c}} d\varphi = \\ \quad = \varphi(\tilde{c}(\beta))-\varphi(\tilde{c}(\alpha)) \equiv \measuredangle_o(U_e(\tilde{c}(\alpha)),U_e(\tilde{c}(\beta))) \mod 2\pi, \end{cases}$$

in particular, if c is closed, i.e. $\tilde{c}(\alpha)=\tilde{c}(\beta)$, then
(2.70_1) *(winding number of* c *with resp. to* **e**) $:= \frac{1}{2\pi}\cdot\int_c \theta \in \mathbb{Z}$.

(viii) Maximal curves of constant oriented curvature in $\mathbb{M}^2_\kappa$.

First we commemorate the following concepts and results about plane curves (from differential geometry):
Suppose (M,g) is a 2-dim. oriented Riemannian C^s manifold, $s\in\{\infty,\omega\}$, J its complex structure (defined m.m. as in (vii)a)). Suppose I is an open interval in $\mathbb{R}$, $c:I \longrightarrow M$ a unit speed C^2 path in M. Then $\dot{c}$ resp. $J\dot{c}$ is the *unit tangent* resp. *principal unit normal* vector field of c and one calls

$$(2.71)\quad \begin{cases} \kappa_c := g(\nabla_\partial \dot{c}, J\dot{c}):I \longrightarrow \mathbb{R} \ \textit{the oriented curvature of } c, \\ \textit{wherefore } c \textit{ satisfies the } \text{FRENET} \textit{ ODE } \ \nabla_\partial \dot{c} = \kappa_c\cdot J\dot{c} \ . \end{cases}$$

Moreover the name "oriented" curvature is motivated by:

$$(2.71_1)\quad \begin{cases} \textit{The reverse path } c^\vee \textit{ of } c \text{ (see (2.41)) } \textit{has oriented} \\ \textit{curvature } \ \kappa_{(c^\vee)} = -\kappa_c \ . \end{cases}$$

Suppose now, that $\lambda:M \longrightarrow \mathbb{R}$ is a C^s function on M, $s\in\{\infty,\omega\}$, with $\|\mathrm{grad}\lambda\| = 1$, and $c:I \longrightarrow M$ is the maximal C^1 integral curve of the unit vector field $J\mathrm{grad}\lambda$ through some point $p\in M$. Then c is a C^s path and (see (2.71)):

$$(2.72)\quad \begin{cases} c(I) \textit{ is the connected component of the point } p \textit{ in the level set } \lambda^{-1}(\{\lambda(p)\}) \textit{ of } \lambda \textit{ through } p \textit{ and } c \textit{ has the} \\ \textit{curvature (see (2.71)): } \kappa_c = \mathrm{hess}\lambda(\mathrm{J}\,\mathrm{grad}\lambda, \mathrm{J}\,\mathrm{grad}\lambda)\circ c, \\ \textit{where } \quad \mathrm{hess}\lambda(X,Y) := g(\nabla_X(\mathrm{grad}\lambda), Y) \quad \textit{for } X,Y\in\mathfrak{X}(M), \\ \textit{moreover:} \\ \textit{If } \lambda^{-1}(\{\lambda(p)\}) \textit{ is compact or } M \textit{ is complete, then } I=\mathbb{R} \\ \textit{and in the first case } c:\mathbb{R}\longrightarrow M \textit{ is periodic.} \end{cases}$$

After these general remarks we return to the geometry of $(M,g) := \mathbb{M}^2_\kappa$ and we choose for the following sections a),b),c) (see (2.5),(2.3),(2.0), in particular $e_1=(0,1,0)$, $e_2=(0,0,1)$)

$$(2.73)\quad u\in T^1_e\mathbb{M}^2_\kappa\ , \quad \textit{such that} \quad (i_*u)^{\rightarrow}=e_1\ , \textit{ therefore } (i_*Ju)^{\rightarrow}=e_2$$

(see (2.57),(2.18),(2.3),(2.5)).

a) Suppose $\rho\in]0,\pi_\kappa[$. Then the set $r^{-1}(\{\rho\})$ (:=*distance sphere of radius* ρ *in* $\mathbb{M}^2_\kappa$ *with center* e) is closed and bounded in the complete manifold $\mathbb{M}^2_\kappa$, hence compact, and therefore (see (2.72), (2.62),(2.61),(2.34)) the maximal integral C^ω curve $c:\mathbb{R}\longrightarrow \mathbb{M}^2_\kappa(e)$ of $JR = J\mathrm{grad}(r)$ with $c(0) = \exp_\kappa(\rho u)$ satisfies: c is a *periodic* unit speed curve with

$$(2.74)\quad \begin{cases} c(\mathbb{R}) = r^{-1}(\{\rho\}) \quad \textit{and } c \textit{ has constant curvature} \\ \kappa_c=\cot_\kappa(\rho), \quad \textit{where} \quad \cot_\kappa(]0,\pi_\kappa[) = \begin{cases}]\sqrt{-\kappa},\infty[& \textit{for } \kappa\le 0 \\ \mathbb{R} & \textit{for } \kappa>0 \end{cases} \end{cases}$$

(in particular c is for $\kappa>0$ and $\rho=(\pi_\kappa/2)$ a unit speed *geodesic*). Moreover it follows from (2.74) and (2.33), that for $\kappa\neq 0$:

$$(2.74_1)\qquad c(\mathbb{R}) = \mathbb{M}^2_\kappa \cap x_0^{-1}(\{\cos_\kappa(\rho)\})\ ,$$

i.e.: $c(\mathbb{R})$ *is the intersection of* $\mathbb{M}^2_\kappa$ *with "the" affine plane in* $\mathbb{R}^3$, *which is affinely parallel in* $\mathbb{R}^3$ *to the affine tangent plane of* $\mathbb{M}^2_\kappa$ *at the center* e *of the distance sphere* $c(\mathbb{R})$ *(at the Euclidean oriented distance* $\cos_\kappa(\rho)-1$).

Finally (see (2.26)):

$$(2.74_2)\quad \begin{cases} c:\mathbb{R}\longrightarrow \mathbb{M}^2_\kappa \quad \textit{is an orbit of the 1-param. subgroup of} \\ G^2_\kappa \textit{ generated by the } \text{Killing} \textit{ vector field } \dfrac{\sin_\kappa(r)}{\sin_\kappa(\rho)}\cdot JR. \end{cases}$$

b) If $\kappa<0$ and $\beta:\mathbb{M}^2_\kappa \longrightarrow \mathbb{R}$ is the Busemann function of the

unit speed geodesic $t \mapsto \exp_\kappa(tu)$, then – since $\mathbb{M}^2_\kappa$ is complete – it follows from (2.72) and (2.48_3),(2.48_4): The maximal integral C^ω curve $c:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa(e)$ of $J(\mathrm{grad}\beta)$ with $c(0) = \mathbf{e}$ satisfies (see (vi)d), (2.52) and use, that due to (2.52_1),(2.52_2) horospheres are connected):
c is a unit speed curve with

(2.75) $\begin{cases} c(\mathbb{R}) = \beta^{-1}(\{0\}) \text{ (=}\textit{horosphere through } \mathbf{e} \textit{ with limit} \\ \textit{point } \text{“}\exp_\kappa(\infty\cdot u)\text{”) } \textit{and c has const. curvature } \kappa_c = \sqrt{-\kappa}. \end{cases}$

Moreover it follows from (2.52),(2.39) and (2.73) that

(2.75_1) $$c(\mathbb{R}) = \mathbb{M}^2_\kappa \cap (\dot{x}_0 - \sqrt{-\kappa}\cdot x_1)^{-1}(\{1\}) ,$$

i.e.: c(ℝ) is the intersection of $\mathbb{M}^2_\kappa$ *with “the” affine plane in* $\mathbb{R}^3$ *through* **e**, *which is which is parallel to the affine tangent plane of the asymptotic cone* ∂C *of* $\mathbb{M}^2_\kappa$ *in* $\mathbb{R}^3$ (see (2.37)) *at the point* $\mathbf{e} + (1/\sqrt{-\kappa})\cdot(i_*u)^{\rightarrow}$.
Finally (see (2.26)):

(2.75_2) $\begin{cases} c:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa \textit{ is an orbit of the 1-param. subgroup of} \\ G^2_\kappa \textit{ gener. by the } \textsc{Killing} \textit{ vector field } e^{\sqrt{-\kappa}\cdot\beta}\cdot J\mathrm{grad}\beta. \end{cases}$

c) Suppose $\kappa \le 0$, therefore (see (1.1)) $\sin_\kappa:\mathbb{R} \longrightarrow \mathbb{R}$ is a C^ω diffeomorphism and we obtain on $\mathbb{M}^2_\kappa$ the extrinsicly defined

(2.76) $$C^\omega \textit{ function } \delta := \sin_\kappa^{-1}\circ x_2|\mathbb{M}^2_\kappa:\mathbb{M}^2_\kappa \longrightarrow \mathbb{R} .$$

Using (2.73) we get on $\mathbb{M}^2_\kappa(e)$ the following intrinsic description of $x_2|\mathbb{M}^2_\kappa$:

$$\sin_\kappa(r)\cdot\sin\varphi_u \underset{(2.58)}{=} \sin_\kappa(r)\cdot g(Ju,\sin\varphi_u\cdot Ju) \underset{(2.3),(2.5)}{=} \sin_\kappa(r)\cdot g(Ju,U_e) \underset{(2.64)}{=} (i_*Ju^{\rightarrow}|\sin_\kappa(r)\cdot i_*U_e^{\rightarrow})_\kappa \underset{(2.33),(2.76_1)}{=} x_2|\mathbb{M}^2_\kappa ,$$

hence (see (2.76)):

(2.76_1) $$\sin_\kappa(\delta) = \sin_\kappa(r)\cdot\sin\varphi_u \quad \textit{on } \mathbb{M}^2_\kappa(e).$$

The “Law of sines” for rectangular triangles in the Euclidean resp. the hyperbolic plane (of curvature κ) and (2.76_1) tell but, that δ *measures the oriented distance from the geodesic* $\exp_\kappa(\mathbb{R}\cdot u) = \mathbb{M}^2_\kappa \cap x_2^{-1}(\{0\})$, δ being positive in the “half plane” $\mathbb{M}^2_\kappa \cap x_2^{-1}(\mathbb{R}_+)$ bounded by $\exp_\kappa(\mathbb{R}\cdot u)$ in which Ju points, negative in the opposite "half plane"[1]. Then one computes from (2.76_1) and (1.1′),(2.65),(2.62_0),(2.60),(2.34):

$$(2.76_2) \qquad \mathrm{grad}\delta = \frac{1}{\cos_\kappa(\delta)}\cdot(\cos_\kappa(r)(\sin\varphi_u)\cdot R + (\cos\varphi_u)\cdot JR)$$

on $\mathbb{M}^2_\kappa(e)$, from where one gets (observing (2.76_1),(2.64_1), (2.61),(1.6)): $\|\mathrm{grad}\delta\|=1$ on $\mathbb{M}^2_\kappa$, moreover one obtains from (2.76_2) (using (2.65),(2.62),(2.59)):

$$(2.76_3) \qquad \forall_{X\in\mathfrak{X}(\mathbb{M}^2_\kappa)} \quad \nabla_X \mathrm{grad}\delta = -\,\kappa\cdot\tan_\kappa(\delta)\cdot g(J\mathrm{grad}\delta, X)\cdot J\mathrm{grad}\delta \ .$$

Since $\mathbb{M}^2_\kappa$ is complete, we get from (2.72) for any $\rho\in\mathbb{R}$:
If $c:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa(e)$ is the maximal integral C^ω curve of $JR = J\mathrm{grad}\delta$ with $c(0) = \exp_\kappa(\rho Ju)$, then (see (2.72), (2.76_3)): c is a unit speed curve with

$$(2.77) \quad \begin{cases} c(\mathbb{R}) = \delta^{-1}(\{\rho\}) \quad \textit{and } c \textit{ has curvature } \kappa_c = -\kappa\tan_\kappa(\rho), \\ \textit{where} \quad -\kappa\tan_\kappa(\mathbb{R}) = \begin{cases} \{0\} & \textit{for } \kappa=0 \\ ]-\sqrt{-\kappa},\sqrt{-\kappa}[& \textit{for } \kappa<0 \end{cases} \end{cases}$$

(in particular c is a geodesic if $\kappa=0$ or $\rho=0$ and $\kappa<0$). Moreover, we get by the above-mentioned geometric interpretation of δ as the oriented distance from the geodesic $t \mapsto \exp_\kappa(tu)$: $c(\mathbb{R})$ is the connected component of the point $\exp_\kappa(\rho Ju)$ in the boundary of the tubular neighborhood of $\exp_\kappa(\mathbb{R}\cdot u)$ of radius $|\rho|$ and which we call therefore the "*ρ-distant parallel of* $\exp_\kappa(\mathbb{R}\cdot u)$". Furthermore, one obtains from (2.76),(2.77):

$$(2.77_1) \qquad c(\mathbb{R}) = \mathbb{M}^2_\kappa \cap x_2^{-1}(\{\sin_\kappa(\rho)\}) \ ,$$

i.e.: $c(\mathbb{R})$, the ρ-distant parallel of the geodesic $\exp_\kappa(\mathbb{R}\cdot u)$ in $\mathbb{M}^2_\kappa$, is the intersection of $\mathbb{M}^2_\kappa$ with the affine plane of $\mathbb{R}^3$,

---

[1] If one wants to describe the oriented distance δ from the geodesic $\exp_\kappa(\mathbb{R}\cdot u)$ also in the case $\kappa>0$, one only has to replace in (2.76) $\sin_\kappa^{-1}$ by $(\sin_\kappa|[-\frac{1}{2}\cdot\pi_\kappa,\frac{1}{2}\cdot\pi_\kappa])^{-1}$. Then this function δ is continuous on $\mathbb{M}^2_\kappa$, however C^ω only on $\mathbb{M}^2_\kappa\setminus\{\sqrt{\kappa}e_2,-\sqrt{\kappa}e_2\}$.

which is (at the oriented Euclidean distance $\sin_\kappa(\rho)$) *affinely parallel to the 2-dim. vector subspace* $x_2^{-1}(\{0\})$ *of* $\mathbb{R}^3$, *that intersects* (in the sense of (2.17)) $\mathbb{M}_\kappa^2$ *in the geodesic* $\exp_\kappa(\mathbb{R}\cdot u)$. Finally (see (2.26))

$$(2.77_2)\ \begin{cases} c:\mathbb{R} \longrightarrow \mathbb{M}_\kappa^2 \ \textit{is an orbit of the 1-param. subgroup of} \\ G_\kappa^2 \ \textit{gener. by the } \textsc{Killing} \textit{ v. field } \ \dfrac{\cos_\kappa(\delta)}{\cos_\kappa(\rho)}\cdot J\mathrm{grad}\delta \ . \end{cases}$$

Since (see (2.27)) the group G_κ^2 of all orientation preserving isometries of $\mathbb{M}_\kappa^2$ acts transitively on the unit tangent bundle of $\mathbb{M}_\kappa^2$, since $\kappa_{(f\circ c)}=\kappa_c$ for every C^2 unit speed path c in $\mathbb{M}_\kappa^2$ and every $f\in G_\kappa^2$ and since every $f\in G_\kappa^2$ is (see (2.20)) induced by a certain linear map $\mathbb{A}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which evidently maps (affine) planes of $\mathbb{R}^3$ into (affine) planes of $\mathbb{R}^3$, preserving the parallelity between such planes, the results (2.71_1), (2.74), ..., (2.77_2) and the uniqueness theorem for the solutions of the ODE (2.71) allow to state the following résumé:

d) Proposition. For every $p\in\mathbb{M}_\kappa^2$, $u\in T_p^1\mathbb{M}_\kappa^2$ and $\kappa_0\in\mathbb{R}$, there exist exactly one maximally unit speed C^2 path $c:\mathbb{R} \longrightarrow \mathbb{M}_\kappa^2$ of constant oriented curvature $\kappa_c\equiv\kappa_0$ and $\dot{c}(0)=u$. This c is C^ω, $i\circ c:\mathbb{R} \longrightarrow \mathbb{R}^3$ is a plane curve in $\mathbb{R}^3$ and – if $\kappa\neq 0$ – $c(\mathbb{R})$ is the intersection of the hypersurface $\mathbb{M}_\kappa^2$ of $\mathbb{R}^3$ with an affine plane of $\mathbb{R}^3$ (which is the osculating plane of $i\circ c$). More specifically one can say: In case

$\kappa<0$ *and*
- $|\kappa_0|\in[0,\sqrt{-\kappa}[$: $c(\mathbb{R})$ *is a ρ-distant parallel curve to a geodesic with* $\rho\in\mathbb{R}$, *s.t.* $\kappa_0=-\kappa\tan_\kappa(\rho)$ (*in partic.* $c(\mathbb{R})$ *a geodesic for* $\kappa_0=0$).
- $|\kappa_0|=\sqrt{-\kappa}$: $c(\mathbb{R})$ *is a 1-dim. horosphere in* $\mathbb{M}_\kappa^2$.
- $|\kappa_0|\in]\sqrt{-\kappa},\infty[$: $c(\mathbb{R})$ *is a 1-dim. distance sphere of radius* $\rho\in\mathbb{R}_+$, *such that* $\kappa_0=\cot_\kappa(\rho)$.

$\kappa=0$ *and*
- $\kappa_0=0$: $c(\mathbb{R})$ *is a geodesic (=straight line).*
- $\kappa_0\in\mathbb{R}^*$: $c(\mathbb{R})$ *is a 1-dim. distance sphere (=euclidean circle) of radius* $1/|\kappa_0|$.

$\kappa>0$ *and* $\kappa_0\in\mathbb{R}$: $c(\mathbb{R})$ *is a 1-dim. distance sphere of radius* $\rho\in]0,\pi_\kappa[$, *s.t.* $\kappa_0=\cot_\kappa(\rho)$ *(in partic.* $c(\mathbb{R})$ *a geodesic for* $\rho=\frac{1}{2}\cdot\pi_\kappa$*).*

Finally: Each of these unit speed curves of constant oriented curvature is the unit speed orbit of a 1-parameter subgroup of orientation preserving isometries of $\mathbb{M}^2_\kappa$ (the opposite of this statement being trivial).

3. Informations about the maximal solutions of the ODE, describing a mechanical process of one degree of freedom

(i) Data:

$$(3.0)\quad \begin{cases} \textit{Suppose } M \textit{ is a non-empty open interval of } \mathbb{R}, \\ a:M \longrightarrow \mathbb{R} \;\; \textit{is a loc. Lipschitzian function (e.g. } C^1\textit{)}, \\ U:M \longrightarrow \mathbb{R} \;\; \textit{is any primitive funct. for } -a \textit{ (i.e. } U'=-a\textit{)}, \\ (r_0,v_0) \in M\times\mathbb{R} \;\; \textit{and} \;\; E_0 := \frac{1}{2}\cdot v_0^2 + U(r_0) \;. \end{cases}$$

Then consider the following ODE initial value problems:

$$(3.1)\quad y'' = a(y) \qquad \textit{with} \quad y(0) = r_0 \quad \textit{and} \quad y'(0) = v_0 \;,$$

resp.

$$(3.2)\quad \tfrac{1}{2}\cdot(y')^2 + U(y) = E_0 \quad \textit{with} \quad y(0) = r_0 \quad \textit{and} \quad y'(0) = v_0 \;.$$

We call the ODE of (3.1) the "*law of acceleration*" resp. the one of (3.2) the "*law of energy*" for a certain *(mechanical) process of one degree of freedom* with M as its *configuration space*, $M\times\mathbb{R}$ as its *state space* (or *phase space*), (r_0,v_0) its *initial state*, E_0 its *total energy*, a its *acceleration* with U as ("a-*effective*") *potential*, the "*mechanical process*" itself will be viewed as "the" maximal C^2 solution of (3.1) (see below (3.5)).

