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SOME ASPECTS OF d^r- SPACES

1. d^r- spaces

Sikorski's definition of differential spaces is a generalization of real C^∞- manifolds. But in many cases, generalizations of complex manifolds or of C^r- manifolds with $r < \infty$ or $r = \omega$ (analytic manifolds) are needed. This leads to the following natural generalization [1] of Sikorski's definition:

Definition 1. Let M be a set, C a non-empty set of functions $M \longrightarrow \mathbb{K}$ ($= \mathbb{R}$ or $= \mathbb{C}$), and $r \in \mathbb{N}_o \cup \{\infty, \omega\}$. If

a) M carries the initial topology w.r.t. C (i. e. the coarsest topology such that all $f \in C$ are continuous)

b) for each finite number of functions $f_1, \ldots, f_n \in C$, also the function $x \longmapsto \sigma(f_1(x), \ldots, f_n(x))$ is in C for all $\sigma \in C^r(\mathbb{K}^n, \mathbb{K})$

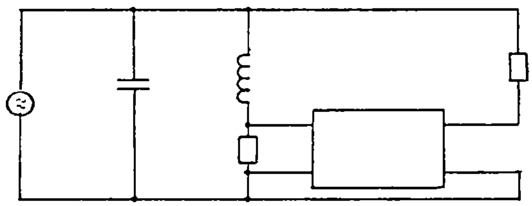
c) $C_M = C$,

then C is called d^r- structure on M and the pair (M, C) d^r- space over K.

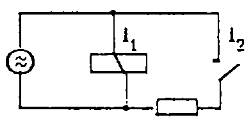
It has been shown in [2] and [3] that the calculus on d^r- spaces is completely analogous to the one on the differential spaces of Sikorski.

2. Examples of d^r- spaces in electrical engineering

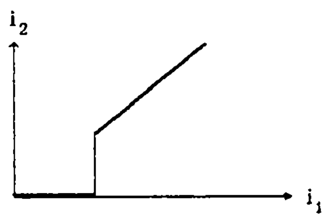
Consider an electrical circuit like



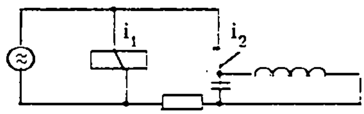
If it has n different wires, one is interested in the voltage (w.r.t. some reference point) and the current in each of them. These 2n variables will be regarded as elements of $\mathbb{R}^{2n}$ (or $\mathbb{C}^{2n}$). But, of course, the possible states of this system are restricted by the well-known equations of electrical circuits: The two laws of Kirchhoff, Ohm's law, the equations for capacitors, inductances, and the other 2-poles, 4-poles etc. By the pre-image theorem, one obtains some submanifold of $\mathbb{R}^{2n}$ (or $\mathbb{C}^{2n}$) as the locus of the possible states [8]. The situation changes, if the circuit contains switches, relays thyristors, or certain operation amplifiers [6]. As a simple example, let us discuss the following diagramm:



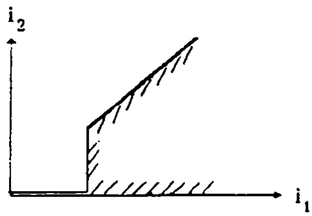
where we assume that the frequency of the voltage source is small as compared with the eigenfrequency of the rest of the circuit. As long as the voltage is small, one has a current i_1 through the relay which is too small to close the switch. But above some threshold, i_2 is different from zero:



In a circuit like



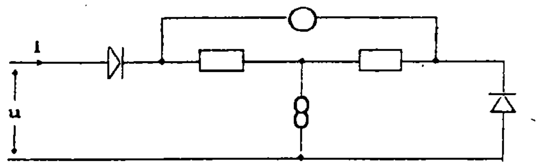
the possible states depend on various parameters, e. g. the frequency of the voltage source. They are described by the following diagram



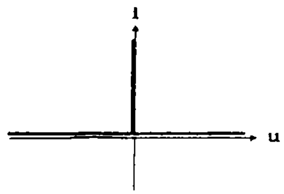
So they do not form a submanifold of $\mathbb{R}^2$, but a differential space. Especially if an alternating current is applied, it is advantageous to consider complex d^r- spaces. If diodes or other nonlinear elements are used, one is usually forced to work with C^r- functions for $r < \infty$. So one is naturally led to d^r- spaces.

