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Michał Heller

ALGEBRAIC FOUNDATIONS OF THE THEORY OF DIFFERENTIAL SPACES

In physics there is an urgent necessity to base some geometric models of physical phenomena on "sufficiently non-smooth" generalizations of the differentiable manifold concept. The theory of differential spaces might provide physics with such a possibility. Algebraic foundations of this theory are discussed. Differential space in the sense of Sikorski turns out to be a "geometric refinement" of the algebraic concept of ringed space, and it naturally generalizes the real manifold concept. However, it proves to be inadequate to deal with complex analytic manifolds. Mostow's theory of differential spaces is a geometric version of the theory of structured spaces (essentially, sheaves of germs of functions on a topological space). It is shown that to naturally generalize the concept of complex analytic manifold one must suitably adapt Mostow's concept of differential space.

0. Introduction

There is a widespread conviction that differential geometry must be constrained to deal with spaces that are "smooth enough", in practice with spaces which are locally diffeomorphic to a Euclidean space. We have learned to live with this view, but in fact it is a cumbersome constraint. The world around us is far from being "smooth enough", and if we would like to have its differentially geometric model, we should ardently look for "sufficiently non-smooth" generalizations of the differentiable manifold concept. It turns out that for several years there are in circulation among mathematicians quite a few such generalizations (for instance [1] - [9], see also [10] and original works quoted therein), although some of them in a seminal state of their development. In almost all of these works it is precisely the assumption of the local resemblance to a Euclidean space that is rejected, and the structure thus obtained is usually called a *differential space*. When one starts to apply these methods to physical problems, they seemed to work well ([12] - [13]); especially encouraging results have been obtained in the field of classical space-time singularities ([14] - [15]). In such circumstances the question is unavoidable: Why do these methods work? Our hitherto views on differential geometry have turned out to be strongly biased. Which is the source of this bias? The aim of the present essay is to attempt at elucidating these question marks.

In the introduction to a modern book on Clifford algebra and geometry one reads: "...the fusion of algebra with geometry is so fundamental that one could well say, 'Geometry without algebra is dump! Algebra without geometry is blind!'" ([16], p.xii). Already in the framework of differentiable manifold theory one can see that algebra is present in differential geometry at least on three levels. The first level is connected with the existence of differentiable manifolds either having an algebraic structure (Lie groups) or admitting actions of Lie groups (fibre bundles). One could also consider here problems related to the existence of a differentiable structure on such objects such as the group of diffeomorphisms of a differentiable manifold, the isometry group of a Riemann manifold, and so forth. The second level refers to methods of the algebraic topology such as Cech or de Rahm cohomology groups, characteristic classes of various types, Sullivan's minimal models, etc. In the following we shall be interested in the third level concerned with using the algebraic language in defining geometrical objects themselves. It was clear from the very beginning of the development of the differential space theory that sooner or later one would have to turn to algebraic foundations of this theory. Some of the above mentioned concepts of differential spaces have already in themselves an algebraic flavour: the structures in question are defined in terms of a ring or a sheave of functions over a set rather than by directly structuring the set itself. A pioneering work of Palais [17], in the field called by him algebraic differential topology, has turned out to be extremely helpful. In fact, his investigations should be placed somewhere between algebraic geometry and differential geometry, and although he never mentions differential spaces, his results can be naturally extended to cover this case. In the present essay I shall amply make use of Palais' work. Proofs of all statements non-documented with references to suitable sources should be looked for in the book by Palais.

In my analyses I shall focus on two concepts of differential spaces: that proposed by Sikorski ([3] and [4], see also [18]) and that introduced by Mostow [9]. As we shall see, these two concepts are closely interrelated. Other versions of differential spaces are studied in [10].

In section 1, some basic algebraic concepts are recalled. To consider rings of functions on certain sets, as it is the case in both Sikorski and Mostow theories, is not a limitation since any abstract (strictly semi-simple) commutatiye ring can be made into a ring of functions via the Gelfand representation. Differential space in the sense of Sikorski is a "geometric refinement" of the algebraic concept of ringed space. Ringed spaces are briefly reviewed in section 3, and Sikorski's theory is analyzed in section 4. Analogously, Mostow's theory of differential spaces is a geometric version of the theory of structured spaces (essentially, sheaves of germs of functions on a topological space); this is dealt with in section 5. Our main conclusions are summarized in section 6. Sikorski's concept of differential space nicely generalizes the notion of real smooth manifold. It is shown that to suitably generalize the concept of complex analytic manifold one must turn to a Mostow-like generalization.