Remark. By a "*solution*" of the ODE (3.1) resp. (3.2) we always mean a twice resp. once differentiable function $r:I \longrightarrow \mathbb{R}$ on an open interval I of $\mathbb{R}$ with $0\in I$ and $r(I)\subseteq M$ such that ($r''=a\circ r$ and $r(0)=r_0$ and $r'(0)=v_0$) resp. ($\frac{1}{2}(r')^2+U(r)=E_0$ and $r(0)=r_0$ and $r'(0)=v_0$), and then one can show, that r has to be C^2 (trivial) resp. C^1 (using DARBOUX's theorem). Such a solution is called "*maximal*", if r is not the restriction of a solution $\tilde{r}:\tilde{I} \longrightarrow \mathbb{R}$ with $I\subseteq\tilde{I}$ and $I\neq\tilde{I}$.

There is a very tight correlation between the two ODE's (3.1),(3.2), essentially well known, and of which correlation we present here a rather subtle (for our purposes useful) version:

(ii) Proposition. (Data as in (i).) For any open interval I of $\mathbb{R}$ with $0\in I$ and every C^2 function $r:I \longrightarrow \mathbb{R}$ with $r(I)\subseteq M$ the following three statements a)-c) are pairwise equivalent:

a) r *is a solution of* (3.1).

b) r *is a solution of* (3.2) *and* (r' *non-constant or* $\mathbf{a}(r_0)=0$).

c) r *is a solution of* (3.2) *and* (r *non-constant or* $\mathbf{a}(r_0)=0$).

Remark. The stated equivalences remain true, if "solution" is substituted each time by "*maximal solution*".

Proof.

a) ⇒ b): That r (being a solution of (3.1)) is as well a solution of (3.2) follows by differentiating $\frac{1}{2}\cdot(r')^2 + \mathbf{U}(r)$. If moreover r' is constant, then $r''=0$, hence

$$\mathbf{a}(r_0) \underset{a}{=} r''(0) = 0 \ .$$

b) ⇒ c): Trivial.

c) ⇒ a): For this we distinguish two cases:

1st case: r *is constant*: Then $r' = r'' = 0$ and therefore

$$r''-\mathbf{a}(r) = -\mathbf{a}(r) = -\mathbf{a}(r_0) \underset{c}{=} 0 \ .$$

2nd case: r *is non-constant*:

Then $H := \{t\in I \mid r'(t)\neq 0\} \neq \emptyset$. From c) (i.e. from (3.2) with y=r) follows by differentiation immediately $r'\cdot(r''-\mathbf{a}(r)) = 0$, which implies by definition of H:

(3.3) $r'' = \mathbf{a}(r)$ (*on* H *and thus by continuity*) *on* $\bar{H}\cap I \neq \emptyset$.

So we are done, if $\bar{H}\cap I = I$. Otherwise denote by

G an arbitrary (non-empty) connected component of $I\setminus\bar{H}$, therefore G is an open, non-empty proper subinterval of I, consequently $\inf I \leq \inf G < \sup G \leq \sup I$ and the equality signs do not hold simultaneously. Therefore there exists

$\tau\in\{\inf G,\sup G\}\cap I$, in particular $\tau\in(\overline{G}\setminus G)\cap I$ and since G is (as a connected component of $I\setminus\overline{H}$) closed in $I\setminus\overline{H}$, i.e. $\overline{G}\cap(I\setminus\overline{H})=G$, it follows, that

(3.4) $\tau \in \overline{H}\cap I$, *whence* (see(3.3)) $r''(\tau) = \mathbf{a}(r(\tau))$.

Moreover $G\subseteq(I\setminus\overline{H})\subseteq(I\setminus H)$, hence r' vanishes on the open interval G of I, in particular $r''-\mathbf{a}(r)$ is constant on G and by continuity actually on $\overline{G}\cap I$ ($\ni\tau$). Therefore (see (3.4)) $r''=\mathbf{a}(r)$ holds on G, which finishes the proof.

(iii) Since the ODE (3.1) of second order is equivalent to the ODE system of first order with locally Lipschitzian **a** (see (3.0)):

$y' = v$ *and* $v' = \mathbf{a}(y)$ *with* $y(0) = r_0$ *and* $v(0) = v_0$,

we obtain from the general theory of ODE's the following

Existence and Uniqueness Theorem for (3.1):
Data as in (i), then:

(3.5) $\left\{\right.$ *There exists exactly one maximal* C^2 *solution of* (3.1) *and any* C^2 *solution of* (3.1) *is a restriction of this maximal one,*

e.g. for any C^2 solution $r:I \longrightarrow \mathbb{R}$ of (3.1) the following five statements a),...,e) are therefore trivially equivalent:

(3.6)
- a) $r:I \longrightarrow \mathbb{R}$ *is constant* ,
- b) $(r,r'):I \longrightarrow \mathbb{R}^2$ *is constant* ,
- c) $r' = \mathbf{a}\circ r = 0$ (*i.e.* $E_0 - (U\circ r) = U'\circ r = 0$),
- d) $v_0 = \mathbf{a}(r_0) = 0$ (*i.e.* $E_0 - U(r_0) = U'(r_0) = 0$), (such a state (v_0,r_0) is called a "*state of equilibrium*" for the law of acceleration $y''=\mathbf{a}(y)$),
- e) *There exists* $t_0\in I$ *with* $r'(t_0) = \mathbf{a}(r(t_0)) = 0$ (*i.e.* $E_0 - U(r(t_0)) = U'(r(t_0)) = 0$) .

The discussion of "the" solution of (3.1) (see (3.5)) is frequently done (e.g. for KEPLER's problem) by discussing "the"(?) solution of the corresponding law of energy (3.2). This procedure deserves some caution (see the "Warning" below), a certain justification for it however is provided by the following

(iv) Corollary. Data as in (i). *Then there exists a* *maximal* C^2 *solution* $r:I \longrightarrow \mathbb{R}$ *of* (3.2), *which is unique among all* C^2 *solutions of* (3.2) *if* $a(r_0) = 0$, *resp. which is non-constant and is at least unique among all non-constant solutions of* (3.2) *if* $a(r_0) \neq 0$. *Moreover this solutions of* (3.2) *is the unique maximal solutions of* (3.1) (see (3.5)).

[Proof: Trivial consequence of Proposition (ii) and (3.6).]

Warning:

a) The C^ω ODE $(y')^2 + y^2 = 1$ *with* $y(0)=1$ *and* $y'(0)=0$, which describes *the "law of energy" for the harmonic oscillator* (with a certain initial state), *has two different maximal* C^ω *solutions* $r = \cos$ *and* $r \equiv 1$ (on $\mathbb{R}$), but here we have $a(r_0) = -1 \neq 0$. Therefore in the last Corollary one can get in general no better results with respect to the uniqueness. Moreover the last example demonstrates that the "law of energy" *alone* is *not sufficient* to single out the proper mechanical process solving (3.1), but that one has in addition to take in account the initial acceleration $a(r_0)$: If $a(r_0)=0$, then *the* constant solution $r \equiv r_0$, resp. if $a(r_0) \neq 0$, then (not the constant but only) *the* non-constant maximal C^2 solution of (3.2) describes the mechanical process determined by (3.1).

b) For the first order ODE (3.2) (law of energy for (3.1)) it makes of course sense to speak of C^1 solutions. However in Corollary (iv) it is essential for the "uniqueness" part to admit only C^2 solutions: Because already in the preceding example a) of the harmonic oscillator there exists e.g. a whole family $(r_\tau)_{\tau \in \mathbb{R}_+}$ of maximal C^1 solutions $r_\tau : \mathbb{R} \longrightarrow \mathbb{R}$ of the corresponding law of energy ((3.2) with $2U(x) := x^2$, $r_0=1$, $v_0=0$), namely

$$r_\tau(t) = 1 \quad \text{for } t \leq \tau \quad \text{and} \quad r_\tau(t) = \cos(t-\tau) \quad \text{for } t \geq \tau.$$

However r_τ is lacking the second order differentiability at (exactly the point) $\tau \in \mathbb{R}_+$, where we have a "jump" in the acceleration: $\lim_{t \nearrow \tau} r''_\tau(t) = 0 > -1 = \lim_{t \searrow \tau} r''_\tau(t)$. Therefore these C^1 solutions of the law of energy correspond to "mechanical processes" (*not admitted as "solutions" in* (3.1)) with *sudden* passages from rest to harmonic oscillation.

For a non-constant process of one degree of freedom $r:I \longrightarrow \mathbb{R}$ with prescribed (acceleration a resp. its) potential U and initial state (r_0,v_0) as in (3.0) one can get *a priori* (i.e. expressible only in terms of the data U,r_0,v_0) qualitative information about I and the behaviour of r without knowing an explicit quantitative description of r (which in general is accessible only through numerical methods for solving an ODE). For the convenience of the reader we give here a rather detailed *summary of these (essentially well-known) results* without proof (a complete proof of which was presented by the first author in his course on "Mechanik" 1982/83 at Köln) in the following

(v) Theorem.

Data: *As in* (i), *in particular*

(3.7$_0$) $\quad E_0 - U(r_0) \geq 0$,

and we make the additional assumption (thereby *excluding a constant process* $r:I \longrightarrow \mathbb{R}$ *solving* (3.1), see (3.6)):

(3.7) $\quad (E_0 - U(r_0), U'(r_0)) \neq (0,0)$.

Consider the following subsets of M:

$$(3.8_0)\quad \begin{cases} M_{\substack{+\\(-)}} := \{\rho\in M \mid E_0 \underset{(<)}{>} U(\rho)\} \quad \textit{and} \\ M_0 := \{\rho\in M \mid E_0 = U(\rho) \textit{ and } U'(\rho)\neq 0\} , \\ \textit{consequently for every } \rho\in M_0 \textit{ there exists } \varepsilon\in\mathbb{R}_+ \textit{ with} \\ ]\rho-\varepsilon,\rho[\subseteq M_{\substack{+\\(-)}} \quad \textit{and} \quad]\rho,\rho+\varepsilon[\subseteq M_{\substack{-\\(+)}} \quad \textit{if} \quad U'(\rho) \underset{(<)}{>} 0 . \end{cases}$$

From (3.7$_0$), (3.7), (3.8$_0$) follows, that $r_0 \in M_+ \cup M_0$ and therefore

$$(3.8)\quad \begin{cases} H := \textit{connected component of the point } r_0 \textit{ in } M_+\cup M_0 \\ \quad \textit{is the largest subinterval of } M_+\cup M_0 \ (\subseteq M) \textit{ with } r_0\in H. \end{cases}$$

For this interval H one obtains from (3.7),(3.8$_0$),(3.8)

(3.9$_0$) $\quad \inf H =: r_{min} \leq r_0 \leq r_{max} := \sup H$ *and* $r_{min} < r_{max}$,

(3.9) $\quad H^\circ =]r_{min}, r_{max}[\neq \emptyset$ *and* $(E_0-U)(H^\circ) \subseteq \mathbb{R}_+ :=]0,\infty[$,

(3.9$_-$) $\quad r_{min}\in M \Rightarrow [E_0=U(r_{min})$ *and* $(r_{min}\in H \Leftrightarrow U'(r_{min})<0)]$,

(3.9_+) $\quad r_{max}\in M \Rightarrow [E_0=U(r_{max})$ *and* $(r_{max}\in H \Leftrightarrow U'(r_{max})>0)]$.

Finally we define the numbers $\tau_-\in[-\infty,0]$ and $\tau_+\in[0,\infty]$ by

$$(3.10) \qquad \tau_- := -\int_{r_{min}}^{r_0} \frac{dx}{\sqrt{2(E_0-U(x))}} \quad \text{and} \quad \tau_+ := \int_{r_0}^{r_{max}} \frac{dx}{\sqrt{2(E_0-U(x))}} ,$$

in particular (see (3.9)):

(3.10_0) $(\tau_-=0 \Leftrightarrow r_{min}=r_0)$ *and* $(\tau_+=0 \Leftrightarrow r_{max}=r_0)$, *hence* $\tau_-<\tau_+$,
and one has the useful information:

$$(3.10_-) \begin{cases} r_{min} \in M\backslash H & \Leftrightarrow \quad r_{min} \in M \text{ and } U'(r_{min}) = 0 , \\ & \Leftrightarrow \quad r_{min} \in M \text{ and } \tau_- = -\infty , \end{cases}$$

$$(3.10_+) \begin{cases} r_{max} \in M\backslash H & \Leftrightarrow \quad r_{max} \in M \text{ and } U'(r_{max}) = 0 , \\ & \Leftrightarrow \quad r_{max} \in M \text{ and } \tau_+ = +\infty . \end{cases}$$

Assertion. If $r:I \longrightarrow \mathbb{R}$ is the maximal C^2 solution of (3.1) (see (3.5)) for the given data $U:M \longrightarrow \mathbb{R}$, $(r_0,v_0) \in M\times\mathbb{R}$ of (3.0),(3.7), then:

a) (3.11) $\begin{cases} r:I \longrightarrow \mathbb{R} \text{ is an open mapping of } I \text{ onto the subspace} \\ H \text{ of } \mathbb{R}, \text{ in particular } r(I) = H . \end{cases}$

b) For every $\tau\in I$ the following is true:

(3.12_0) $\quad r(\tau)\in\{r_{min},r_{max}\} \Leftrightarrow r'(\tau)=0$

$$(3.12) \begin{cases} \Leftrightarrow \ I \text{ is symmetric and } r \text{ is even w.r.t. } \tau, \\ \quad i.e. \ \sigma_\tau(I) = I \text{ and } r\circ\sigma_\tau = r , \end{cases}$$

where $\sigma_\tau(x) := 2\tau-x : \mathbb{R} \longrightarrow \mathbb{R}$ is the reflection of $\mathbb{R}$ at τ.

c) With $\operatorname{sgn}(v_0) := \begin{cases} +1 \\ -1 \end{cases}$ *iff* $v_0 \begin{cases} \geq 0 \\ < 0 \end{cases}$ one gets:

(1) If $H =]r_{min},r_{max}[$, then $v_0\neq 0$ and

$$(3.13) \begin{cases} r:I \longrightarrow H \text{ is bijective with } r'(I) \subseteq \operatorname{sgn}(v_0)\cdot\mathbb{R}_+ \\ \text{and } I = \operatorname{sgn}(v_0)\cdot]\tau_-,\tau_+[. \end{cases}$$

(2) If $H = [r_{min},r_{max}[$, then

$$(3.14)\quad \begin{cases} \tau_0 := \mathrm{sgn}(v_0)\cdot\tau_- \in I\ , & \textit{moreover}\quad r(\tau_0) = r_{min}\ , \\ r'(I\cap]-\infty,\tau_0[) \subseteq \mathbb{R}_-\ , & r'(I\cap]\tau_0,\infty[) \subseteq \mathbb{R}_+ \\ \textit{and}\quad I = \mathrm{sgn}(v_0)\cdot]2\tau_- -\tau_+,\tau_+[\ . \end{cases}$$

(3) If $H =]r_{min},r_{max}]$, then

$$(3.15)\quad \begin{cases} \tau_0 := \mathrm{sgn}(v_0)\cdot\tau_+ \in I\ , & \textit{moreover}\quad r(\tau_0) = r_{max}\ , \\ r'(I\cap]-\infty,\tau_0[) \subseteq \mathbb{R}_+\ , & r'(I\cap]\tau_0,\infty[) \subseteq \mathbb{R}_- \\ \textit{and}\quad I = \mathrm{sgn}(v_0)\cdot]\tau_-,2\tau_+ -\tau_-[\ . \end{cases}$$

(4) If $H = [r_{min},r_{max}]$, then

$$(3.16_0)\quad \begin{cases} I = \mathbb{R}\ ,\quad \tau_-,\tau_+\in\mathbb{R}\quad \textit{and}\quad r:\mathbb{R}\longrightarrow\mathbb{R}\quad \textit{is periodic} \\ \textit{with smallest period}\quad 2\hat{T} := 2\cdot(\underset{+}{\tau}-\underset{-}{\tau}) \in \underset{+}{\mathbb{R}}\ , \end{cases}$$

moreover one has for all $\tau\in I$:

$$(3.16)\quad \begin{cases} r(\tau) = r_{min} \iff r(\tau+\hat{T}) = r_{max} \\ \qquad\qquad \Rightarrow\ r'(]\tau,\tau+\hat{T}[) \subseteq \mathbb{R}_+ \quad \textit{and}\quad r'(]\tau-\hat{T},\tau[) \subseteq \underline{\mathbb{R}}\ , \\ \textit{finally:}\quad (r(\tau),r'(\tau)) = (r_0,v_0) \iff \tau \in (2\hat{T})\cdot\mathbb{Z}\ . \end{cases}$$

4. Conics in $\mathbb{M}^2_\kappa$

We use the notations introduced in § 2 and we study now the geometry of certain subsets of $\mathbb{M}^2_\kappa$, which will later occur as the possible orbit sets of a point moving in a central force field with a NEWTON's potential, see below § 6.

(i) Definition. A subset K of $\mathbb{M}^2_\kappa$ is called a *conic* (*section*) in $\mathbb{M}^2_\kappa$ iff K is a connected component of the intersection of $\mathbb{M}^2_\kappa$ ($\subseteq\mathbb{R}^3$, see (2.5)) with the zero-level set $\rho^{-1}(\{0\})$ of a polynomial function $\rho:\mathbb{R}^3\longrightarrow\mathbb{R}$, which is homogeneous of degree two.

[Evidently $\rho^{-1}(\{0\})$ is a quadric cone in $\mathbb{R}^3$ and recalling, that $\mathbb{M}^2_0$ is the affine hyperplane $x_0=1$, we see that in case $\kappa=0$ this definition coincides with the concept of a conic in classical Euclidean geometry.]

(ii) Definition.

Suppose K is a subset of $\mathbb{M}^2_\kappa$. Then we define:

a) K is called an *ellipse* iff there exist points $p,\tilde{p}\in\mathbb{M}^2_\kappa$ and a number $a\in\mathbb{R}$ with

(4.1_0) $\quad 0 \le f := \frac{1}{2}\cdot d_\kappa(p,\tilde{p}) < a < \frac{1}{2}\cdot\pi_\kappa$ (see (1.9)),

such that

(4.1) $\quad K = \{q\in\mathbb{M}_\kappa^2 \mid d_\kappa(q,p)+d_\kappa(q,\tilde{p})=2a\}$.

Correspondingly we call p and $\tilde{p}$ the *focal points* of K, a the *major semiaxis* of K and f (see (4.1_0)) the *focal length* of K.

b) K is called a *hyperbola* iff there exist points $p,\tilde{p}\in\mathbb{M}_\kappa^2$ and a number $a\in\mathbb{R}$ with

(4.2_0) $\quad 0 < a < f := \frac{1}{2}\cdot d_\kappa(p,\tilde{p}) < \frac{1}{2}\cdot\pi_\kappa$ (see (1.9)),

such that

(4.2) $\quad K = \{q\in\mathbb{M}_\kappa^2 \mid d_\kappa(q,\tilde{p})=d_\kappa(q,p)+2a\}$.

Correspondingly we call p and $\tilde{p}$ the *focal points* of K, p the *closer* one, a the *major semiaxis* of K and f (see (4.2_0)) the *focal length* of K.