Sasin [7] has shown that in a differential space, the exterior derivative cannot be defined on differential forms, but only on certain equivalence classes of differential forms. This is so, because there may be differential forms that vanish in all points of a differential space. But if the formal rules of exterior differentiation are applied, they would yield a form different from zero. In the theory of electrical circuits, such forms occur e. g. if ideal diodes with the following current / voltage diagramm are considered:

These forms occur also in circuits like [6]



where —— —— denotes a nullator (i. e. an element with $u = 0$ and $i = 0$) and | a norator (an element for which all points in the u / i plane describe possible states). The current / voltage diagramm of this circuit is



In the equations of motion [8] of such a cirquit, the 1-form $\alpha := u \cdot di$ occurs which is evidently zero. According to the formal rules of exterior differentiation, one would obtain $d\alpha = du \wedge di$ which is different from zero at the origin. Now Smale's theory of electrical circuits shows that these forms may simply be ignored in order to get the correct equations of motion. This is the engineer's argument for passing over to the above-mentioned equivalence classes.

3. d^r- and C^r- tensor fields

Kowalczyk [5] and Sasin [7] have defined smooth forms on a differential space. In the spirit of d^r- spaces, this concept must be generalized. First, we need the definition of a C^s- vector field (cf. also [1]):

Definition 2. A vector field X on a d^r- space (M, C) is a C^s- vector field ($s \le r$), if

$$\forall_{f \in C} \; Xf \in H_s := (sc^s(C))_M ,$$

where sc^s denotes the closure by C^s- functions $\mathbb{K}^n \longrightarrow \mathbb{K}$ (see axiom b) of definition 1). The elements of H_s will be called C^s- functions, and the H_s- module of the C^s- vector fields on (M, C) will be denoted by $\mathfrak{X}^s M$.

Next, we consider a d^r- space (M, C) over $\mathbb{K}$, and two numbers p, q $\in \mathbb{N}$. Then we define

$$T^{p,q}_x M := \{(x,\overset{1}{\sigma},\dots,\overset{p}{\sigma}, \overset{1}{X},\dots,\overset{q}{X}) \mid \overset{i}{\sigma} \in T^*_x M;\ \overset{k}{X} \in T_x M\}.$$

$T^{p,q} M_s$ denotes the d^r- space with the point set $\bigcup_{x\in M} T^{p,q}_x M$ and the d^s- structure generated by

$$\{f\circ\pi_0 \mid f \in C\} \cup \bigcup_{i=1}^{p} \{X\circ\pi_1 \mid X \in \mathfrak{X}^s\mathfrak{M}\} \cup \bigcup_{i=p+1}^{q} \{df\circ\pi_1 \mid f \in C\}.$$

Here the projections π_i are defined by

$$(x,\overset{1}{\sigma},\dots,\overset{p}{\sigma}, \overset{1}{X},\dots,\overset{q}{X}) \longmapsto x \quad \text{for } i = 0$$
$$(x,\overset{1}{\sigma},\dots,\overset{p}{\sigma}, \overset{1}{X},\dots,\overset{q}{X}) \longmapsto \overset{i}{\sigma} \quad \text{for } 1 \le i \le p$$
$$(x,\overset{1}{\sigma},\dots,\overset{p}{\sigma}, \overset{1}{X},\dots,\overset{q}{X}) \longmapsto \overset{i-p}{X} \quad \text{for } i > p.$$

This allows us to define d^s- tensor fields:

Definition 3. A $\mathbb{K}$-valued tensor field t over (M, C) is a d^s- tensor field, if the map

$$T^{p,q}M_s \longrightarrow (\mathbb{K},\ C^s(\mathbb{K},\ \mathbb{K}))$$

is smooth. Here $s \le r$ must be assumed. In the case $s = r$, the tensor field t is called smooth. If W is a Banach space and t a W-valued tensor field, the above condition has to be replaced by

$$T^{p,q}M_s \longrightarrow (W,\ C^s(W,\ W)).$$

Although this definition is rather complicated, the d^s- tensor fields are very convenient to work with, because one has [2], [3], [5]:

Lemma 1. Let (M, C) be a d^r- space over $\mathbb{K}$, and $s \in \mathbb{N}_0 \cup \{\infty, \omega\}$; $s \le r$. Then each W-valued (p, q)-tensor field may be locally written as

$$t = t^{k_1 \ldots k_p}_{i_1 \ldots i_q} X_{k_1} \otimes \ldots \otimes X_{k_p} \otimes df^{i_1} \otimes \ldots \otimes df^{i_q}.$$

Here $X_k \in \mathfrak{X}^s M$ and all sums are finite.

This lemma is a simple consequence of the definition of a smooth map of a differential space [2].

But the concept of a d^r- tensor is too restrictive: On an infinite dimensional manifold, not even the identity $T_x M \longrightarrow T_x M$ is a d^r- tensor for any r. This statement is independent of the generalization to $r \neq \infty$; it follows from the fact that the identity at $x \in M$ is a (1, 1)-tensor (or a vector valued 1-form) that can be written as

$$\text{(1)} \qquad id_{T_x M} = e_i \otimes \vartheta^i$$

for any Hamel base $\{e_i\}_{i \in I}$ of $T_x M$ and the covectors ϑ^i; $i \in I$ defined by $\vartheta^i(e_j) = \delta^i_j$. Here the summation may be over a non countable set I. But applied to any $X = X^j e_j \in T_x M$, this makes sense, because the summation over j is finite. Therefore

$$e_i \otimes \vartheta^i(X^j e_j) = e_i X^j \delta^i_j$$

contains only finitely many terms different from zero.

Clearly, (1) is not of the form required by lemma 1, as the summation is not finite, if $\dim M = \infty$. Therefore the identity is not a d^r- tensor.

In the same way, one shows that any metric on an infinite dimensional Hilbert space cannot be a d^r- tensor for any r. Similar difficulties arise for infinite dimensional symplectic or complex manifolds. Therefore one defines

Definition 4. A (p, q)-tensor field t on a d^r- space (M, C) is called C^s- tensor field ($s \leq r$), if locally (i. e. in some neighbourhood U of each $x \in M$) it can be written as

$$(2) \qquad t^{k_1 \ldots k_p}_{i_1 \ldots i_q} X_{k_1} \otimes \ldots \otimes X_{k_p} \otimes df^{i_1} \otimes \ldots \otimes df^{i_q}$$

with $t^{k_1 \ldots k_p}_{i_1 \ldots i_q} \in H_s$; $X_k \in \mathfrak{X}^s U$; $f^i \in C_U$. The summation is not necessarily finite. (Cf. the remark after equation (1)).

Note that in (2), the linear independence of the X^k and of the df^i is not required. So in general, the representation (2) is not unique.

This definition includes a large class of tensor fields. Trivially, all d^s- tensor fields are C^s, too. But also the metric of a Hilbert space, a symplectic form, a complex structure can be C^r- tensor fields, if they are sufficiently often differentiable. Also the identity can be a C^r- tensor field (but in general it is not C^r. E. g. consider a point x, where the C^r- vector fields do not span T_xM.)

It is immediately seen from (2) that a C^r- tensor field t of type (p, q) satisfies

$$(3) \qquad \begin{array}{l} t(u) \in H_s \\ \text{for all } u = (df^1, \ldots, df^p, Y_1, \ldots, Y_q) \\ \text{with } f^i \in C;\ Y_i \in \mathfrak{X}^s M. \end{array}$$

Under certain additional conditions (e. g. if d^r- spaces of constant differential dimension carry a base $(X_i)_{i \in I}$ of $\mathfrak{X}^s M$ such that $\forall_{i,j} [X_i, X_j] = 0$), condition (3) is also sufficient for t to be a C^s- tensor field.