In order to compare ours with the classical notation of a hyperbola, we point out, that the set (4.2) is only *one* branch of the classical hyperbola

(4.2_1) $\quad \tilde{K} = \{q\in\mathbb{M}_\kappa^2 \mid |d_\kappa(q,p)-d_\kappa(q,\tilde{p})|=2a\}$,

namely that branch, which faces p.

c) K is called a *semihyperbola* iff there exists a point $p\in\mathbb{M}_\kappa^2$, a unit speed geodesic $\gamma:\mathbb{R}\longrightarrow\mathbb{M}_\kappa^2$ and a number $a\in\mathbb{R}$ with

(4.3_0) $\quad -\frac{1}{4}\cdot\pi_\kappa < a < f := \frac{1}{2}\cdot\delta_\gamma(p) \quad + \frac{1}{4}\cdot\pi_\kappa$ (see (1.9)),

such that

(4.3) $\quad K = \{q\in\mathbb{M}_\kappa^2 \mid \delta_\gamma(q)-d_\kappa(q,p)=2a\}$,

where $\delta_\gamma:\mathbb{M}_\kappa^2\longrightarrow\mathbb{R}$ denotes the *oriented distance* from $\gamma(\mathbb{R})$, defined by $\delta_\gamma := \delta\circ\mathfrak{f}$ with δ from (2.76) and $\mathfrak{f}\in G_\kappa^2$ being the orientation preserving isometry of $\mathbb{M}_\kappa^2$, such that $\mathfrak{f}_*\dot{\gamma}(0) = u$ with u from (2.73). Correspondingly we call p the *focal point* of K, γ the *focal line* of K, a the *major semiaxis* of K and f (see (4.3_0)) the *focal length* of K.

d) K is called a *horoellipse* iff $\kappa<0$ and there exists a point $p\in\mathbb{M}_\kappa^2$, a unit speed geodesic $\gamma:\mathbb{R}\longrightarrow\mathbb{M}_\kappa^2$ with $\gamma(0)=p$ and a positive number $s\in\mathbb{R}_+$, such that

(4.4) $$K = \{q\in\mathbb{M}^2_\kappa \mid d_\kappa(q,p)+\beta_\gamma(q)=2s\} ,$$

where $\beta_\gamma:\mathbb{M}^2_\kappa \longrightarrow \mathbb{R}$ denotes the BUSEMANN function of γ (see (2.47), (2.47$_1$)). Correspondingly we call p the *focal point* of K, γ the *axis* of K and s the *pericentral distance* of K (see below (4.10$_1$)).

e) K is called a *horohyperbola* iff $\kappa<0$ and there exists a point $p\in\mathbb{M}^2_\kappa$, a unit speed geodesic $\gamma:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa$ with $\gamma(0)=p$ and a positive number $s\in\mathbb{R}_+$, such that

(4.5) $$K = \{q\in\mathbb{M}^2_\kappa \mid d_\kappa(q,p)-\beta_\gamma(q)=2s\} ,$$

where $\beta_\gamma:\mathbb{M}^2_\kappa \longrightarrow \mathbb{R}$ denotes the BUSEMANN function of γ (see (2.47), (2.47$_1$)). Correspondingly we call p the *focal point* of K, γ the *axis* of K and s the *pericentral distance* of K (see below (4.11$_1$)).

(iii) Remarks.

a) The Euclidean *parabola* is obviously a semihyperbola (see (ii)c)) with the special data $\kappa=0$ and $a=0$. Oppositely one checks easily, that in case $\kappa=0$ every semihyperbola can be interpreted as a (Euclidean) parabola (with the same focal point).

b) In case $\kappa>0$ hyperbolas and semihyperbolas can be interpreted as ellipses. To see this, choose for the set (4.2) as focal points **p** and $-\tilde{p}$ and as major semiaxis $\frac{1}{2}\cdot\pi_\kappa - a$, for the set (4.3) choose as focal points p and $(1/\sqrt{\kappa})\cdot i_* J\dot{\gamma}(0)^{\rightarrow}$ and as major semiaxis $\frac{1}{4}\cdot\pi_\kappa - a$.

c) We can unify the metric descriptions of conics in (ii) as follows: If the subset K of $\mathbb{M}^2_\kappa$ is an ellipse or a hyperbola or a semihyperbola or a horoellipse or a horohyperbola (see (ii)), then there exists a maximal unit speed C^2 path $c:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa$ *of constant oriented curvature* κ_c (see § 2.viii), such that

(4.6) $$K = \{q\in\mathbb{M}^2_\kappa \mid d_\kappa(q,p)=\mathrm{dist}(q,c(\mathbb{R}))\} ,$$

where $p\in\mathbb{M}^2_\kappa$ is the focal point of K resp. in case K is an ellipse any of the two focal points resp. in case K is a

hyperbola the closer focal point. This path c of constant curvature has the following image:
If K is an ellipse or a hyperbola, then $c(\mathbb{R})$ is the 1-dim. distance sphere of radius two times the major semiaxis and with center at the other focal point $\bar{p}$. In case of the ellipse both focal points are contained in the same connected component of $\mathbb{M}^2_\kappa \setminus c(\mathbb{R})$ (namely in the "interior", i.e. in the bounded one, if $\kappa \le 0$). In case K is a hyperbola the two focal points are contained in different connected components of $\mathbb{M}^2_\kappa \setminus c(\mathbb{R})$ (hence p lies in the "exterior", i.e. in the unbounded one, if $\kappa \le 0$).

If K is a semihyperbola, then $c(\mathbb{R})$ is the (2a)-distant parallel curve to the geodesic, which is the focal line of K, and a being the major semiaxis (see § 2.viii.c,d).

If K is a horoellipse, then $c(\mathbb{R})$ is the 1-dim. horosphere $\beta_\gamma^{-1}(\{2s\})$, and if K is a horohyperbola, then $c(\mathbb{R})$ is the 1-dim. horosphere $\beta_\gamma^{-1}(\{-2s\})$, where γ is the axis and s the pericentral distance of K. These two cases differ only with respect to the connected component of $\mathbb{M}^2_\kappa \setminus c(\mathbb{R})$ which contains the focal point p.

Remark. The idea to characterize the conics uniformly as sets of points in $\mathbb{M}^2_\kappa$ which have equal distance from a certain "*leading curve*" and one "*focal point*" (not on that curve) is used (for $\kappa=-1$) by LIEBMANN (see [Li],p.184-186), however the *uniform interpretation* of the "leading curves" *as curves of constant curvature* is lacking in LIEBMANN's book.

(iv) Proposition.

a) Let K denote the *ellipse* in $\mathbb{M}^2_\kappa$ (see (ii)a)) with focal length f, major semiaxis a (see (4.1_0)) and focal points e and $f^u(2f,\pi)$ (see (2.33_0), (2.64_0), (2.67)). Then K is the regular C^ω submanifold of $\mathbb{M}^2_\kappa$ defined by the equation (see (1.2), (2.33), (2.64_0), (2.64), (4.1_0))

$$(4.7) \qquad \cot_\kappa(r) = \frac{\sin_\kappa(2a) + \sin_\kappa(2f)\cdot\cos\varphi_u}{2\cdot\sin_\kappa(a+f)\cdot\sin_\kappa(a-f)}$$

(for $\kappa=-1$ see [Li],p.184) and the function r measuring the distance from e has the following values on K:

(4.7_1) $r(K) = [a-f, a+f]$ (see (2.33), (4.1_0)).

b) Let K denote the *hyperbola* in $\mathbb{M}^2_\kappa$ (see (ii)b)) with focal length f, major semiaxis a (see (4.2_0)) and focal points e and $f^u(2f,0)$, e being the closer one (see (2.33_0), (2.64_0), (2.67)). Then K is the regular C^ω submanifold of $\mathbb{M}^2_\kappa$ defined by the equation (see (1.2), (2.33), (2.64_0), (2.64), (4.2_0))

$$(4.8) \qquad \cot_\kappa(r) = \frac{\sin_\kappa(2a) + \sin_\kappa(2f)\cdot\cos\varphi_u}{2\cdot\sin_\kappa(f+a)\cdot\sin_\kappa(f-a)}$$

(for $\kappa=-1$ see [Li],p.184) and the function r measuring the distance from e attains on K the minimum (see (2.33), (4.2_0)):

(4.8_1) $\min(r|K) = f-a$.

c) Let K denote the *semihyperbola* in $\mathbb{M}^2_\kappa$ (see (ii)c)) with focal length f, major semiaxis a (see (4.2_0)), focal point e (see (2.33_0)) and focal line γ orthogonal to the geodesic $\exp_\kappa(\mathbb{R}\cdot u)$ through $\exp_\kappa(2f\cdot u)$ (see (2.64_0)). Then K is the regular C^ω submanifold of $\mathbb{M}^2_\kappa$ defined by the equation (see (1.2),(2.33), (2.64_0),(2.64),(4.3_0))

$$(4.9) \qquad \cot_\kappa(r) = \frac{\cos_\kappa(2a) + \cos_\kappa(2f)\cdot\cos\varphi_u}{\sin_\kappa(2f) - \sin_\kappa(2a)}$$

(for $\kappa=-1$ see [Li],p.185-186) and the function r measuring the distance from e attains on K the minimum (see (2.33),(4.2_0)):

(4.9_1) $\min(r|K) = f-a$.

d) Suppose $\kappa<0$ and let K denote the *horoellipse* in $\mathbb{M}^2_\kappa$ (see (ii)d)) with focal point e (see (2.33_0)), pericentral distance s ($\in\mathbb{R}_+$) and axis $\exp_\kappa(-xu)$ (see (2.64_0)). Then K is the regular C^ω submanifold of $\mathbb{M}^2_\kappa$ defined by the equation (see (1.2),(2.33),(2.64_0),(2.64))

$$(4.10) \qquad \cot_\kappa(r) = \frac{e^{\sqrt{-\kappa}\cdot s} + e^{-\sqrt{-\kappa}\cdot s}\cdot\cos\varphi_u}{2\cdot\sin_\kappa(s)}$$

(for $\kappa=-1$ see [Li],p.186) and the function r measuring the distance from **e** attains on K the minimum (see (2.33),(4.2_0)):

$$(4.10_1)\qquad \min(r|K) = s\ .$$

e) Suppose $\kappa<0$ and let K denote the *horohyperbola* in $\mathbb{M}^2_\kappa$ (see (ii)e)) with focal point **e** (see (2.33_0)), pericentral distance s ($\in\mathbb{R}_+$) and axis $\exp_\kappa(xu)$ (see (2.64_0)). Then K is the regular C^ω submanifold of $\mathbb{M}^2_\kappa$ defined by the equation (see (1.2),(2.33),(2.64_0),(2.64))

$$(4.11)\qquad \cot_\kappa(r) = \frac{e^{-\sqrt{-\kappa}\cdot s} + e^{\sqrt{-\kappa}\cdot s}\cdot\cos\varphi_u}{2\cdot\sin_\kappa(s)}$$

(for $\kappa=-1$ see [Li],p.186) and the function r measuring the distance from **e** attains on K the minimum (see (2.33),(4.2_0)):

$$(4.11_1)\qquad \min(r|K) = s\ .$$

Remark. The equations (4.7)-(4.11) evidently have a common functional structure, namely they all can be written as linear equations in $\cot_\kappa(r)$ and $\cos\varphi_u$:

$$(4.12)\qquad \cot_\kappa(r) = \alpha + \beta\cdot\cos\varphi_u \quad \textit{with} \quad \alpha\in\mathbb{R}_+ \quad \textit{and} \quad \beta\in[0,\infty[\ .$$

We will use this unified version of the defining equations below in (v) for our further investigations on the geometry of the subsets introduced in (ii).

Sketch of the proof. Using the "κ-geometric functions" introduced in § 1 it is possible, to formulate the trigonometry in $\mathbb{M}^2_\kappa$ uniformly for all values of κ. For example the "law of cosines" takes the form (see [Fe],p.55):

$$(4.13)\quad \begin{cases} \psi_\kappa(c) = \psi_\kappa(a)\cos_\kappa(b) + \psi_\kappa(b) - \sin_\kappa(a)\sin_\kappa(b)\cos(\gamma), \\ \textit{if } a,b,c\in]0,\pi_\kappa[\textit{ are the length's of the sides of a} \\ \textit{geodesic triangle in } \mathbb{M}^2_\kappa \textit{ and } \gamma \textit{ is the (inner) angle} \\ \textit{opposite to the side } c. \end{cases}$$

Ad a): Due to the definition (ii)a) and our assumption, K is the set of all solutions of the equation

$$(4.14_1)\qquad d_\kappa(\mathbf{e},..) + d_\kappa(f^u(2f,\pi),..) = 2a\ .$$

If we define (see (2.66),(2.16),(2.64_0))

(4.14_2) $\qquad \bar{\varphi} := \angle_o(u, U_e(..)) : \mathbb{M}^2_\kappa(e) \longrightarrow]-\pi, \pi]$,

then $\bar{\varphi}$ is not continuous on $\mathbb{M}^2_\kappa(e)$, nevertheless (see (2.66_2), (4.14_2)):

(4.14_3) $\qquad \cos\circ\bar{\varphi} = \cos\varphi_u$, *hence* $\cos(\pi-\bar{\varphi}) = -\cos\varphi_u$.

Using (4.13) and (2.33), (2.64_0), (2.67), (4.14_2) we get, that the solutions of (4.14_1) are exactly the ones of the equation

$$(4.14_4) \quad \begin{cases} \psi_\kappa(2a-r) = \psi_\kappa(2f)\cos_\kappa(r) + \psi_\kappa(r) - \\ \qquad\qquad - \sin_\kappa(2f)\sin_\kappa(r)\cos(\pi-\bar{\varphi}) \ . \end{cases}$$

Moreover, using (1.3), (1.5), (1.7), (1.6), we get

$$(4.14_5) \qquad \psi_\kappa(2a-r) = 2\sin^2_\kappa(a)\cos_\kappa(r) - \sin_\kappa(2a)\sin_\kappa(r) + \psi_\kappa(r)$$

and

$$(4.14_6) \qquad \sin^2_\kappa(a) - \sin^2_\kappa(f) = \sin_\kappa(a+f)\cdot\sin_\kappa(a-f) \ .$$

Because of (4.1_0) and (1.3), (4.14_3), (4.14_5), (4.14_6) the solutions of (4.14_4) are exactly the ones of (4.7).

Finally (1.2′), (1.9) imply, that

(4.14_7) $\qquad \cot_\kappa$ *is strictly monotonic decreasing on* $]0, \pi_\kappa[$.

We get from (4.7) and (2.16), (4.14_2), (4.14_3), (4.14_7), that $r|K$ can attain extremal values only at the points of $K \cap \exp_\kappa(\mathbb{R}\cdot u)$, and this intersection can be computed explicitely from (4.14_1) and (2.67), which leads to (4.7_1).

Ad b): Similar to a).

Ad c): Due to the definition (ii)c) and our assumption, K is the set of all solutions of the equation

(4.15_1) $\qquad \delta_\gamma(..) - d_\kappa(e, ..) = 2a$,

where γ is a geodesic in $\mathbb{M}^2_\kappa$ orthogonal to $\exp_\kappa(\mathbb{R}\cdot u)$ and $\delta_\gamma(e) = 2f$. Without loss of generality we assume $\vec{u} = \mathbf{e}_1$ and $\gamma(0) = \exp_\kappa(2f\cdot u)$. Then the isometry $\mathfrak{f}$ from the definition (ii)c) is in the sense of (2.20), (2.27) induced by the matrix

$$\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\cdot\begin{pmatrix}\cos_\kappa(-2f)&-\kappa\sin_\kappa(-2f)&0\\\sin_\kappa(-2f)&\cos_\kappa(-2f)&0\\0&0&1\end{pmatrix}=\begin{pmatrix}\cos_\kappa(2f)&\kappa\sin_\kappa(2f)&0\\0&0&1\\\sin_\kappa(2f)&-\cos_\kappa(2f)&0\end{pmatrix},$$

hence according to (2.76) and the definition of δ_γ:

$$(4.15_2)\qquad \sin_\kappa\circ\delta_\gamma = x_2\circ\bar{f} = [\sin_\kappa(2f)\cdot x_0 - \cos_\kappa(2f)\cdot x_1]\,|\,\mathbb{M}^2_\kappa\ .$$

From (2.33),(2.64),(2.16),(2.5),(2.3),(2.1)) and our special choice of u we get

$$(4.15_3)\quad \begin{cases}\cos_\kappa(r) = x_0\,|\,\mathbb{M}^2_\kappa\ ,\quad \sin_\kappa(r) = \sqrt{x_1^2+x_2^2}\,|\,\mathbb{M}^2_\kappa\ ,\\ \cos\varphi_u = (x_1/\sqrt{x_1^2+x_2^2})\,|\,\mathbb{M}^2_\kappa(e)\ .\end{cases}$$

Now we can prove, that the solutions of (4.15_1) are exactly the ones of the equation

$$(4.15_4)\quad \sin_\kappa(r+2a) = \sin_\kappa(2f)\cos_\kappa(r) - \cos_\kappa(2f)\cos(\tilde{\varphi})\sin_\kappa(r).$$

In case $\kappa\le 0$ this follows from (2.33),(4.15_2),(4.15_3) and the injectivity of $\sin_\kappa$ (see (1.1)). The case $\kappa>0$ requires some additional arguments (but we will not use it in this article). Because of (1.5),(4.3_0) the solutions of (4.15_4) are exactly the ones of (4.9). (4.9_1) is proved analogously to (4.7_1).

Ad d),e): If we introduce for brevity

$$(4.16_1)\qquad \sigma := \begin{cases}-1 & \textit{if K is a horoellipse}\ ,\\ +1 & \textit{if K is a horohyperbola}\ ,\end{cases}$$

and if we define

$$(4.16_2)\qquad \gamma(x) := \exp_\kappa(xu)\ :\ \mathbb{R}\longrightarrow\mathbb{M}^2_\kappa\ ,$$

we can prove the assertions d),e) simultaneously, namely: Due to the definition (ii)d),e) and (2.33) and our assumption, K is the set of all solutions of the equation

$$(4.16_3)\qquad r - \sigma\cdot\beta_{\gamma(\sigma x)} = 2s\ ,$$

where according to (4.16_2),(4.16_1),(2.64_0),(2.47),(2.47_4), (2.33),(2.64) we have

$$(4.16_4)\qquad e^{(\sqrt{-\kappa}\cdot\beta_{\gamma(\sigma x)})} = \cos_\kappa(r) - \sqrt{-\kappa}\cdot\sin_\kappa(r)\cdot\cos\varphi_u$$

and according to (2.40) and $\kappa<0$

$$(4.16_5)\qquad e^{\sigma\cdot\sqrt{-\kappa}\cdot(r-2s)} = e^{-2\sigma\cdot\sqrt{-\kappa}\cdot s}\cdot[\cos_\kappa(r) + \sqrt{-\kappa}\cdot\sin_\kappa(r)]\ .$$

Because of (2.40), (4.16_1), (4.16_4), (4.16_5) and $\kappa<0$ the solutions of (4.16_3) are exactly the ones of (4.10) resp. (4.11). (4.10_1) and (4.11_1) are proved analogously to (4.7_1), using (2.50).

(v) Proposition. Suppose $\alpha\in\mathbb{R}_+$ and $\beta\in[0,\infty[$ and consider the level set (see (1.2), (2.33_0), (2.33), (2.64_0), (2.64))

$$(4.12')\qquad K := [\cot_\kappa(r) - (\alpha+\beta\cdot\cos\varphi_u)]^{-1}(\{0\}) \subseteq \mathbb{M}^2_\kappa(e)\ .$$

Then the following is true:

(4.17_1) $\left\{\begin{array}{l}\textit{K is a 1-dim. regular } C^\omega \textit{ submanifold of } \mathbb{M}^2_\kappa,\\ \textit{connected and closed in } \mathbb{M}^2_\kappa.\end{array}\right.$

(4.17_2) *If K is the ellipse as in (iv)a), then K is compact.*

(4.17) *K is a conic (see (i)), more precisely it is isometric to one connected component of the intersection of $\mathbb{M}^2_\kappa$ with the quadric cone in $\mathbb{R}^3$ (see § 2.i, (4.12')):* $\alpha^2\cdot(x_1^2+x_2^2) = (x_0-\beta x_1)^2$.

(4.18) *K is a 1-dim. integral manifold of the Pfaffian form* $dr - \mathcal{F}\cdot\theta$ *with* $\mathcal{F} := \beta\cdot\sin^2_\kappa(r)\cdot\sin\varphi_u$

(see (2.33), (2.62_0), (2.62), (2.64), (4.12')). For the last function we find (see (4.18), (4.12')):

$$(4.19)\qquad \mathcal{F}^2|K = \mathcal{G}\circ r|K \quad \textit{with}\quad \mathcal{G} := \sin^4_\kappa\cdot[\beta^2-(\cot_\kappa-\alpha)^2]\ .$$

For further applications we specify the constants α,β in the latter function $\mathcal{G}$ for the different metric types of conics, which we had introduced in (ii):

If K is the *ellipse* as in (iv)a), then the function $\mathcal{G}$ of (4.19) can be written as

$$(4.19_1)\qquad \left\{\begin{aligned}\mathcal{G} = \sin^2_\kappa\cdot\Bigg[\Bigg(&\frac{-\cos_\kappa(2a)}{\sin_\kappa(a+f)\cdot\sin_\kappa(a-f)} +\\ &+\frac{\sin_\kappa(2a)}{\sin_\kappa(a+f)\cdot\sin_\kappa(a-f)}\cdot\cot_\kappa\Bigg)\cdot\sin^2_\kappa - 1\Bigg]\ .\end{aligned}\right.$$

If K is the *hyperbola* as in (iv)b), then the function $\mathfrak{g}$ of (4.19) can be written as

$$(4.19_2)\quad \begin{cases} \mathfrak{g} = \sin_\kappa^2 \cdot \Bigg[\Bigg(\dfrac{\cos_\kappa(2a)}{\sin_\kappa(f+a)\cdot\sin_\kappa(f-a)} + \\ \qquad + \dfrac{\sin_\kappa(2a)}{\sin_\kappa(f+a)\cdot\sin_\kappa(f-a)}\cdot\cot_\kappa\Bigg)\cdot\sin_\kappa^2 - 1\Bigg] . \end{cases}$$

If K is the *semihyperbola* as in (iv)c), then the function $\mathfrak{g}$ of (4.19) can be written as

$$(4.19_3)\quad \begin{cases} \mathfrak{g} = \sin_\kappa^2 \cdot \Bigg[\Bigg(\dfrac{-\,2\kappa\cdot\sin_\kappa(2a)}{\sin_\kappa(2f) - \sin_\kappa(2a)} + \\ \qquad + \dfrac{2\cdot\cos_\kappa(2a)}{\sin_\kappa(2f) - \sin_\kappa(2a)}\cdot\cot_\kappa\Bigg)\cdot\sin_\kappa^2 - 1\Bigg] . \end{cases}$$

If K is the *horoellipse* as in (iv)d), then the function $\mathfrak{g}$ of (4.19) can be written as

$$(4.19_4)\qquad \mathfrak{g} = \sin_\kappa^2 \cdot \Bigg[\Bigg(\frac{-\sqrt{-\kappa}\cdot e^{\sqrt{-\kappa}\cdot s}}{\sin_\kappa(s)} + \frac{e^{\sqrt{-\kappa}\cdot s}}{\sin_\kappa(s)}\cdot\cot_\kappa\Bigg)\cdot\sin_\kappa^2 - 1\Bigg] .$$

If K is the *horohyperbola* as in (iv)e), then the function $\mathfrak{g}$ of (4.19) can be written as

$$(4.19_5)\qquad \mathfrak{g} = \sin_\kappa^2 \cdot \Bigg[\Bigg(\frac{\sqrt{-\kappa}\cdot e^{-\sqrt{-\kappa}\cdot s}}{\sin_\kappa(s)} + \frac{e^{-\sqrt{-\kappa}\cdot s}}{\sin_\kappa(s)}\cdot\cot_\kappa\Bigg)\cdot\sin_\kappa^2 - 1\Bigg] .$$

We go back now to the general situation of (4.12'):
Suppose $c: I \longrightarrow \mathbb{M}_\kappa^2$ is a path with the following 4 properties:

(4.20_1) $\quad c(I) \subseteq \mathbb{M}_\kappa^2(e)$, *c is* C^2 *and* $\theta(\dot{c})^2 > 0$

(see (2.33_0),(2.62_0)),

(4.20) $\quad ((r\circ c)')^2 = \mathfrak{g}(r\circ c)\cdot\theta(\dot{c})^2$

(see (2.33),(2.62_0),(4.19)),

(4.20_2) $\quad$ *there exists* $\tau\in I$ *with* $c(\tau) \in K$ *and* $\dot{c}(\tau) \in T_{c(\tau)}K$

(see (4.12'),(4.17_1)) and

$$(4.20_3)\quad \begin{cases} \textit{either } [\beta=0 \textit{ and } r\circ c \textit{ constant}] \textit{ or} \\ \textit{or } [\beta\neq 0 \textit{ and } r\circ c \textit{ non-constant}] . \end{cases}$$

Then it follows from (4.20_1),(4.20),(4.20_2),(4.20_3), that

(4.21) $c(I) \subseteq K$,

in particular c is a C^2 map into the submanifold K of $\mathbb{M}^2_\kappa$.
Oppositely every C^1 path $c: I \longrightarrow \mathbb{M}^2_\kappa$ with $c(I) \subseteq K$ is a solution of the differential equation (4.20).
With respect to the condition (4.20_2) we make the remark:

$$(4.20_4) \begin{cases} \text{If } \tau\in I \text{ fulfills } c(\tau) \in K \cap \exp_\kappa(\mathbb{R}\cdot u) , \\ \text{then (4.20) implies } \dot{c}(\tau) \in T_{c(\tau)}K . \end{cases}$$

Remark. In [Li],p.192-193 was proved the special case $\kappa=-1$ of the following result (valid for all $\kappa\in\mathbb{R}$, see $[\mathrm{Zi}_1]$,p.163):

The *ellipse* of (iv)a) is isometric to one connected component of the intersection of $\mathbb{M}^2_\kappa$ with the quadric cone in $\mathbb{R}^3$ defined by

$$(4.17_3) \qquad \frac{x_1^2}{\tan_\kappa^2(a)} + \frac{x_2^2}{\tan_\kappa^2(b)} = x_0^2$$

(see § 2.i, (4.12′) and let b denote the minor semiaxis of K, defined by $\sin_\kappa^2(a) = \sin_\kappa^2(b) + \sin_\kappa^2(f) - \kappa\sin_\kappa^2(b)\sin_\kappa^2(f)$. and $b\in]0,a]$).

The *hyperbola* of (iv)b) is isometric to one connected component of the intersection of $\mathbb{M}^2_\kappa$ with the quadric cone in $\mathbb{R}^3$ defined by

$$(4.17_4) \qquad \frac{x_1^2}{\tan_\kappa^2(a)} - \frac{x_2^2}{\sin_\kappa^2(b)} = x_0^2$$

(see § 2.i, (4.12′) and let b denote the minor semiaxis of K, defined by $\sin_\kappa^2(f) = \sin_\kappa^2(a) + \sin_\kappa^2(b) - \kappa\sin_\kappa^2(a)\sin_\kappa^2(b)$ and $b\in]0,f[$), and moreover the other connected component of this intersection is the image of the second branch of the classical hyperbola (4.2_1) under the same isometry.

Sketch of the proof. From (1.2′),(2.65) we get

$$d[\cot_\kappa(r)-(\alpha+\beta\cdot\cos\varphi_u)] = -\sin_\kappa^{-2}(r)\cdot(dr-\mathcal{F}\cdot\theta)$$

with $\mathcal{F}$ as in (4.18). This implies (4.18) and (because of (2.62),(4.12′)) the property of K being a 1-dim. regular C^ω submanifold of $\mathbb{M}^2_\kappa$. Using (2.64_1) and (4.12′) we conclude (4.19) and by (4.19),(1.2),(1.6) we get

$$\mathcal{G} = \sin_\kappa^2 \cdot [[(\beta^2-\alpha^2+\kappa) + 2\alpha\cdot\cot_\kappa]\cdot\sin_\kappa^2 - 1] .$$

Now we compare (4.7)-(4.11) with (4.12') and use the identities from § 1 to get (4.19_1)-(4.19_5).

Because of (1.2'),(1.9),(4.14_7) we have the

$$(4.22_1) \quad \begin{cases} \textit{inverse } C^\omega \textit{ function } \operatorname{arcot}_\kappa := (\cot_\kappa |]0,\pi_\kappa[)^{-1} , \\ \textit{defined on } \begin{cases}]\sqrt{-\kappa},\infty[& \textit{if } \kappa\le 0, \\ \mathbb{R} & \textit{if } \kappa>0, \end{cases} \end{cases}$$

and we consider the C^ω path $\tilde{c}:\tilde{I} \longrightarrow \mathbb{M}_\kappa^2$ (see (2.67))

$$(4.22) \qquad \tilde{c}(x) := f^u(\operatorname{arcot}_\kappa(\alpha+\beta\cdot\cos(x)),x)|\tilde{I} , \quad \textit{where}$$

$$(4.22_2) \quad \begin{cases} \tilde{I} \textit{ is the connected component of zero in the maximal} \\ \textit{domain of definition of } f^u(\operatorname{arcot}_\kappa(\alpha+\beta\cdot\cos(x)),x) . \end{cases}$$

Now we check, that $\{t\in[-\pi,\pi] | \alpha+\beta\cdot\cos(t)\in\cot_\kappa(]0,\pi_\kappa[)\}$ is an interval which is contained in $\tilde{I}$, and we use this and (4.12'), (4.22),(2.67_2) to prove

$$(4.22_3) \qquad \tilde{c}(\tilde{I}) = K , \qquad \textit{hence } K \textit{ is connected.}$$

According to § 2.iv isometries of $\mathbb{M}_\kappa^2$ are restrictions of certain linear mappings of $\mathbb{R}^3$ and therefore they map conics of $\mathbb{M}_\kappa^2$ again onto such conics. Hence we may assume for the proof of (4.17) without loss of generality $\vec{u} = e_1$. Then we get from (4.15_3) and (1.2),(4.12'), that K is contained in the quadric cone given by the equation $\alpha^2\cdot(x_1^2+x_2^2) = (x_0-\beta x_1)^2$. Then we prove, that $(x_1^2+x_2^2)|K > 0$ and conclude (together with (4.22_3)), that K is a connected component of the intersection of this quadric cone with $\mathbb{M}_\kappa^2$.

From (4.12'),(4.18),(4.19) we get immediately the validity of (4.20_4) and that any C^1 path c with $c(I)\subseteq K$ satisfies the ODE (4.20). We now return to (4.22) and prove

(4.22_4) $\tilde{c}$ *is the universal covering of the 1-dim. manifold* K (see (2.67), (2.67_2), (4.12'), (4.22), (4.22_1)-(4.22_3)) and moreover, that

$$(4.22_5) \qquad \theta(\tilde{c}^{\,\cdot}) = 1 \quad \textit{and} \quad r\circ\tilde{c} \textit{ satisfies } (4.20_3).$$

Let $c:I \longrightarrow \mathbb{M}^2_\kappa$ be a path with the properties (4.20_1), (4.20), (4.20_2), (4.20_3).

If $\beta=0$, then K is a distance sphere with center e (see (2.33),(4.12′)). Hence (4.20_2) and $r\circ c$ constant imply $c(I)\subseteq K$.

Suppose now $\beta\neq 0$, choose $\tau\in I$ as in (4.20_2) and consider

(4.22_6) $\quad \varphi_0 := \angle_o(u,U_e(c(\tau))) \in \mathbb{I}$ *and*

(4.22_7) $\quad \varphi_c(x) := \int_\tau^x \theta(\dot{c}(t))dt : I \longrightarrow \mathbb{R}$

(see (2.64_0),(2.66),(2.67_0),(2.67), (2.67_3), (2.67_5), (4.20_2), (4.22_2),(4.22_3)). Then we conclude from (4.20_1), that φ_c is a C^2 diffeomorphism from I onto the interval $\varphi_c(I)$ of $\mathbb{R}$, and we prove (using (4.12′),(4.18), (4.19), (4.20), (4.20_3), (4.22_1), (4.22),(4.22_2)-(4.22_7)):

$r\circ\tilde{c}\circ(x+\varphi_0)$ and $r\circ c\circ\varphi_c^{-1}$ both are non-constant C^2 solutions of the ODE initial value problem

(4.22_8) $\quad (y')^2 = \mathfrak{F}(y)$ *with* $y(0)=(r\circ\tilde{c})(\varphi_0)$, $y'(0)=(r\circ\tilde{c})'(\varphi_0)$,

where $\mathfrak{F}$ is as in (4.19), and moreover $r\circ\tilde{c}\circ(x+\varphi_0)$ is a maximal solution of this problem.

Hence $r\circ c = r\circ\tilde{c}\circ(\varphi_c+\varphi_0)$ by the uniqueness result § 3.iv. Using (2.67_0),(2.67),(2.67_1),(2.67_3),(2.67_5), (4.22), (4.22_6), (4.22_7) we conclude

(4.22_9) $\quad c = \tilde{c}\circ(\varphi_c+\varphi_0)$,

hence $c(I)\subseteq K$ according to (4.22_3).

5. Motion of a point-like particle with p-central acceleration, KEPLER's second law and NEWTON's potential in the space $\mathbb{M}^n_\kappa$ of constant curvature $\kappa\in\mathbb{R}$ $(n\geq 2)$

(i) Data and notations for § 5: As in § 2 and $n\in\mathbb{N}$, $n\geq 2$.

(ii) Motion of a point-like particle in $\mathbb{M}^n_\kappa$ with p-central acceleration.

Suppose

(5.0) *we have fixed a point* p *in* $\mathbb{M}^n_\kappa$

(e.g. the position of the sun).

a) Let $r_p := d_\kappa(p,..):\mathbb{M}^n_\kappa \longrightarrow \mathbb{R}$ denote the continuous function (C^ω on $\mathbb{M}^n_\kappa(p):=\mathbb{M}^n_\kappa\backslash\{p,-p\}$) measuring the distance of points of $\mathbb{M}^n_\kappa$ from p in the intrinsic metric $d_\kappa:\mathbb{M}^n_\kappa\times\mathbb{M}^n_\kappa \longrightarrow \mathbb{R}$ of the riemannian manifold $\mathbb{M}^n_\kappa$ of constant curvature κ ($\in\mathbb{R}$) (see (2.15),(2.16)). Then according to (2.15),(2.12) for every point $q\in\mathbb{M}^n_\kappa(p)$ there exists a unique unit speed geodesic $\gamma_q:\mathbb{R} \longrightarrow \mathbb{M}^n_\kappa$ with $\gamma_q(0) = p$ and $\gamma_q(r_p(q)) = q$ (the "light ray" from the sun to q), and we call

(5.1) $R_p(q) := \dot\gamma_q(r_p(q)) \in T^1_q\mathbb{M}^n_\kappa$ *the* p*-radial direction at* q,

and – as for (2.34) – one proves

(5.2) $R_p = \operatorname{grad}(r_p)$ *is a* C^ω *unit vector field on* $\mathbb{M}^n_\kappa(p)$.

b) Let be given $s\in\mathbb{N}\cup\{\infty,\omega\}$, $s\geq 2$, and let (see (5.0),(2.16))

(5.3) I *be an open interval of* $\mathbb{R}$ *and* $c:I \longrightarrow \mathbb{M}^n_\kappa(p)$ *a* C^s *path*

describing the motion of a point-like particle in $\mathbb{M}^n_\kappa(p)$.

Definition (see (2.10)).

$$(5.4_0)\quad \begin{cases} c:I \longrightarrow \mathbb{M}^n_\kappa(p) \quad \textit{has p-central acceleration} \\ \textit{iff} \quad \nabla_\partial \dot c \in \mathbb{R}\cdot(R_p\circ c) \end{cases}$$

[i.e. for every time $t\in I$ the acceleration vector $(\nabla_\partial \dot c)(t)$ of c is proportional to the p-central direction $R_p(c(t))$ at the position of the particle at this time t], and which evidently is equivalent to saying (choose $\alpha:=g(\nabla_\partial \dot c, R\circ c)$):

$$(5.4_p)\quad \begin{cases} c:I \longrightarrow \mathbb{M}^n_\kappa(p) \quad \textit{has p-central acceleration} \\ \textit{iff there exists a } C^{s-2} \textit{ function } \alpha:I \longrightarrow \mathbb{R} \textit{ with} \\ (\nabla_\partial \dot c)(t) = \alpha(t)\cdot R_p(c(t)) \quad \textit{for all } t\in I, \textit{ i.e.} \\ c \textit{ is a } C^2 \textit{ solution of the ODE } \nabla_\partial \dot c - \alpha\cdot(R_p\circ c) = o \,. \end{cases}$$

[In the terminology of dynamics this means (the particle having unit mass), that *the acceleration of* c *is induced by the time-dependent* p*-central force field* $\alpha\cdot R_p:I\times\mathbb{M}^n_\kappa(p) \longrightarrow T\mathbb{M}^n_\kappa$ $((t,q) \mapsto (\alpha(t),R_p(q))$.]

c) **Remark.** If $f\in G^n_\kappa$ is an orientation preserving isometry of $\mathbb{M}^n_\kappa$ such that $f(p)=e$ (see (2.0)), then evidently $c:I \longrightarrow \mathbb{M}^n_\kappa$ is a C^s path in $\mathbb{M}^n_\kappa(p)$ iff $f\circ c:I \longrightarrow \mathbb{M}^n_\kappa$ is a C^s path in $\mathbb{M}^n_\kappa(e)$ and c *has* p*-central acceleration iff* $f\circ c$ *has* e*-central acceleration*. [Because one checks (using $f(p)=e$ and f distance preserving), that $r_p = r_e \circ f$, therefore (see (5.2)) $f_* R_p = R_e \circ f$, finally $\nabla_\partial (f \circ c)^\cdot = f_* \nabla_\partial \dot{c}$.]

Since G^n_κ acts transitively on $\mathbb{M}^n_\kappa$ (cf. (2.27)) one sees, that studying the motion of a point-like particle with p-*central* acceleration is – up to an isometry – the same as studying the motion of a point-like particle with e-*central* acceleration. Therefore from now on we will consider only motions of the latter type, in which case we have with the notations of (2.33),(2.34): $r_e = r$ and $R_e = R$, and (5.4_p) is reduced (for $s \in \mathbb{N} \cup \{\infty, \omega\}$, $s \geq 2$) to:

For every C^s *path* $c: I \longrightarrow \mathbb{M}^n_\kappa(e) := \mathbb{M}^n_\kappa \setminus \{e, -e\}$ *one defines:*

$$(5.4) \quad \begin{cases} c \textit{ has e-central acceleration} & :\Leftrightarrow \quad \nabla_\partial \dot{c} \in \mathbb{R} \cdot (R \circ c) \\ & \Leftrightarrow \quad \textit{There exists a } C^{s-2} \textit{ function } \alpha: I \longrightarrow \mathbb{R} \\ & \qquad \textit{with } \nabla_\partial \dot{c} = \alpha \cdot (R \circ c) \ . \end{cases}$$

d) Since $i: \mathbb{M}^n_\kappa(e) \hookrightarrow \mathbb{R} \times (\mathbb{R}^n \setminus \{o\})$ (see (2.5),(2.16)) the ODE of (5.4) admits the following extrinsic version: For the

(5.5_0) C^s *path* $(y_0, y) := i \circ c : I \longrightarrow \mathbb{R} \times (\mathbb{R}^n \setminus \{o\})$ $(t \mapsto (y_0(t), y(t)))$

the ODE (5.4) is equivalent to:

$$(5.5) \quad \begin{cases} y_0'' = ((y_0')^2 + \kappa \langle y', y' \rangle) \cdot y_0 - \kappa \cdot \alpha \cdot \sqrt{\langle y, y \rangle} \ , \\ y'' = ((y_0')^2 + \kappa \langle y', y' \rangle) \cdot y + \alpha \cdot y_0 \cdot \langle y, y \rangle^{-1/2} \cdot y \ , \end{cases}$$

which is (even $\alpha: I \longrightarrow \mathbb{R}$ might be only C^0 !) *locally Lipschitzian* in $(y_0, y, y_0', y') \in \mathbb{R}^{2(n+1)}$. Therefore we get from the well-known uniqueness theorem:

If $\alpha: I \longrightarrow \mathbb{R}$ is a C^0 function, $t_0 \in I$, G a neighborhood of t_0 in I and $c, \tilde{c}: G \longrightarrow \mathbb{M}^n_\kappa(e)$ are two C^2 path's satisfying

$$(5.6) \quad \begin{cases} \nabla_\partial \dot{c} = \alpha \cdot (R \circ c) \ , \quad \nabla_\partial \dot{\tilde{c}} = \alpha \cdot (R \circ \tilde{c}) \quad \textit{and} \\ c(t_0) = \tilde{c}(t_0) \ , \quad \dot{c}(t_0) = \dot{\tilde{c}}(t_0) \ , \\ \textit{then } c \textit{ and } \tilde{c} \textit{ coincide on a neighborhood of } t_0 \textit{ in } I. \end{cases}$$

(iii) Theorem (*Motions of point-like particles in* $\mathbb{M}^n_\kappa$ $(n \geq 2)$, *which have e-central acceleration, are "plane".*).