The calculus of C^s- fields is completely analogous to the one on d^s- fields. Here we only repeat the definition of the exterior derivative: For $p \in \mathbb{N}$ define $\mathfrak{M}^p_s(M)$ as the set of all C^s- p-forms α such that each $x \in M$ has a neighbourhood U and a set of C^{s+1}- functions $\alpha_{i_1 \ldots i_{p-1}}$, $f^{i_1}, \ldots, f^{i_{p-1}}$ satisfying

$$\alpha|_U := d\alpha_{i_1 \ldots i_{p-1}} \wedge df^{i_1} \wedge \ldots \wedge df^{i_{p-1}}$$

and

$$\alpha_{i_1 \ldots i_{p-1}} \wedge df^{i_1} \wedge \ldots \wedge df^{i_{p-1}} = 0.$$

Put $\mathfrak{M}^o_s(M) = \{0\}$, where 0 is the zero function $M \longrightarrow \mathbb{K}$. Then one considers, in the usual way, the equivalence classes $[\sigma] := \{\sigma+\omega \mid \omega \in \mathfrak{M}^p_s(M)\}$, where σ is any C^s- p-form. The set of these equivalence classes is denoted by $A^p_s(M)$.

Theorem 1. There is precisely one operator

$$d: A^p_s(M) \longrightarrow A^{p+1}_{s-1}(M); \; p \geq 0; \; s \geq 1$$

such that the following conditions are satisfied for all $\alpha, \beta \in A^p_s(M)$ and $\gamma \in A^q_s(M)$:

(a) $d(\alpha + \beta) = d\alpha + d\beta$

(b) $d \circ d = 0$

(c) $d(\alpha \wedge \gamma) = (d\alpha) \wedge\gamma + (-1)^p\alpha \wedge d\gamma$

(d) If $p = 0$ (i.e. $\alpha \in H_s$), then $d[\alpha] = [d\alpha]$.

The proof can be taken verbatim from [2] or from [7], where it is given for d^r- and for d^∞- tensor fields, respectively.

4. Remarks on the tangent bundle

It is well known that the tangent bundle of a d^r- space of constant differential dimension n is a vector bundle in the sense of differential spaces (i. e. the local trivializations and their inverses are smooth), if the dimension n is finite. This is no longer true, if n is infinite. Gerstner [4] has proved the following two lemmas:

Lemma 2. Let $M := \ell^2(\mathbb{R})$ be the Hilbert space with scalar product $<a, b> := \sum a_k b_k$ for $a = (a_k)$ and $b = (b_k)$. Define C as the d^∞- structure on M generated by $C^\infty(M, \mathbb{R})$; further set $F := \ell^2(\mathbb{R})$ and let its d^o- structure $\underline{F}$ be generated by the dual space (space of all continuous linear forms on F). Then a bundle map (ρ, U) of the tangent bundle cannot be defined in any $x \in M$ such that ρ and ρ^{-1} are smooth. (Here $\rho: TU \longrightarrow U\times F$.)

The main point in the proof of this lemma is that the usual definition of the Cartesian product does not contain enough functions to include the pullback of the differential $d<a, a>$. Even if the differential structure $\underline{F}$ of F is enlarged so that it contains all continuous functions $F \longrightarrow \mathbb{R}$, ρ^{-1} is not smooth:

Lemma 3. Lemma 2 holds true if $\underline{F}$ is the set of all continuous functions $F \longrightarrow \mathbb{R}$.

So in order to make TM a vector bundle, it would be necessary to take out some of the functions of its differential structure. But this is definitely not what one wants, as its differential structure contains only the functions generated by $(p, X) \longmapsto f(p)$ and $(p, X) \longmapsto Xf(p)$ for all $f \in C$. Therefore, the notion of vector bundles seems not to be suitable for tangent bundles of differential spaces. But, of course, all tangent bundles are bundles in the usual sense.

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