Suppose $n \geq 2$, I is an open interval of $\mathbb{R}$, $0 \in I$ and the C^2 path $c: I \longrightarrow \mathbb{M}^n_\kappa(e)$ (see (2.16)) describes the motion of a point-like particle, c having e-central acceleration (see (5.4)). Then there exists a C^2 path $c_0:I \longrightarrow \mathbb{M}^2_\kappa(e_0)$ ($e_0:=(1,0,0)$) which has e_0-central acceleration and a distance preserving isometric immersion $f:(\mathbb{M}^2_\kappa,e_0) \longrightarrow (\mathbb{M}^n_\kappa,e)$ onto a 2-dim. totally geodesic submanifold P of $\mathbb{M}^n_\kappa$, such that $c = f\circ c_0$, in particular c(I) is contained in the 2-dim. "plane" P of $\mathbb{M}^n_\kappa$.

Remark. This theorem justifies, to study motions of point-like particles with e-central acceleration in $\mathbb{M}^n_\kappa$ with $n\geq 2$ only in the *2-dim. case of* $\mathbb{M}^2_\kappa$.

Proof. Denote by

$$(5.7)\quad \begin{cases} V \textit{ a 3-dim. vector subspace of } \mathbb{R}^{n+1} \textit{ containing} \\ e,\ c(0) \textit{ and } c'(0)\ (:=i_*\dot{c}(0)^{\rightarrow}, \text{ see (2.11)}). \end{cases}$$

Then (see (2.17)) one has:

$$(5.8)\quad \begin{cases} P := \mathbb{M}^n_\kappa\cap V \textit{ is a 2-dim. totally geodesic submanifold} \\ \textit{with inclusion } j:P \hookrightarrow \mathbb{M}^n_\kappa \textit{ and which is } C^\omega \textit{ isometric} \\ \textit{to } \mathbb{M}^2_\kappa. \end{cases}$$

We want to show first

$$(5.9)\qquad c(I) \subseteq P ,$$

and we introduce for that purpose the following subset H of I:

$$(5.10)\qquad H := \{t\in I\,|\,c(t)\in P \textit{ and } \dot{c}(t)\in j_*T_{c(t)}P\} .$$

By the choice of P (see (5.7),(5.8)) it follows $0\in H$ and by its definition (5.10) and continuity of $c,\dot{c}$ it follows that H is closed in I. For proving (5.9) it suffices therefore to show:

$$(5.11)\qquad H \textit{ is open in } I.$$

Ad (5.11): By hypothesis and (5.4) there exists a

$$(5.12)\qquad C^0 \textit{ function } \alpha:I \longrightarrow \mathbb{R} \textit{ such that } \nabla_\partial\dot{c} = \alpha\cdot(R\circ c) .$$

Choose next (see (5.8) and (5.7) for "$e\in P$") a

$$(5.13)\qquad C^\omega \textit{ isometry } h:(\mathbb{M}^2_\kappa,e_0) \longrightarrow (P,e) \textit{ with } e_0:=(1,0,0)\in\mathbb{R}^3.$$

Now, since $h:\mathbb{M}^2_\kappa \longrightarrow P$ is an isometry, since P is totally geodesic in $\mathbb{M}^n_\kappa$ and because of (2.15),(2.14$_1$) it follows that

$$(5.14)\quad \begin{cases} f := j\circ h:(\mathbb{M}^2_\kappa,e_0) \longrightarrow (\mathbb{M}^n_\kappa,e) \textit{ is a distance preserving} \\ \qquad\qquad\qquad\qquad \textit{isometric } C^\omega \textit{ immersion,} \end{cases}$$

in particular $r_0 := d_\kappa(e_0,..) = d_\kappa(e,f(..)) = r\circ f$, therefore if (see (2.34))

$$(5.15)\qquad R_0:=\text{grad}(r_0) \textit{ resp. } R:=\text{grad}(r), \textit{ then } f_*R_0 = R\circ f .$$

Suppose now $t_0 \in H$. Then there exists by the existence theorem for ODE's (cf. (5.5)) a neighborhood G of t_0 in I and a

$$(5.16)\quad \begin{cases} C^2 \textit{ path } c_0:G \longrightarrow \mathbb{M}^2_\kappa(e_0) \textit{ with } \nabla_\partial \dot{c}_0 = \alpha\cdot(R_0\circ c_0) \textit{ and} \\ f(c_0(t_0)) = c(t_0)\ ,\quad f_*\dot{c}_0(t_0) = \dot{c}(t_0) \end{cases}$$

($c(t_0)\in P$ and $\dot{c}(t_0)\in j_*T_{c(t_0)}P$ by (5.10)!). Therefore (see (5.14) and use, that h is isometric, j totally geodesic):

$$(5.17)\quad \begin{cases} \nabla_\partial(f\circ c_0)^\cdot \underset{(5.14)}{=} \nabla_\partial(j\circ h\circ c_0)^\cdot = \nabla_\partial j_*h_*\dot{c}_0 = j_*\nabla_\partial h_*\dot{c}_0 = \\ = j_*h_*\nabla_\partial\dot{c}_0 = f_*\nabla_\partial\dot{c}_0 \underset{(5.16)}{=} \alpha\cdot(f_*R_0\circ c_0) \underset{(5.15)}{=} \alpha\cdot(R\circ f\circ c_0) \\ \textit{and } f(c_0(t_0)) = c(t_0)\ ,\quad f_*\dot{c}_0(t_0) = \dot{c}(t_0) \end{cases}$$

From (5.12),(5.17) and (5.6) follows but: c and $f\circ c_0$ coincide on a neighborhood of t_0 in I, i.e. without loss of generality (G so small that)

$c|G = f\circ c_0 \underset{(5.14)}{=} j\circ h\circ c_0$, in particular $c(G)\subseteq h(c_0(G))\subseteq P$ and

for all $t\in G$: $\dot{c}(t) = j_*(h_*\dot{c}_0(t)) \in j_*T_{h\circ c_0(t)}P = j_*T_{c(t)}P$.

This proves $G\subseteq H$ (see (5.10)) and therefore (5.11).

Therefore (5.9) is true, and since P is by (5.8) a regular submanifold of $\mathbb{M}^n_\kappa$ it follows from (5.9) that there exists a

$$(5.18)\quad C^2 \textit{ path } c_1:I \longrightarrow P \textit{ with } c = j\circ c_1\ ,$$

and if we define (see (5.13),(5.18)) the C^2 path

$$(5.19)\quad c_0 := h^{-1}\circ c_1:I \longrightarrow \mathbb{M}^2_\kappa\ ,\quad \textit{then one has}\quad c = f\circ c_0$$

(see (5.14),(5.18)) and one proves (using (5.19),(5.15), (5.12)) analogously to (5.17) $f_*(\nabla_\partial\dot{c}_0) = f_*(\alpha\cdot(R_0\circ c_0))$, from where one concludes by the immersion property of f (see (5.14)) that $\nabla_\partial\dot{c}_0 = \alpha\cdot(R_0\circ c_0)$, i.e. c_0 has e_0-central acceleration. This, together with (5.14),(5.19) finishes the proof of the theorem.

(iv) The law of acceleration for motions of point-like particles in $\mathbb{M}^2_\kappa$ expressed in polar coordinates.

Applying the orthonormal frame field (R,JR) on $\mathbb{M}^2_\kappa(e)$ of § 2.vii we are going to give now a (more explicit, but still intrinsic) transcription of the ODE of (5.4), which will be useful for further discussions of the properties of the solutions of (5.4). First we have for any C^s path $c:I \longrightarrow \mathbb{M}^2_\kappa(e)$ with $s \in \mathbb{N}\cup\{\infty,\omega\}$ and $s\geq 2$ (see (2.61)):

$$\dot{c} = g(R\circ c,\dot{c})\cdot(R\circ c) + g(JR\circ c,\dot{c})\cdot(JR\circ c) \ ,$$

therefore (see (2.34),(2.60),(2.62$_0$))

$$(5.20_0) \qquad \dot{c} = (r\circ c)'\cdot(R\circ c) + \sin_\kappa(r\circ c)\cdot\theta(\dot{c})\cdot(JR\circ c) \ ,$$

from where we get by covariant differentiation (using the ∇-parallelity of J (see (2.59)) and (2.62) for computing $\nabla_{\dot{c}}R$):

$$(5.20) \left\{ \begin{array}{l} \nabla_\partial \dot{c} = [(r\circ c)'' - \sin_\kappa(r\circ c)\cos_\kappa(r\circ c)(\theta(\dot{c}))^2]\cdot(R\circ c) + \\ \quad + [\sin_\kappa(r\circ c)(\theta(\dot{c}))' + 2(r\circ c)'\cos_\kappa(r\circ c)\theta(\dot{c})]\cdot(JR\circ c). \end{array} \right.$$

Since $\sin_\kappa(r)$ is strictly positive on $\mathbb{M}^2_\kappa(e)$ (see (1.9), (2.14$_1$),(2.16)), the ODE of (5.4), which characterizes c as a path having e-central acceleration, becomes via (5.20) equivalent to the following ODE system:

$$(5.21) \left\{ \begin{array}{lr} (5.21_1) & (r\circ c)'' - \sin_\kappa(r\circ c)\cdot\cos_\kappa(r\circ c)\cdot(\theta(\dot{c}))^2 = \alpha \ , \\ (5.21_2) & (\theta(\dot{c}))' + 2\cdot(r\circ c)'\cdot\cot_\kappa(r\circ c)\cdot\theta(\dot{c}) = 0 \ . \end{array} \right.$$

Here the C^s function $r\circ c:I \longrightarrow \,]0,\pi_\kappa[$ *measures the distance of the moving particle from the center* **e**, whereas (due to (2.70))

(5.22) $\quad \theta(\dot{c}):I \longrightarrow \mathbb{R}$ *is the* (C^{s-1}) *angular velocity of* $U_e\circ c$,

i.e. of the *direction map* $U_e\circ c:I \longrightarrow T^1_e\mathbb{M}^2_\kappa$, which assigns to each $t\in I$ the initial vector $U_e\circ c(t) := \dot{\gamma}_t(0) \in T^1_e\mathbb{M}^2_\kappa$ of the unique unit speed geodesic $\gamma_t:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa$ (the “sunbeam”) joining $e = \gamma_t(0)$ (the “sun”) with $c(t) = \gamma_t(r\circ c(t))$ (the “planet” at the time t).

(v) Theorem (*The constancy of the scalar angular momentum and* Kepler*'s second law for motions of point-like particles in* $T^1_e\mathbb{M}^2_\kappa$ *having* **e**-*central acceleration*).

Suppose I is an open interval of $\mathbb{R}$ and let the C^2 path $c:I \longrightarrow \mathbb{M}^2_\kappa(e)$ (see (2.16)) describe the motion of a point-like particle in $\mathbb{M}^2_\kappa(e)$, having e-central acceleration, i.e. (see (5.4)) there exists a

(5.23) $\quad C^0$ *function* $\alpha:I \longrightarrow \mathbb{R}$ *with* $\nabla_\partial\dot{c} = \alpha\cdot(R\circ c)$.

Let $r := d_\kappa(e,..):\mathbb{M}_\kappa^2 \longrightarrow \mathbb{R}$ denote the function, which measures the distance of the points of $\mathbb{M}_\kappa^2$ from the center e (see (2.33)) and let θ denote the polar angle (Pfaffian) form on $\mathbb{M}_\kappa^2(e)$ (see (2.62_1)).

Assertion.

a) The so-called *scalar angular momentum (function)* of c

$$(5.24) \quad \begin{cases} \sin_\kappa^2(r\circ c)\cdot\theta(\dot{c}):I \longrightarrow \mathbb{R} \ \ \textit{is constant on } I, \\ \textit{say of value } L\in\mathbb{R}. \end{cases}$$

b) Since (see (1.9), (2.14_1))

$$(5.25) \qquad (\sin_\kappa\circ r)(\mathbb{M}_\kappa^2(e)) \subseteq \mathbb{R}_+$$

the two scalar ODE's (5.21) (which transcribe the vector ODE of (5.23) into radial and angular components, see (5.20)) are equivalent to the following two ones:

$$(5.26) \begin{cases} (5.26_1) \qquad (r\circ c)'' - \dfrac{\cos_\kappa(r\circ c)}{\sin_\kappa^3(r\circ c)}\cdot L^2 = \alpha \quad , \\ (5.26_2) \qquad \theta(\dot{c}) = \dfrac{L}{\sin_\kappa^2(r\circ c)} \quad , \end{cases}$$

which – in contrast to (5.21) – is no more a "coupled" system of ODE's.

c) ("KEPLER's second law"): Consider the

$$(5.27) \quad \begin{cases} C^2 \ \textit{map} \ \ F:\mathbb{R}\times I \longrightarrow \mathbb{M}_\kappa^2 \ \ \textit{with} \\ F(s,t) := \exp_\kappa(s\cdot(r\circ c)(t)\cdot(U_e\circ c)(t)) \ \ \textit{for} \ (s,t)\in\mathbb{R}\times I. \end{cases}$$

[Geometric interpretation of F: For every fixed $t\in I$ the map $F(..,t):\mathbb{R} \longrightarrow \mathbb{M}_\kappa^2$ is the constant speed geodesic $\gamma_{c(t)}$ of $\mathbb{M}_\kappa^2$ with $\gamma_{c(t)}(0) = e$ and $\gamma_{c(t)}(1) = c(t)$. Hence for any $\lambda\in\mathbb{R}_+$ and all $t,\mathfrak{t}\in I$ with $t\le\mathfrak{t}$ the map $F|[0,\lambda]\times[t,\mathfrak{t}]$ describes the C^2 surface in $\mathbb{M}_\kappa^2$, which is "swept out" by the geodesic segments $\gamma_{c(\tau)}([0,\lambda])$ for τ varying in $[t,\mathfrak{t}]$ (with $\gamma_{c(\tau)}([0,\lambda])$ being "λ-homothetic" to the geodesic segment $\gamma_{c(\tau)}([0,1])$ joining e and $c(\tau)$ as shortest path).]

If σ denotes the area form of $\mathbb{M}_\kappa^2$ (see (2.60)) and if we define for $\lambda\in\mathbb{R}_+$ the function $A_\lambda:\{(t,\mathfrak{t})\in I\times I\,|\,t\le\mathfrak{t}\} \longrightarrow \mathbb{R}$ by

$$(5.28_0) \qquad A_\lambda(t,\mathfrak{t}) := \int_{[0,\lambda]\times[t,\mathfrak{t}]} F^*\sigma \quad \textit{for } t,\mathfrak{t}\in I \textit{ with } t\le\mathfrak{t},$$

which is the oriented area of the C^2 *surface* $F|[0,\lambda]\times[t,\mathfrak{t}]$ *in* $\mathbb{M}^2_\kappa$, then for all $t,\mathfrak{t}\in I$ with $t\le\mathfrak{t}$:

$$(5.28) \quad \begin{cases} A_\lambda(t,\mathfrak{t}) = 2L\cdot\displaystyle\int_t^{\mathfrak{t}} \frac{\sin_\kappa^2(\frac{\lambda}{2}\cdot(r\circ c)(\tau))}{\sin_\kappa^2((r\circ c)(\tau))}\,d\tau = \\ = \begin{cases} \frac{\lambda^2 L}{2}\cdot(\mathfrak{t}-t) & , \textit{ if } \kappa=0 \\ 2L\cdot(\mathfrak{t}-t) & , \textit{ if } \lambda=2 \\ \textit{not constantly proportional to } (\mathfrak{t}-t), \textit{ if} \\ \quad (\kappa\ne0 \textit{ and } \lambda\ne2 \textit{ and } r\circ c \textit{ not constant}). \end{cases} \end{cases}$$

Commentary. The last result *only* in case $\kappa=0$ happens to express (with $\lambda=1$) KEPLER's second law, namely: "The shortest geodesic segments $\gamma_{c(\tau)}([0,1])$ joining e and $c(\tau)$ sweep out within equally long time intervals equally large oriented areas in $\mathbb{M}^2_\kappa$." This law has (according to (5.28)) for arbitrary $\kappa\in\mathbb{R}$ to be substituted by: "The two times lengthened geodesic segments $\gamma_{c(\tau)}([0,1])$, i.e. the segments $\gamma_{c(\tau)}([0,2])$, starting at e and having $c(\tau)$ as their midpoint, sweep out within equally long time intervals equally large oriented areas in $\mathbb{M}^2_\kappa$." [From this result, where one has $\lambda=2$, the classical "KEPLER's second law" (i.e. $\lambda=1$) follows *in case* $\kappa=0$ (without the explicit result (5.28)) directly via the property $\mu_2(\lambda N) = \lambda^2\cdot\mu_2(N)$ for the Lebesgue measure μ_2 and all μ_2-measurable sets N in the Euclidean 2-space $\mathbb{M}^2_0 \cong \mathbb{E}^2$.]

This version of KEPLER's second law was stated in $[Ki_2]$,p.9 for $\kappa\in\mathbb{R}^*$ and proved in [Li],p.234 for $\kappa=-1$ and conservative forces.

Proof. (5.24) follows by differentiating $\sin_\kappa^2(r\circ c)\cdot\theta(\dot c)$ and using (5.21_1). From (5.24) and (5.25) one obtains (5.26_2). Substituting the value of $\theta(\dot c)$ from (5.26_2) into the equation (5.21_1) gives (5.26_1).

Ad c): Choose a unit vector $u\in T^1_e\mathbb{M}^2_\kappa$. Let $\rho,\varphi:\mathbb{R}^2\longrightarrow\mathbb{R}$ denote the first resp. second projection of $\mathbb{R}^2$ onto $\mathbb{R}$. Then we get the (see (2.13))

$$(5.29)\quad \begin{cases} C^{\omega} \textit{ map } \ f := \exp_{\kappa}(\rho\cdot(\cos(\varphi)\cdot u + \sin(\varphi)\cdot Ju)) : \mathbb{R}^2 \longrightarrow \mathbb{M}^2_{\kappa}, \\ \textit{e.g. } \ r\circ f = \rho \ \textit{ on } [0,\pi_{\kappa}[\times\mathbb{R}. \end{cases}$$

The comparison of (5.29) with (2.67) yields

$$(5.30)\qquad f|\tilde{M} = f^{u} \ \textit{ with } \ \tilde{M} =]0,\pi_{\kappa}[\times\mathbb{R} \subseteq \mathbb{R}^2 ,$$

and therefore we obtain from (5.30), (2.62_1), (2.67_1), (2.67_3):

$$(5.31_0)\quad (f^*\theta)|\tilde{M} = d\varphi|\tilde{M} \ \textit{ resp. } \ (f^*\sigma)|\tilde{M} = (\sin_{\kappa}(\rho)\cdot(d\rho\wedge d\varphi))|\tilde{M}.$$

But since $f^*\theta$ and $d\varphi$ resp. $f^*\sigma$ and $\sin_{\kappa}(\rho)\cdot(d\rho\wedge d\varphi)$ are *real-analytic* differential forms on $\mathbb{R}^2$ and $\tilde{M}$ open in $\mathbb{R}^2$ (see (5.30)) it follows from (5.31_0):

$$(5.31)\qquad f^*\theta = d\varphi \ \textit{ resp. } \ f^*\sigma = \sin_{\kappa}(\rho)\cdot(d\rho\wedge d\varphi) \ \textit{ on } \mathbb{R}^2.$$

Suppose now (see (2.68_0), (2.68)), that we have chosen a

$$(5.32_0)\quad \begin{cases} C^2 \textit{ lift } \ \tilde{c}: I \longrightarrow \tilde{M} \ \textit{ of } \ c: I \longrightarrow \mathbb{M}^2_{\kappa}(e) \\ \textit{with respect to } \ f^{u}:\tilde{M} \longrightarrow \mathbb{M}^2_{\kappa}(e) \ , \textit{ i.e. } \ c = f^{u}\circ\tilde{c} . \end{cases}$$

Then due to (5.32_0), (5.30), (2.68_1):

$$U_e\circ c = \cos(\varphi\circ\tilde{c})u + \sin(\varphi\circ\tilde{c})Ju ,$$

which together with (5.29) and (5.27) implies

$$(5.32_1)\quad \begin{cases} F(s,t) = f\circ g(s,t) \ , \ \textit{ where} \\ g(s,t) := (s\cdot(r\circ c)(t), (\varphi\circ\tilde{c})(t)) \ \textit{ for } (s,t)\in\mathbb{R}\times I, \end{cases}$$

therefore (if $x:\mathbb{R}\times I \longrightarrow \mathbb{R}$, $\tau:\mathbb{R}\times I \longrightarrow I$ are canonical), then

$$(5.32)\quad \begin{cases} g \textit{ is } C^2 \textit{ with } \ \rho\circ g = x\cdot(r\circ c\circ\tau) \ \textit{ and } \ \varphi\circ g = \varphi\circ\tilde{c}\circ\tau , \\ \textit{thus } \ d(\rho\circ g) = (r\circ c)(\tau)dx + x\cdot(r\circ c)'(\tau)d\tau \ \textit{ and} \\ d(\varphi\circ g) = \tau^*d(\varphi\circ\tilde{c}) \underset{(2.68_1)}{=} \theta(\dot{c}(\tau))d\tau , \end{cases}$$

consequently we get from (5.32_1), (5.31), (5.32):

$$\begin{aligned} F^*\sigma &= g^*(\sin_{\kappa}(\rho)\cdot(d\rho\wedge d\varphi)) = \\ &= \sin_{\kappa}(x\cdot(r\circ c)(\tau))\cdot(r\circ c)'(\tau)\cdot\theta(\dot{c}(\tau))\cdot(dx\wedge d\tau) = \\ &\underset{1.3),(5.26_2)}{=} 2\cdot\frac{\partial}{\partial x}[\sin^2_{\kappa}(\tfrac{x}{2}\cdot(r\circ c)(\tau))]\cdot\frac{L}{\sin^2_{\kappa}((r\circ c)(\tau))}\cdot(dx\wedge d\tau) . \end{aligned}$$

From the last equation, from (5.28_0) and FUBINI's theorem the assertion (5.28) follows immediately.

(vi) Motions of particles in $\mathbb{M}^n_{\kappa}$ with accelerations induced by a potential.

a) If (M,g) is any C^s Riemannian manifold ($s\in\mathbb{N}\cup\{\infty,\omega\}$, $s\geq 2$), ∇ its LEVI-CIVITA covariant derivative, then we say that the *acceleration* $\nabla_{\partial}\dot{c}$ of a C^2 path $c:I \longrightarrow M$ *is induced by a* *potential* iff there exists a C^1 function

$$(5.33)\quad \begin{cases} \mathcal{V} \textit{ defined on an open neighborhood } U \textit{ of } c(I) \textit{ in } \dot{M}, \\ \textit{such that } \nabla_\partial \dot{c} = -(\operatorname{grad}\mathcal{V})\circ c\ , \end{cases}$$

and any C^k function $\mathcal{V}:U \longrightarrow \mathbb{R}$ $(1\le k\le s)$ satisfying (5.33) is called *a* (C^k) *potential for* $\nabla_\partial \dot{c}$. – From (5.33) follows the well-known *Law of energy*:

$$(5.34)\qquad E_c := \frac{1}{2}\cdot g(\dot{c},\dot{c}) + (\mathcal{V}\circ c):I \longrightarrow \mathbb{R} \textit{ is constant,}$$

where the functions E_c , $\frac{1}{2}\cdot g(\dot{c},\dot{c})$, $\mathcal{V}\circ c$ on I are called then *total, kinetic, potential energy of* c respectively. [(5.34) follows just by differentiating E_c and using then (5.33). Moreover (5.34) motivates the (at first sight strange) choice of the minus sign in (5.33): Gains (resp. losses) in kinetic energy should be compensated by losses (resp. gains) in potential energy so as to balance the total energy.]

b) If in particular (M,g) is $\mathbb{M}^n_\kappa$ and $r := d_\kappa(e,..):\mathbb{M}^n_\kappa \longrightarrow \mathbb{R}$ the function measuring the distance of the points of $\mathbb{M}^n_\kappa$ from **e** (being C^ω on $\mathbb{M}^n_\kappa(\mathbf{e})$, see (2.33_0),(2.33)), then one says, that the acceleration of a C^2 path $c:I \longrightarrow \mathbb{M}^n_\kappa(\mathbf{e})$ is induced by a (C^k) *potential only depending on* r, iff there exists a (C^k) function $(1\le k)$ of one real variable

$$(5.35)\quad \begin{cases} V:H \longrightarrow \mathbb{R}\ , \textit{ defined on an open interval } H \textit{ of } \mathbb{R} \\ \textit{containing } r(c(I)), \textit{ such that } V\circ r \textit{ is a potential for} \\ \nabla_\partial \dot{c}, \textit{ i.e.} \text{ (see (5.33),(2.34)):} \\ \nabla_\partial \dot{c} = -[\operatorname{grad}(V\circ r)]\circ c = -(V'\circ(r\circ c))\cdot(R\circ c)\ . \end{cases}$$

The comparison of (5.35) and (5.4) shows for any C^2 path $c:I \longrightarrow \mathbb{M}^n_\kappa(\mathbf{e})$: *If the acceleration of* c *is induced by a potential* $(V\circ r)$ *depending only on* r, *then* c *has* **e**-*central acceleration (with* $\alpha := g(\nabla_\partial \dot{c}, R\circ c) = -(V'\circ(r\circ c))$*)*.

c) With respect to the question, whether the inverse of the last conclusion is true, we have the following information: Suppose, that $s\in\mathbb{N}\cup\{\infty,\omega\}$, $s\ge 2$ and that we are given a

(5.36) $\begin{cases} C^s \textit{ path } c:I \longrightarrow \mathbb{M}^n_\kappa(e) \textit{ with e-central acceleration} \\ \textit{(see (5.4)) and let us assume in addition, that} \\ (r\circ c)' \textit{ has no zero's on I. Then } \nabla_\partial \dot{c} \textit{ is induced by a} \\ C^{s-1} \textit{ potential } V\circ r \textit{ only depending on r.} \end{cases}$

[A construction of such a V is very direct: First, by assumption, $r\circ c:I \longrightarrow \mathbb{R}$ is a C^s diffeomorphism onto an open interval $H := r(c(I))$ of $\mathbb{R}$. If $\alpha := g(\nabla_\partial \dot{c}, R\circ c)$, then $-\alpha\circ(r\circ c)^{-1}:H \longrightarrow \mathbb{R}$ is a C^{s-2} function and if V denotes any primitive function of it, then $V:r(c(I)) \longrightarrow \mathbb{R}$ is a C^{s-1} function with $-V'\circ(r\circ c) = \alpha = g(\nabla_\partial \dot{c}, R\circ c)$, which together with (5.4), (5.35) shows, that $V\circ r$ is a C^{s-1} potential for $\nabla_\partial \dot{c}$.]

d) **Remark.** The result c) guarantees, that for any arbitrary C^s path $c:I \longrightarrow \mathbb{M}^n_\kappa(e)$ *with e-central acceleration* and for every (open!) connected component G of $\{t\in I \mid (r\circ c)'(t)\neq 0\}$ the acceleration of $c|G$ is induced by a potential $V_G\circ r$ depending only on r. However in general for two such components G and $\tilde{G}$ the ranges of definition $r^{-1}(c(G))$ resp. $r^{-1}(c(\tilde{G}))$ of V_G resp. $V_{\tilde{G}}$ overlap and V_G and $V_{\tilde{G}}$ don't fit together, in particular $V_G\circ r$ and. $V_{\tilde{G}}\circ r$ are not restrictions of a potential for $\nabla_\partial \dot{c}$ on I [e.g.: The C^ω path $c:I \longrightarrow \mathbb{M}^n_\kappa(e)$ *with e-central acceleration*, defined by $c(t) := (1, 1+\sin(1)t^2-\sin(t), 0)$ for $t\in\mathbb{R}$, has only one zero t_0 of $(r\circ c)'$, and (because $2\sin(1)>1$) $t_0\in]0,1[$. So 0 and 1 are from two different connected components of $\{t\in\mathbb{R} \mid (r\circ c)'(t)\neq 0\}$, but $c(0) = c(1)$ *and* $(\nabla_\partial \dot{c})(0) \neq (\nabla_\partial \dot{c})(1)$, which excludes in view of (5.33) the existence of any potential for $\nabla_\partial \dot{c}$ on all of $\mathbb{R}$ whatsoever!]. Nevertheless: If e.g. $c:I \longrightarrow \mathbb{M}^n_\kappa(e)$ is C^ω, then the zero's of $(r\circ c)'$ are isolated and hence any compact subinterval [a,b] of I is covered by only finitely many of these connected components G, i.e. the study of $c|[a,b]$ is then reduced to the study of finitely many C^ω path's with accelerations induced by C^ω potentials depending only on r (see (5.35)).

(vii) Newton's potential in $\mathbb{M}^n_\kappa(e)$ (i.e. one, depending only on r and having divergence free gradient).

a) If (M,g) is a n-dim. C^s ($s\geq 2$) Riemannian manifold, ∇ its Levi-Civita covariant derivative, then for every C^k ($1\leq k\leq s-1$) vector field $X\in\mathcal{X}(M)$ its C^{k-1} *divergence* (function) divX is defined by

$$(5.37_0)\quad \begin{cases} \mathrm{div}X := \mathrm{trace}(\nabla X) \ (\nabla X \textit{ viewed as } (1,1) \textit{ tensor field}) \\ \quad = \sum_{i=1}^{n} g(\nabla_{E_i}X, E_i) \ \textit{ locally,} \\ \quad\quad \textit{if } (E_1,\dots,E_n) \textit{ is an ON frame field.} \end{cases}$$

Then for every C^k function λ of M one has

$$(5.37)\qquad \mathrm{div}(\lambda X) = (X\cdot\lambda) + \lambda\mathrm{div}X = g(\mathrm{grad}\lambda, X) + \lambda\mathrm{div}X$$

and for any C^{k+1} function ψ of M one defines its Laplacian $\Delta\psi$ by

$$(5.37_1)\qquad \Delta\psi = \mathrm{divgrad}\psi \underset{(5.37_0)}{=} \mathrm{trace}(\mathrm{Hess}\psi)\ .$$

b) All classical authors agree that "the" distinguishing property of Newton's gravitational potential $\tilde{V}$ in $\mathbb{E}^3\setminus\{o\}$ is the fact, that it first depends only on r (= distance from the origin), i.e. $\tilde{V} = V\circ r$ with decreasing $V:\mathbb{R}_+ \longrightarrow \mathbb{R}$ and that it has divergence free gradient, i.e. (see (5.37_1)) $\Delta\tilde{V} = \Delta(V\circ r) = 0$, and which proves to be equivalent to the condition that there exists $k\in\mathbb{R}_+$, such that for all $p\in\mathbb{E}^3\setminus\{o\}$ the "force vector" $-(\mathrm{grad}\tilde{V})(p)$ points towards the origin (i.e. the center "attracts") and the length of the "force vector" $\|\mathrm{grad}(V\circ r)\|(p)$ equals k divided by the normalized area $r(p)^2$ of the sphere through p with center *o*.

This definition is adopted always for $\mathbb{E}^n\setminus\{O\}$ with arbitrary $n\geq 2$ and was for the hyperbolic space $\mathbb{M}^3_{-1}(e)$ already proposed by J. Bolyai between 1848 and 1851 (see [Bo],p.156, line 7-) in order to study motions of celestial bodies in hyperbolic 3-space. It was then extensively used by W. Killing (see [Ki_2],p.7) for $\mathbb{M}^3_\kappa(e)$ with $\kappa\neq 0$, later on (about

1905) it was discussed widely by H. LIEBMANN (see e.g. [Li],p.224, § 49, section 1) and applied by many others: We shall follow this convention and define for $n \geq 2$ (see (1.9) and see (2.36) for the volume of spheres in $\mathbb{M}^n_\kappa$):

There exists $k \in \mathbb{R}_+$, such that the following holds:

$$(5.38_0) \quad \left\{ \begin{array}{l} \textit{(Gradient of NEWTON's potential } \mathcal{V}_n \textit{ in } \mathbb{M}^n_\kappa(e)) = \\ \qquad = k \cdot \sin_\kappa^{1-n}(r) \cdot R \end{array} \right.$$

(k including the gravitational constant and the mass of the attracting "sun" at e). If we denote therefore by

$$(5.38_1) \qquad V_n :]0,\pi_\kappa[\longrightarrow \mathbb{R} \quad \textit{a primitive function of} \quad k \cdot \sin_\kappa^{1-n}$$

(for an explicit description of such a primitive function V_n for arbitrary n see e.g. [Sch_2],p.153 resp. [Ki_2],p.27), then trivially

$$(5.38) \qquad \operatorname{grad}(V_n \circ r) = (V_n' \; r)\operatorname{grad} r \underset{\uparrow (5.38),(2.34)}{=} k \cdot \sin_\kappa^{1-n}(r) \cdot R \; .$$

Now we have due to (2.35): $\nabla_X R = \cot_\kappa(r) \cdot [X - g(X,R) \cdot R]$. Therefore choosing a local orthonormal frame field $(E_1,\ldots,E_n)$ of $\mathbb{M}^n_\kappa(e)$ with E_1 coinciding with R, then due to (5.37_0):

$$(5.39_0) \quad \left\{ \begin{array}{l} \operatorname{div}(R) = \sum_{i=1}^{n} \cot_\kappa(r) g(E_i - \delta_{i1} E_1, E_i) = \\ \qquad = \cot_\kappa(r) \cdot \sum_{i=1}^{n} (1-\delta_{i1}^2) = (n-1) \cdot \cot_\kappa(r) \; . \end{array} \right.$$

Consequently one computes using (5.38),(5.37),(5.39_0),(2.34), (1.1) easily

$$(5.39) \qquad \Delta(V_n \circ r) = \operatorname{div}\operatorname{grad}(V_n \circ r) = 0 \; ,$$

and (5.38),(5.39) prove, that $V_n \circ r$ satisfies m.m. the classical requirements for the NEWTON's potential in $\mathbb{E}^3 \setminus \{o\}$.

From (5.38_1) and (1.2′) we get in case n=3 explicitely:

$$(5.40) \quad \left\{ \begin{array}{l} \text{NEWTON's } \textit{potential} \quad V_3 \circ r = -\, k \cdot (\cot_\kappa \circ r) \quad \textit{on } \mathbb{M}^3_\kappa(e), \\ \textit{with} \quad V_3 = -\, k \cdot \cot_\kappa \quad \textit{and} \quad V_3' = +\, k \cdot \sin_\kappa^{-2} \; . \end{array} \right.$$

(viii) C^2 paths with accelerations admitting NEWTON's 3-dim. potential.

According to (vi)b) every C^2 path $c:I \longrightarrow \mathbb{M}^3_\kappa(e)$ with an acceleration induced by NEWTON's potential $V_3 \circ r$ has e-central acceleration, hence [according to theorem (iii) and the following remark (of § 5)] c is a "plane" curve and it can be studied through a

$$(5.41)\quad \begin{cases} C^2 \textit{ path } c:I \longrightarrow \mathbb{M}^2_\kappa(e) \textit{ with an acceleration induced} \\ \textit{by NEWTON's (3-dim.) potential} \\ \qquad V\circ r = -\cot_\kappa(r):\mathbb{M}^2_\kappa(e) \longrightarrow \mathbb{R}\ . \\ \textit{Here: } k\in\mathbb{R}_+\ ,\ \ V := -k\cdot\cot_\kappa\ ,\ \ V' := k\cdot\sin_\kappa^{-2}\ ,\ \textit{and} \\ \textit{therefore } \alpha = g(\nabla_\partial \dot{c}, R\circ c) = -k\cdot\sin_\kappa^{-2}(r\circ c)\ . \end{cases}$$

Combining this statement (5.41) with (5.26) (which is via (5.21),(5.4) equivalent to c having e-central acceleration with $\alpha:=g(\nabla_\partial \dot{c},R\circ c)$) gives the following

(ix) Proposition. A C^2 path $\tilde{c}:I \longrightarrow \mathbb{M}^3_\kappa(e)$ has an acceleration induced by NEWTON's 3-dim. potential $-k\cdot\cot_\kappa(r)$ with $k\in\mathbb{R}_+$ iff there exists an $L\in\mathbb{R}$ such that $\tilde{c}$ is congruent in $\mathbb{M}^3_\kappa(e)$ to a C^2 path $c:I \longrightarrow \mathbb{M}^2_\kappa(e)$ ($\hookrightarrow \mathbb{M}^3_\kappa(e)$) which satisfies the ODE's:

$$(5.42)\qquad (r\circ c)'' = \sin_\kappa^{-2}(r\circ c)\cdot[\cot_\kappa(r\circ c)\cdot L^2 - k]\ ,$$

$$(5.43)\qquad \theta(\dot{c}) = L\cdot\sin_\kappa^{-2}(r\circ c)\ .$$

Moreover, if (5.42),(5.43) are satisfied, then we get from (5.33) the *law of energy*:

$$(5.44)\quad \left\{ \frac{1}{2}\cdot[((r\circ c)')^2 + \sin_\kappa^2(r\circ c)\cdot\theta(\dot{c})^2] - k\cdot\cot_\kappa(r\circ c) = \right. \\ \qquad\qquad = E_c = \textit{constant}\ .$$

(x) In the Euclidean case one more constant of the motion in NEWTON's (3-dim.) potential is known, the so-called Lenz vector. This concept has the following generalization for arbitrary κ (see [Zi_1],p.51):

If $L\in\mathbb{R}$ and $c:I \longrightarrow \mathbb{M}^2_\kappa(e)$ is a solution of the ODE's (5.42),(5.43), then the *Lenz vector* (see (2.16),(2.57),(5.24))

(5.45) $\ell_c := -L\cdot J\mathfrak{v}_c - [k + \kappa\cdot L^2\cdot \tan_\kappa(\frac{r\circ c}{2})]\cdot(U_e\circ c):I \longrightarrow T_e\mathbb{M}^2_\kappa$

is constant on I, where (see (2.16),(2.57),(2.62$_0$))

(5.46) $\mathfrak{v}_c := (r\circ c)'\cdot(U_e\circ c) + \sin_\kappa(r\circ c)\cdot\theta(\dot{c})\cdot(JU_e\circ c):I \longrightarrow T_e\mathbb{M}^2_\kappa$

can be characterized by the ∇-parallel-transport along $\exp_\kappa(x\cdot U_e(c(t)))\,|\,[0,1]$ mapping $\mathfrak{v}_c(t)$ onto $\dot{c}(t)$ for $t\in I$.

6. Motion of a point-like particle in $\mathbb{M}^3_\kappa$ with acceleration induced by NEWTON's gravitational potential

Now we are able to classify all possible orbits in NEWTON's 3-dim. potential and to discuss the geometry of these orbits. According to § 5.iii,viii we have to consider only "plane" curves and hence we have the following

(i) Data: We use for the 2-dim. standard space $\mathbb{M}^2_\kappa$ of constant curvature $\kappa\in\mathbb{R}$ the notations introduced in § 2, in particular $e := (1,0,0) \in \mathbb{M}^2_\kappa$, $r := d_\kappa(e,..):\mathbb{M}^2_\kappa \longrightarrow \mathbb{R}$ the function measuring the distance from e and θ the polar angle form on $\mathbb{M}^2_\kappa(e):=\mathbb{M}^2_\kappa\setminus\{e,-e\}$.

Suppose $c:I \longrightarrow \mathbb{M}^2_\kappa$ is a *maximally defined path describing the motion of a point-like particle*, c *having an acceleration induced by* NEWTON'*s 3-dim. potential*, i.e. (see § 5.vi-viii) $c:I \longrightarrow \mathbb{M}^2_\kappa$ is a maximal solution of the ODE

(6.1) $\nabla_\partial\dot{c} = -[\mathrm{grad}(-k\cdot\cot_\kappa(r))]\circ c = -k\cdot\sin_\kappa^{-2}(r\circ c)\cdot(R\circ c)$

with a fixed positive number $k\in\mathbb{R}_+$.

Then I is an open interval of $\mathbb{R}$,

(6.1$_1$) c *is a* C^∞ *path with* $c(I)\subseteq\mathbb{M}^2_\kappa(e)$

and according to (5.24),(5.44) we introduce the numbers $E,L\in\mathbb{R}$, where

(6.2) E *is the total energy of* c, *i.e. the constant value of* $\frac{1}{2}\cdot[((r\circ c)')^2 + \sin^2_\kappa(r\circ c)\cdot\theta(\dot{c})^2] - k\cdot\cot_\kappa(r\circ c)$,

(6.3) L *is the scalar angular momentum of* c, *i.e. the constant value of* $\sin^2_\kappa(r\circ c)\cdot\theta(\dot{c})$.

(ii) Proposition. Data as in (i) and consider the function

(6.4) $$U := \frac{L^2}{2\cdot\sin_\kappa^2} - k\cdot\cot_\kappa \mid]0,\pi_\kappa[$$

(see (1.1),(1.2),(1.9),(6.3)). Then (see § 3.i,(6.2),(6.4))

(6.5) $\left\{\right.$ *$(r\circ c)$ is a maximally defined process of one degree of freedom with the (effective) potential* U *and total energy* E,

and this function U (see (6.4)) has the following properties:

(6.6_1) U *is a* C^ω *function on* $]0,\pi_\kappa[$,

(6.6_2) $$U(x) \xrightarrow[x\searrow 0]{} \begin{cases} -\infty & \text{if } L=0 \\ +\infty & \text{if } L\neq 0 \end{cases},$$

(6.6_3) $$U(x) \xrightarrow[x\nearrow \pi_\kappa]{} \begin{cases} +\infty & \text{if } \kappa>0 \\ -k\cdot\sqrt{-\kappa} & \text{if } \kappa\leq 0 \end{cases},$$

(6.6_4) *if* L=0, *then* U *is strictly monotonic increasing,*

(6.6_5) $\left\{\right.$ *if* $\kappa<0$ *and* $L^2\geq k/\sqrt{-\kappa}$, *then* U *is strictly monotonic decreasing,*

(6.6_6) $\left\{\right.$ *otherwise, i.e. if* $L\neq 0$ *and* $[\kappa<0 \Rightarrow L^2<k/\sqrt{-\kappa}]$, *there exists exactly one critical point* ρ *of* U *in* $]0,\pi_\kappa[$ *and* U *is strictly monotonic decreasing on* $]0,\rho]$, *strictly monotonic increasing on* $[\rho,\pi_\kappa[$ *and attains at* ρ *its absolute minimum of value*

(6.6_7) $$\min(U) = -\frac{k^2}{2\cdot L^2} + \frac{\kappa}{2}\cdot L^2 .$$

Remark. For the later applications of (6.6_1)-(6.6_7) it is helpful to "see" these properties of U by drawing a picture of the graph of U.

Proof. According to § 5.ix the ODE (6.1) is equivalent to the ODE-system (5.42),(5.43), where (using (6.4)) the equation (5.42) can be written as

(6.7_1) $$(r\circ c)'' = -U'(r\circ c) ,$$

and using (6.3),(6.4) the law of energy (6.2) takes the form

(6.7_2) $$E = \frac{1}{2}\cdot((r\circ c)')^2 + U(r\circ c) .$$

Since c is a maximal solution of (6.1), we can conclude, that $r \circ c$ is a maximal solution of (6.7_1). This completes the proof of (6.5).

From (6.4) and (1.1),(1.2),(1.1'),(1.2') we get immediately (6.6_1),(6.6_2),(6.6_3) and

(6.7_3) $\quad [U'(t) = 0 \iff k = L^2 \cdot \cot_\kappa(t)]$ *for all* $t \in]0,\pi_\kappa[$.

This implies: In case L=0 resp. in case $\kappa<0$ and $L^2 \geq k/\sqrt{-\kappa}$ there exists no critical point of U. Otherwise there exists exactly one critical point ρ of U in $]0,\pi_\kappa[$ and (using (1.6))

(6.7_4) $$U(\rho) = -\frac{k^2}{2 \cdot L^2} + \frac{\kappa}{2} \cdot L^2 \ .$$

The preceding remarks imply (6.6_4) and (6.6_5) (using (6.6_1), (6.6_2)). Moreover we investigate the rational function

(6.8_1) $$\ell(x) := -\frac{k^2}{2x} + \frac{\kappa}{2} \cdot x \ .$$

Since $\ell'(x) = \dfrac{k^2}{2 \cdot x^2} + \dfrac{\kappa}{2}$, we have

(6.8_3) $\kappa \geq 0 \Rightarrow \ell$ *strictly monotonic increasing on* $\mathbb{R}_+$,
(6.8_4) $\kappa < 0 \Rightarrow \ell$ *strictly monotonic increasing on* $]0, k \cdot \sqrt{-\kappa}]$.

From (6.6_3),(6.7_4),(6.8_1)-(6.8_3) we get $U(\rho) < \lim\limits_{x \nearrow \pi_\kappa} U(x)$ and thereby we can complete the proof of (6.6_6).

(iii) Proposition. Data as in (i).

a) The constants E,L (see (6.2),(6.3)) fulfill:

(6.9) $$\begin{cases} r \circ c \textit{ is bounded} \iff \\ \quad \iff [(\kappa>0 \textit{ and } E \textit{ arbitrary}) \textit{ or } (\kappa \leq 0 \textit{ and } E < -k \cdot \sqrt{-\kappa})] \end{cases}$$

(6.10) $$r \circ c \textit{ bounded} \Rightarrow L^2 \leq \begin{cases} -\dfrac{k^2}{2E} & \textit{if } \kappa = 0 \ , \\ \dfrac{1}{\kappa} \cdot (E + \sqrt{E^2 + \kappa k^2}) & \textit{if } \kappa \neq 0 \ . \end{cases}$$

b) If $L \neq 0$ and $r \circ c$ is bounded (see (6.3),(6.9)), then the following is true:

If (6.10) holds with "=", then the equation $U(x) = E$ (see (6.4),(6.2)) has exactly one solution in $]0,\pi_\kappa[$ and $r \circ c$ is constant of this value.

If (6.10) holds with "<", then the equation $U(x) = E$ (see (6.4),(6.2)) has in $]0,\pi_\kappa[$ exactly two solutions $r_{min} < r_{max}$ and $r\circ c$ is periodic with $(r\circ c)(I) = [r_{min}, r_{max}]$ and has the behaviour as indicated in (3.16).

c) If $L\neq 0$ and $r\circ c$ is unbounded (see (6.3),(6.9)), then the equation $U(x) = E$ (see (6.4),(6.2)) has exactly one solution $r_{min} \in \mathbb{R}_+$ and $r\circ c$ has the image $(r\circ c)(I) = [r_{min},\infty[$ and behaves as described in (3.14), in particular $(r\circ c)(t) \longrightarrow \infty$ for $t \longrightarrow \inf I$ and for $t \longrightarrow \sup I$.

Proof.

Ad a): Because of (6.5) we can apply § 3.iii,v to $r\circ c$. Hence we can choose

$$(6.11_1)\quad \begin{cases} t_0\in I \text{ and } r_0 := (r\circ c)(t_0) \text{ with} \\ r\circ c \text{ constant or } \sup(r\circ c)(I) > r_0 > \inf(r\circ c)(I) \geq 0 \end{cases}$$

and get (see (6.4),(6.11_1),(6.2))

$$(6.11_2)\qquad U(r_0) \leq E \text{ and } (r\circ c \text{ not constant } \Leftrightarrow U(r_0) < E)$$

and (see (3.9_+) for "$\Rightarrow$" and (3.9_0),(3.8),(3.8_0) for "$\Leftarrow$"):

$$(6.11_3)\qquad r\circ c \text{ bounded } \Leftrightarrow \exists_{\tilde\rho\in[r_0,\pi_\kappa[}\ U(\tilde\rho) \geq E .$$

Using (6.6_1)-(6.6_7) we see the equivalence of the right hand sides of (6.11_3) and (6.9).

Now suppose $r\circ c$ is bounded. Then according to (6.9) the right hand side of the inequality in (6.10) is positive. Hence (6.10) holds for $L=0$ and we can assume $L\neq 0$. Then we conclude from (6.6_1)-(6.6_7),(6.8_1),(6.9),(6.11_2):

$$(6.11_4)\qquad E \geq \ell(L^2) \text{ with } \ell \text{ as in } (6.8_1).$$

Now we consider (see (6.8_1)-(6.8_3))

$$(6.8_4)\qquad \bar\ell := \begin{cases} (\ell|\mathbb{R}_+)^{-1} & \text{if } \kappa\geq 0 \\ (\ell|\,]0,k/\sqrt{-\kappa}])^{-1} & \text{if } \kappa<0 \end{cases}$$

and we obtain

$$(6.8_5)\qquad \bar\ell = \begin{cases} \frac{1}{\kappa}\ (x + \sqrt{x^2+\kappa k^2}) & \text{if } \kappa>0\ , \\ -\ \frac{k^2}{2x}|\mathbb{R}_- & \text{if } \kappa=0\ , \\ \frac{1}{\kappa}\ (x + \sqrt{x^2+\kappa k^2})\,|\,]-\infty,-k\cdot\sqrt{-\kappa}] & \text{if } \kappa<0\ . \end{cases}$$

Then (6.11_4) implies because of (6.8_1)-(6.8_5),(6.9): $L^2 \le \mathfrak{l}(E)$. Because of (6.8_5) this proves (6.10).

Ad b),c): Because of (6.5) a direct consequence of § 3.iii,v and (6.6_1)-(6.6_7), notice

$$(\text{equality in } (6.10)) \iff E = \min U .$$

(iv) Proposition. Data as in (i), assume $L \neq 0$ and consider the function

$$(6.12) \qquad \mathcal{G} := \sin_\kappa^2 \cdot \left[\left(\frac{2E}{L^2} + \frac{2k}{L^2}\cdot \cot_\kappa\right)\cdot \sin_\kappa^2 - 1\right] .$$

Then c is a C^∞ solution of the first order ODE

$$(6.13) \qquad ((r\circ c)')^2 = \mathcal{G}(r\circ c)\cdot\theta(\dot c)^2 \quad \textit{and} \quad \theta(\dot c)^2 > 0$$

(see (2.33),(6.12),(2.62_0)).

Remark. In [Li] (case $\kappa=-1$) and E. WHITTAKER's classical book on analytical dynamics (case $\kappa=0$) the equation (6.13) is formulated as $(\frac{\partial r}{\partial \varphi})^2 = \mathcal{G}(r)$.

Proof. This is a consequence of the law of energy, namely

$$((r\circ c)')^2 \underset{(6.2)}{=} 2E + 2k\cdot\cot_\kappa(r\circ c) - \sin_\kappa^2(r\circ c)\cdot\theta(\dot c)^2 =$$
$$\underset{(6.3)}{=} [2E + 2k\cdot\cot_\kappa(r\circ c) - L^2\cdot\sin_\kappa^{-2}(r)]\circ c ,$$

where in case $L\neq 0$

$$2E + 2k\cdot\cot_\kappa(r\circ c) - L^2\cdot\sin_\kappa^{-2}(r) \underset{(6.12)}{=} L^2\cdot\sin_\kappa^{-4}(r)\cdot\mathcal{G}(r)$$

and (6.3) completes the proof.

(v) Theorem (*All radial orbits in* NEWTON's *3-dim. potential*).

Data as in (i) and in addition

$$(6.14) \qquad \theta(\dot c) = 0 .$$

Assertion.

a) There exists a unit speed geodesic $\gamma:\mathbb{R} \longrightarrow \mathbb{M}_\kappa^2$ with $\gamma(0) = \mathbf{e}$ and $c(I) \subseteq \gamma(]0,\pi_\kappa[)$.

b) Suppose $t_0 \in I$ and $r_0 := (r\circ c)(t_0)$, $r_0' := (r\circ c)'(t_0)$. If

$$(6.15) \qquad \kappa \le 0 \quad \textit{and} \quad r_0 \ge v_\infty(r_0) := \sqrt{2k\cdot[\cot_\kappa(r_0) - \sqrt{-\kappa}]} ,$$

then $r\circ c$ is strictly monotonic increasing with $(r\circ c)(t) \longrightarrow \infty$

for $t \longrightarrow \sup I$. In all cases different from (6.15) one has $(r\circ c)(t) \longrightarrow 0$ for $t \longrightarrow \sup I$, i.e. the particle falls into the "sun".

Hence the number $v_\infty(r_0)$ is the *"escaping velocity"* (for the distance r_0) and this effect occurs only if $\kappa\le 0$.

Sketch of the proof.

Choose $\gamma(x) := \exp_\kappa(x\cdot U_e(c(t_0)))$ with a $t_0\in I$, check $E \ge \sup U(]r_0,\pi_\kappa[) \iff [\kappa\le 0 \text{ and } |r_0'|\ge v_\infty(r_0)]$ (see (6.2),(6.4),(6.15)) and apply § 3.v, § 6.ii.

(vi) Theorem *(All bounded non-radial orbits in* NEWTON's *3-dim. potential,* KEPLER's *first and third law).*

Data as in (i) and in addition

(6.16) $\quad \theta(\dot c) \ne 0$ *and* $r\circ c$ *bounded.*

Assertion.

a) ("KEPLER's first law"): $c:I \longrightarrow M_\kappa^2$ is *periodic* (in particular $I=\mathbb{R}$) and $c(I)$ is an *ellipse* (see § 4.ii.a) with one focal point at e.

b) Between the "geometrical" parameters of this ellipse – major semiaxis a and focal length f – and the "physical" parameters of c – total energy E and scalar angular momentum L – the following relations hold:

(6.17) $$E = -k\cdot \cot_\kappa(2a)$$

(6.18) $$\frac{L^2}{2k}\cdot \sin_\kappa(2a) = \sin_\kappa^2(a) - \sin_\kappa^2(f) \; .$$

c) ("KEPLER's third law"): The *periodic time* T of c (i.e. the smallest positive period of c) satisfies

(6.19) $$T^2 = \frac{(2\pi)^2}{k}\cdot \cos_\kappa(a)\cdot \sin_\kappa^3(a) \; .$$

With fixed major semiaxis a the periodic time T is a strictly monotonic decreasing function of the curvature constant κ.

Remarks.

a) For KEPLER's second law see above § 5.v.c.

b) These results were essentially formulated for $\kappa\in\mathbb{R}^*$ in [Ki_2],p.7-9 and proved for $\kappa=-1$ in [Li],p.234-236.

c) We try to give a numerical estimate for the influence of the curvature of the universe onto the planetary year, i.e. the periodic time of our earth. The major semiaxis of the elliptic orbit of our earth around the sun is well known and equals approximately $a \sim 150\cdot 10^6$ km $= 1{,}5\cdot 10^{13}$ cm . Due to [HE](preface) the radius of our universe is of size 10^{28} cm . If our universe would be the $\mathbb{M}^3_\kappa$ with $\kappa>0$, then we can assume $\pi_\kappa = \operatorname{diam}(\mathbb{M}^3_\kappa) \geq 10^{28}$ cm , hence

$$(6.20_1) \qquad \rho := a/\pi_\kappa \leq 1{,}5\cdot 10^{-15} .$$

For the version (6.19) of KEPLER's third law one finds the power series expansion

$$(6.20_2) \quad \begin{cases} T(a,\kappa)^2 = \frac{(2\pi)^2}{k}\cdot a^3\cdot \\ \qquad \cdot[1 - (\kappa a^2) + \frac{2}{5}\cdot(\kappa a^2)^2 - \frac{17}{189}\cdot(\kappa a^2)^3 + \ldots] , \end{cases}$$

which leads in case $\kappa>0$ to the approximation (see (6.20_1))

$$(6.20_3) \qquad T(a,\kappa) \sim T(a,0)\cdot[1 - \tfrac{\pi\rho}{2}] \quad \textit{if} \quad 0 < \pi\rho \ll 1 .$$

Using $T(a,0) \sim 365$ days $= 3{,}1536\cdot 10^6$ sec and (6.20_1),(6.20_3) we get:

$$(6.20_4) \qquad |T(a,\kappa) - T(a,0)| \leq 7{,}58\cdot 10^{-9} \text{ sec} < 10^{-8} \text{ sec} ,$$

i.e.: If our universe is an $\mathbb{M}^3_\kappa$ with $\kappa>0$, then the "planetary year" of our earth differs from the Euclidean value at most by 10^{-8} sec !

Proof.

Ad a),b): First we get from (6.16),(6.9)

$$(6.21_1) \qquad \kappa > 0 \quad \textit{or} \quad [\ \kappa \leq 0 \quad \textit{and} \quad E < -k\cdot\sqrt{-\kappa}\ ]$$

and from (6.16),(6.3)

$$(6.21_2) \qquad L \neq 0 .$$

Because of (6.21_1) the equation (6.17) has exactly one solution a in $]0,\frac{1}{2}\cdot\pi_\kappa[$ and, since $r\circ c$ is bounded, we conclude from (6.10),(6.17) (using (1.1),(1.2),(1.6),(1.7)) for this solution

$$(6.21_3) \qquad \frac{L^2}{2k}\cdot\sin_\kappa(2a) \leq \sin^2_\kappa(a)$$

with equality in (6.21_3) iff equality in (6.10). Because of

(6.21_3) and $a\in]0,\frac{1}{2}\cdot\pi_\kappa[$ the equation (6.18) has exactly one solution f in $[0,a[$ and $f=0$ iff equality in (6.10). Hence (using (1.2), (4.14_6), (6.21_2)) we have found unique numbers $a,f\in\mathbb{R}$ with the following three properties:

$$(6.21_4) \qquad 0 \le f < a < \frac{1}{2}\cdot\pi_\kappa \quad ,$$

$$(6.21_5) \qquad \frac{2E}{L^2} = \frac{-\cos_\kappa(2a)}{\sin_\kappa(a+f)\cdot\sin_\kappa(a-f)} \quad ,$$

$$(6.21_6) \qquad \frac{2k}{L^2} = \frac{\sin_\kappa(2a)}{\sin_\kappa(a+f)\cdot\sin_\kappa(a-f)} \quad .$$

These numbers a,f satisfy (6.17),(6.18) and fulfill (because of (iii)b))

$$(6.21_7) \qquad f=0 \Rightarrow r\circ c \textit{ constant.}$$

Moreover we compute using (6.4), (6.21_6), (1.1), (1.2), (1.5), (1.7), (6.17)

$$(6.21_8) \qquad U(a-f) = U(a+f) = E \quad \textit{with } U \textit{ from } (6.4).$$

Because of (6.16), (6.21_2), (6.21_4), (6.21_7), (6.21_8) the proposition (iii)b) yields

$$(6.21_9) \begin{cases} I=\mathbb{R} \textit{ and } r\circ c \textit{ periodic with} \\ (r\circ c)(I) = [a-f,a+f] \ . \end{cases}$$

Hence we can choose

$$(6.22_1) \qquad \tau \in \mathbb{R} \quad \textit{with} \quad (r\circ c)(\tau) = a-f = \min(r\circ c)$$

and (see (2.16))

$$(6.22_2) \qquad u := U_e(c(\tau)) \in T_e^1 M_\kappa^2 \quad \textit{with } \tau \textit{ from } (6.22_1).$$

Now let K denote the ellipse in $\mathbb{M}_\kappa^2$ (see § 4.ii.a) with focal length a, major semiaxis a (see (6.21_4)-(6.21_6)) and focal points e and $f^u(2f,\pi)$ (see (2.33_0), (6.22_2), (2.67)). Then due to § 4.iv.a we have

$$(6.22_3) \begin{cases} K = [\cot_\kappa(r) - (\alpha+\beta\cdot\cos\varphi_u)]^{-1}(\{0\}) \\ \textit{with } \alpha\in\mathbb{R}_+,\ \beta\in[0,\infty[\textit{ and } (\beta=0 \Leftrightarrow f=0) \end{cases}$$

and according to (4.19_1), (6.21_5), (6.21_6)

$$(6.22_4) \qquad \textit{the functions } \mathfrak{G} \textit{ from (4.19) and (6.12) coincide.}$$

Moreover (4.7),(4.7_1),(6.22_1),(6.22_2) imply

(6.22_5) $\quad c(\tau) \in K \cap \exp_\kappa(\mathbb{R}\cdot u)$

and (6.3),(6.21_2) yield

(6.22_6) $\quad \theta(\dot{c})^2 > 0$.

We can conclude from (6.1_1),(6.21_7),(6.21_9),(6.22_3)-(6.22_6) by using proposition § 4.v

(6.22_7) $\begin{cases} c(\mathbb{R}) \subseteq K \\ \textit{and } c \textit{ is an open mapping into the 1-dim. manifold } K \end{cases}$

(since c is a submersion into K because of (6.22_6)).

We now introduce (see (6.21_9),(6.22_1))

(6.23_1) $\quad \varphi_c(x) := \int_\tau^x \theta(\dot{c}(t))dt$.

Then (see (6.1_1),(6.22_6) and use (6.3),(6.21_9),(6.21_4),(1.9) for surjectivity)

(6.23_2) $\quad \varphi_c:\mathbb{R} \longrightarrow \mathbb{R}$ *is a* C^∞ *diffeomorphism*

with (use (2.67_3),(2.67_5),(6.22_2))

(6.23_3) $\quad \varphi_c \equiv \angle_o(u,U_e\circ c) \mod 2\pi$.

Hence we can introduce the number (see (6.23_1),(6.23_2))

(6.23_4) $\quad T := (\varphi_c)^{-1}(\mathrm{sgn}(\theta(\dot{c}))\cdot 2\pi) - (\varphi_c)^{-1}(0) \in \mathbb{R}_+$

and find (use (2.16),(6.22_2),(6.23_1)-(6.23_4)): $\tau+T$ is the first moment in time after τ, at which c intersects the geodesic segment $\exp_\kappa(xu)\,|\,]0,\pi_\kappa[$. Using (6.22_1),(6.22_7), (6.22_3),(4.7_1) we conclude $(r\circ c)(\tau+T) = (r\circ c)(\tau) = a-f$. Hence according to (6.21_9) and proposition (iii)b) the number T must be a period of $r\circ c$ and because of (6.3) a period of $\theta(\dot{c})$ too, which implies (see (6.23_1),(6.23_4)) $\varphi_c(x+T) \equiv \varphi_c(x)$ mod 2π. Now we can conclude:

(6.23_5) $\quad$ c *is periodic with smallest positive period* T.

Hence $c(\mathbb{R}) = c([0,T])$ is compact and with (6.22_7), (6.22_3), (4.17_1) follows $c(\mathbb{R}) = K$.

Ad c): Because of (6.23_5),(6.23_4),(6.23_1) the periodic time T of c can be computed by

$$T = \left|\int_0^{2\pi} ((\varphi_c)^{-1})'(t)dt\right| = \left|\int_0^{2\pi} (\frac{1}{\theta(\dot c)}\circ(\varphi_c)^{-1})(t)dt\right| =$$

$$\underset{(6.3)}{=} \frac{1}{|L|}\cdot\int_0^{2\pi} (\underbrace{\sin_\kappa^2(r)}_{=1/(\cot_\kappa^2(r)+\kappa)}\circ c\circ(\varphi_c)^{-1})(t)dt =$$

$$= \frac{1}{|L|}\cdot\int_0^{2} \frac{4\sin_\kappa^2(a+f)\sin_\kappa^2(a-f)}{[\sin_\kappa(2a)+\sin_\kappa(2f)\cos(t)]^2+4\kappa\sin_\kappa^2(a+f)\sin_\kappa^2(a-f)}\,dt =$$

$(6.22_7),(4.7),(6.22_3),(2.66_2)$

$$\underset{(6.21_6)}{=} (1/\sqrt{\kappa})\cdot\cos_\kappa^{1/2}(a)\cdot\sin_\kappa^{1/2}(a)\cdot 2\vartheta$$

with

$$\vartheta := \int_0^{\pi} \frac{4\sin_\kappa^{3/2}(a+f)\sin_\kappa^{3/2}(a-f)}{[\sin_\kappa(2a)+\sin_\kappa(2f)\cos(t)]^2+4\kappa\sin_\kappa^2(a+f)\sin_\kappa^2(a-f)}\,dt\ .$$

Hence for a proof of (6.19) we have to show

$$\vartheta = \pi\cdot\sin_\kappa(a)\ .$$

This integration is the only part of the proof, which requires different argumentation in case $\kappa>0$ resp. $\kappa=0$ resp. $\kappa<0$. For details see [Zi_1],p.36-40.

Because of (6.19),(6.21_4),(1.9) it suffices for the proof of the monotony statement to show:

$$(6.24_1)\quad \begin{cases} \frac{\partial}{\partial x}[\cos_x(y)\sin_x^3(y)] < 0 \quad \textit{on} \\ \Omega := \{(\kappa,a)\in\mathbb{R}^2\,|\,a>0 \textit{ and } \kappa<(\frac{\pi}{2a})^2\}\ . \end{cases}$$

Since (see (1.9),(6.24_1))

$$(6.24_2)\quad \cos_x(y) > 0 \ \textit{ and } \ \sin_x(y) > 0 \ \textit{ on } \Omega\ ,$$

it suffices for (6.24_1) to show

$$(6.24_3)\quad \frac{\partial}{\partial x}[\cos_x(y)] < 0 \ \textit{ and } \ \frac{\partial}{\partial x}[\sin_x(y)] < 0 \ \textit{ on } \Omega\ .$$

For this purpose we deduce from (1.1):

$$(6.24_4)\quad \begin{cases} \frac{\partial}{\partial x}[\cos_x(y)] = -\frac{y}{2}\cdot\sin_x(y) \quad \textit{on } \mathbb{R}^2 \textit{ and} \\ \frac{\partial}{\partial x}[\sin_x(y)] = 0 \quad \textit{on } y^{-1}(\{0\})\ . \end{cases}$$

(vii) Theorem (*All unbounded non-radial orbits in* NEWTON's *3-dim. potential*).

Data as in (i) and in addition

(6.25) $\theta(\dot{c}) \neq 0$ *and* $r \circ c$ *unbounded.*

Assertion. Then we have $\kappa \leq 0$ and $E \geq -k \cdot \sqrt{-\kappa}$ and for c(I) holds:

a) If $E > +k \cdot \sqrt{-\kappa}$,
then c(I) is a *hyperbola* (see § 4.ii.b) with closer focal point e. Between the "geometrical" parameters of this hyperbola – major semiaxis a and focal length f – and the "physical" parameters of c – total energy E and scalar angular momentum L – the following relations hold:

$$(6.26_1) \qquad E = k \cdot \cot_\kappa(2a) \ ,$$

$$(6.26_2) \qquad \frac{L^2}{2k} \cdot \sin_\kappa(2a) = \sin_\kappa^2(f) - \sin_\kappa^2(a) \ .$$

b) If $\kappa = 0$ and $E = 0$,
then c(I) is a *parabola* (see § 4.iii.a) with focal point e. Between the "geometrical" parameter of this parabola – pericentral distance s (see (4.9_1)) – and the "physical" parameter of c – scalar angular momentum L – the following relation holds:

$$(6.27_1) \qquad L^2 = k \cdot 2s \ .$$

c) If $\kappa < 0$ and $E = +k \cdot \sqrt{-\kappa}$,

then c(I) is a *horohyperbola* (see § 4.ii.e) with focal point e. Between the "geometrical" parameter of this horohyperbola – pericentral distance s – and the "physical" parameter of c – scalar angular momentum L – the following relation holds:

$$(6.28_1) \qquad L^2 = 2k \cdot \sin_\kappa(s) \cdot e^{\sqrt{-\kappa} \cdot s} \ .$$

d) If $\kappa < 0$ and $-k \cdot \sqrt{-\kappa} < E < +k \cdot \sqrt{-\kappa}$,
then c(I) is a *semihyperbola* (see § 4.ii.c) with focal point e. Between the "geometrical" parameters of this semihyperbola – major semiaxis a and focal length f – and the "physical" parameters of c – total energy E and scalar angular momentum L – the following relations hold:

$$(6.29_1) \qquad E = -k\kappa \cdot \tan_\kappa(2a) \ ,$$

(6.29_2) $(L^2/k)\cdot\cos_\kappa(2a) = \sin_\kappa(2f) - \sin_\kappa(2a)$.

e) If $\kappa < 0$ and $E = -k\cdot\sqrt{-\kappa}$,
then c(I) is a *horoellipse* (see § 4.ii.d) with focal point e. Between the "geometrical" parameter of this horoellipse – pericentral distance s – and the "physical" parameter of c – scalar angular momentum L – the following relation holds:

(6.30_1) $L^2 = 2k\cdot\sin_\kappa(s)\cdot e^{-\sqrt{-\kappa}\cdot s}$.

Proof. First we get from (6.25),(6.9),(6.3) the first assertion and

(6.31_1) $L \neq 0$.

Because of (6.31_1),(6.25) we can apply proposition (iii)c) to $r\circ c$ and hence choose a

(6.31_2) $\tau \in I$ *with* $(r\circ c)(\tau) = \min(r\circ c)$

and (see (2.16))

(6.31_3) $u := U_e(c(\tau)) \in T^1_e M^2_\kappa$ *with* τ *as in* (6.31_2).

Moreover we get from the limes statements in (iii)b): In the one point compactification $M^2_\kappa \dot\cup \{\infty\}$ of M^2_κ one has $c(t) \longrightarrow \infty$ for $t \longrightarrow \inf I$ and for $t \longrightarrow \sup I$, hence $c(I) \cup \{\infty\}$ compact in $M^2_\kappa \dot\cup \{\infty\}$ and consequently

(6.31_4) $c(I)$ *is closed in* M^2_κ .

Ad a): Because of $\kappa \le 0$ and $E > +k\cdot\sqrt{-\kappa}$ the equation (6.26_1) has exactly one solution a in $\mathbb{R}_+$. For this number a the equation (6.26_2) has exactly one solution f in $]a,\infty[$. Hence (using (1.2),(4.14_6),(6.31_1)) we have found unique numbers $a,f\in\mathbb{R}$ with the following three properties:

(6.32_1) $0 < a < f < \infty$,

(6.32_2) $$\frac{2E}{L^2} = \frac{\cos_\kappa(2a)}{\sin_\kappa(f+a)\cdot\sin_\kappa(f-a)} ,$$

(6.32_3) $$\frac{2k}{L^2} = \frac{\sin_\kappa(2a)}{\sin_\kappa(f+a)\cdot\sin_\kappa(f-a)} .$$

These numbers fulfill (6.26_1),(6.26_2) and we compute using

(6.4),(6.32_3),(1.2),(1.5),(1.7),(6.26_1)

(6.32_4) $U(f-a) = E$ *with* U *as in* (6.4).

Because of (6.25),(6.31_1),(6.32_1),(6.32_4) the proposition (iii)c) yields

(6.32_5) $(r\circ c)(I) = [f-a,\infty[$.

Now let K denote the hyperbola in $\mathbb{M}^2_\kappa$ (see § 4.ii.b) with focal length f, major semiaxis a (see (6.32_1)-(6.32_3)) and focal points e and $f^u(2f,0)$, e being the closer one (see (2.33_0), (6.31_3),(2.67)). Then due to § 4.iv.b we have

(6.32_6) $K = [\cot_\kappa(r) - (\alpha+\beta\cdot\cos\varphi_u)]^{-1}(\{0\})$ *with* $\alpha,\beta\in\mathbb{R}_+$

and according to (4.19_2),(6.32_2),(6.32_3)

(6.32_7) *the functions* $\mathcal{G}$ *from* (4.19) *and* (6.12) *coincide.*

Moreover (4.8),(4.8_1),(6.31_2),(6.31_3),(6.32_5) imply

(6.32_8) $c(\tau) \in K \cap \exp_\kappa(\mathbb{R}\cdot u)$

and (6.3),(6.31_1) yield

(6.32_9) $\theta(\dot{c})^2 > 0$.

We can conclude from (6.1_1),(6.32_6)-(6.32_9) by using proposition § 4.v: $c(I) \subseteq K$ and c is an open mapping into the 1-dim. manifold K (since c is a submersion into K because of (6.32_9)). Together with (6.31_4),(6.32_6),(4.17_1) it follows then $c(I) = K$.

Ad b)-e): Analogous to a), use (4.19_3)-(4.19_5) and conclude from (6.6_2)-(6.6_7) and (6.11_2), that ($\kappa<0$ and $E = -k\cdot\sqrt{-\kappa}$) is possible only if $L^2 < k/\sqrt{-\kappa}$.

(viii) Remark. In case $\kappa<0$ one has (see § 6.vi,vii) rather many geometrically different types of orbits of point-like particles moving in a central force field with NEWTON's potential, which correspond to the various geometric types of the geometry of curves of constant (oriented) curvature in the hyperbolic plane (see § 2.viii.d and § 4.iii.c). For a better understanding of this geometric diversity of "KEPLER orbits" the following consideration might be helpful:

Choose a fixed negative value κ of the curvature constant, a fixed unit speed geodesic $\gamma:\mathbb{R} \longrightarrow \mathbb{M}^2_\kappa$ with $\gamma(0)=e$, a fixed value L of the scalar angular momentum with $0 < L^2 < k/\sqrt{-\kappa}$ and study then the family of curves c_E characterized by the following three properties:

a) $c_E:I \longrightarrow \mathbb{M}^2_\kappa$ is a maximally defined path in $\mathbb{M}^2_\kappa$ with acceleration induced by NEWTON's 3-dim. potential,

b) c_E has total energy E and scalar angular momentum equal to the given L,

c) $0\in I$ and $c_E(0) = \gamma(\min(r\circ c_E))$ (the minimum exists according to § 6.iii.b,c).

From condition c) and the proofs of (vi),(vii) we get the following additional information in Theorem (vi),(vii):

In case $E < -k\cdot\sqrt{-\kappa}$ the ellipse $c_E(I)$ (see § 4.ii.a, § 6.vi.a,b) has the second focal point $\gamma(-2f_E)$, where a_E, f_E are the unique solutions of (6.17),(6.18),(6.21_4).

In case $E = -k\cdot\sqrt{-\kappa}$ the horoellipse $c_E(I)$ (see § 4.ii.d, § 6.vii.e) has the axis $\gamma(-x)$.

In case $-k\cdot\sqrt{-\kappa} < E < +k\cdot\sqrt{-\kappa}$ the semihyperbola $c_E(I)$ (see § 4.ii.c, § 6.vii.d) has the focal line with initial velocity vector $J\dot{\gamma}(2f_E)$, where a_E, f_E are the unique solutions of (4.3_0),(6.29_1),(6.29_2).

In case $E = +k\cdot\sqrt{-\kappa}$ the horohyperbola $c_E(I)$ (see § 4.ii.e, § 6.vii.c) has the axis $\gamma(x)$.

In case $E > +k\cdot\sqrt{-\kappa}$ the hyperbola $c_E(I)$ (see § 4.ii.b, § 6.vii.a) has the second focal point $\gamma(2f_E)$, where a_E, f_E are the unique solutions of (6.26_1),(6.26_2),(6.32_1).

Now suppose $E \nearrow -k\cdot\sqrt{-\kappa}$. Then one checks $f_E \longrightarrow +\infty$ (use (4.7_1),(6.17)). Hence the focal point $\gamma(-2f_E)$ of the ellipse $c_E(I)$ different from e converges in $\overline{\mathbb{M}^2_\kappa}$ to $\gamma(-\infty)$ (see § 2.vi.b) and for the equation characterizing $c_E(I)$ we get (see (2.47_1), (4.7_1) and (4.1),(4.4)):

$$d_\kappa(e,..) + \underbrace{[d_\kappa(\gamma(-2f_E),..) - 2f_E]}_{\longrightarrow\ \beta_{\gamma(-x)}(..)} = 2\cdot\underbrace{(a_E - f_E)}_{\longrightarrow\ s} .$$

For $E \searrow +k\cdot\sqrt{-\kappa}$ one again checks $f_E \longrightarrow +\infty$ (use (4.8_1), (6.26_1)). Hence the focal point $\gamma(2f_E)$ of the hyperbola $c_E(I)$ different from e converges in $\overline{\mathbb{M}^2_\kappa}$ to $\gamma(+\infty)$ (see § 2.vi.b) and for the equation characterizing $c_E(I)$ we get (see (2.41_1), (4.8_1) and (4.2),(4.5)):

$$d_\kappa(e,..) - \underbrace{[d_\kappa(\gamma(2f_E),..) - 2f_E]}_{\longrightarrow\ \beta_\gamma(..)} = 2\cdot\underbrace{(f_E - a_E)}_{\longrightarrow\ s} .$$

Suppose $-k\cdot\sqrt{-\kappa} < E < +k\cdot\sqrt{-\kappa}$ and $E \longrightarrow \sigma\cdot k\cdot\sqrt{-\kappa}$ with $\sigma\in\{1,-1\}$. Then one checks $f_E \longrightarrow \sigma\cdot\infty$ (use (4.9_1),(6.29_1)). Hence the intersection point $\gamma(2f_E)$ of the focal line of the semihyperbola $c_E(I)$ with the given geodesic γ has the same limes in $\overline{\mathbb{M}^2_\kappa}$ as the focal point different from e of the ellipse resp. hyperbola when approaching $\sigma\cdot k\cdot\sqrt{-\kappa}$ from the other side.

Acknowledgement: The first author wishes to thank sincerely Professor Z. Zekanowski (Warszawa) and Professor M. Heller (Kraków) for inviting him to their Seminar on Differential Spaces (in September 1990 at Pasierbiec) to give there lectures on the main topics of this article and some topics of the subsequent article (this Journal) by the second author.

REFERENCES

[Ba] **E. Beltrami:** Teoria fondamentale degli spazii die curvatura constante, Annali di Math. II, 62 (1868), 232-255.

[Bo] **W. Bolyai, J. Bolyai:** Geometrische Untersuchungen, Hrsg. P. Stäckel, Teubner, Leipzig/Berlin, 1913.

[Do] **P. Dombrowski:** Krümmungsgrößen gleichungsdefinierter Untermannigfaltigkeiten Riemann'scher Mannigfaltigkeiten, Math. Nachr. 38 (1968), 133-180.

[Fe] **Hans-Joachim Feldhoff:** Periodische Bewegungsvorgänge zweidimensionaler Riemannscher Mannigfaltigkeiten und die Analoga der Sätze von HOLDITCH und WOOLHOUSE in zweidimensionalen Räumen konstanter Krümmung, Doctoral thesis, Köln 1984.

[HE] **S.W.Hawking, G.F.R.Ellis:** The large scale structure of space-time, Cambridge University Press, 1973.

[Ki_1] **W. Killing:** Über zwei Raumformen constanter positiver Krümmung, J. Reine Angew. Math. 86 (1879), 72-83.

[Ki_2] **W. Killing:** Die Mechanik in den Nicht-Euklidischen Raumformen, J. Reine Angew. Math. 98 (1885), 1-48.

[Li] **H. Liebmann:** Nichteuklidische Geometrie, G.J.Göschen, Leipzig, 1905.

[Lp_1] **R. Lipschitz:** Fortgesetzte Untersuchungen in Betreff der ganzen homogenen Funktionen von n Differentialen, J. Reine Angew. Math. 72 (1870), 1-56.

[Lp_2] **R. Lipschitz:** Untersuchung eines Problems der Variationsrechnung, in welcem das Problem der Mechanik enthalten ist, J. Reine Angew. Math. 74 (1872), 116-149.

[Sch_1] **E. Schering:** Die Schwerkraft im Gaussischen Raume, Nachr. Königl. Ges. Wiss. Göttingen (1870), 311-321.

[Sch_2] **E. Schering:** Die Schwerkraft in mehrfach ausgedehnten Gaussischen und Riemannschen Räumen, Nachr. Königl. Ges. Wiss. Göttingen (1873), 149-159.

[St] **P. Stäckel:** Bericht über die Mechanik mehrfacher Mannigfaltigkeiten, Jahresber. DMV 12 (1903), 469-481.

[Zi_1] **J. Zitterbarth:** Zur klassischen Mechanik in Räumen konstanter Krümmung, Doctoral thesis, Köln 1990.

[Zi_2] **J. Zitterbarth:** Some remarks on the motion of a rigid body in a space of constant curvature without external forces, Demonstratio Math. 24 (1991)

MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, 5000 KÖLN 41, GERMANY

Received May 9, 1991.