1. Gelfand representation of algebras

In the following $\mathbb{K}$ will always denote a fixed commutative field. By a non-zero algebra $\mathcal{A}$ we mean a commutative $\mathbb{K}$-algebra with identity such that $\dim_{\mathbb{K}}\mathcal{A} \geq 1$. The zero algebra 0 is defined to be the trivial $\mathbb{K}$-algebra consisting of the single element $0\in\mathbb{K}$ which is regarded to be the zero element and the identity element, simultaneously. In what follows, by an algebra we shall mean a non-zero algebra or the zero algebra. An algebra $\mathcal{A}$ is the zero algebra if and only if its identity and zero elements coincide. By a homomorphism from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ we shall mean a usual algebra homomorphism which maps the identity of $\mathcal{A}$ into the identity of $\mathcal{B}$ if $\mathcal{A}$ is a non-zero algebra, and the zero homomorphism if $\mathcal{A}$ is the zero algebra. In this way, the category of commutative $\mathbb{K}$-algebras with identity has been defined.

Let $\mathcal{A}^*$ be the dual of $\mathcal{A}$ as a vector space over $\mathbb{K}$, then $\mathcal{A}^\wedge\subset\mathcal{A}^*$ will denote the "algebraic dual" of $\mathcal{A}$, i.e. the set of all homomorphisms $\phi: \mathcal{A} \longrightarrow \mathbb{K}$. If $\mathbb{K}^M$ denotes the algebra of all $\mathbb{K}$-valued functions on a set M with the algebraic operations defined in a pointwise manner, one has a canonical mapping Ev: $M \longrightarrow (\mathbb{K}^M)^\wedge$ such that $Ev(p): \mathbb{K}^M \longrightarrow \mathbb{K}$, $p\in M$, is defined by $f \longmapsto f(p)$, $f\in\mathbb{K}^M$.

A homomorphism $\rho: \mathcal{A} \longrightarrow \mathbb{K}^M$ is said to be a *representation* of $\mathcal{A}$. It will be assumed that ρ *separates points* in M, i.e. that given two distinct points p and q of M there exists x in $\mathcal{A}$ such that $\rho(x)(p) \neq \rho(x)(q)$. In fact, it is not a limitation since we can always define an equivalence relation # such that the representation $\rho^{\#}: \mathcal{A} \longrightarrow \mathbb{K}^{M^{\#}}$, where $M^{\#} = M/\#$, will separate points in $M^{\#}$.

Let $\rho: \mathcal{A} \longrightarrow \mathbb{K}^M$ be a representation of $\mathcal{A}$ and $T\subset M$. A representation $\rho|T: \mathcal{A} \longrightarrow \mathbb{K}^T$ of $\mathcal{A}$, defined by $(\rho|T)(x) = \rho(x)|T$, $x\in\mathcal{A}$, is called a *subrepresentation* of $\mathcal{A}$ (defined by T).

For any representation $\rho: \mathcal{A} \longrightarrow \mathbb{K}^M$ we have an *associated* map ψ: M $\longrightarrow \mathcal{A}^\wedge$ defined by p $\longmapsto \psi_p$, i.e. ψ_p: $\mathcal{A} \longrightarrow \mathbb{K}$ which is given by $\psi_p(x) = \rho(x)(p)$. One can easily see that $\psi = Ev \circ \rho$. Since ρ is a representation, ψ is injective. Therefore, M can be naturally identified with a subset of $\mathcal{A}^\wedge$.

Every algebra $\mathcal{A}$ admits a representation as a function algebra. Such a representation can be achieved with the help of the so-called Gelfand representation. A representation $\rho^{\mathcal{A}}$: $\mathcal{A} \longrightarrow \mathbb{K}^{\mathcal{A}^\wedge}$ of $\mathcal{A}$ is called the *Gelfand representation* of $\mathcal{A}$ if it is given by $\rho^{\mathcal{A}}(x)(\phi) = \phi(x)$, $x \in \mathcal{A}$, $\phi \in \mathcal{A}^\wedge$.

Two representations ρ_1: $\mathcal{A} \longrightarrow \mathbb{K}^M$ and ρ_2: $\mathcal{A} \longrightarrow \mathbb{K}^N$ of $\mathcal{A}$ are said to be *equivalent* if there is a bijection s: M $\longrightarrow$ N such that $\rho_2 = K^s \circ \rho_1$, where K^s: $\mathbb{K}^N \longrightarrow \mathbb{K}^M$ is a homomorphism induced by s, i.e. given by $g \longmapsto g \circ s$, $g \in \mathbb{K}^N$.

A representation ρ of an algebra $\mathcal{A}$ is said to be a *universal representation* of $\mathcal{A}$ if every representation of $\mathcal{A}$ is equivalent to a subrepresentation of ρ. One can show [17] that the Gelfand representation of an algebra $\mathcal{A}$ is universal. It is also natural in the sense of the category theory.

An ideal M in an algebra $\mathcal{A}$ is said to be *strictly maximal* if dim $(\mathcal{A}/M) = 1$. In the following the set of all strictly maximal ideals of $\mathcal{A}$ will be denoted by Spec $\mathcal{A}$.

The mapping

$$\mathcal{A}^\wedge \longrightarrow \text{Spec } \mathcal{A}$$

defined by $\phi \longmapsto \ker \phi$ is a bijection. Indeed, if $\phi \in \mathcal{A}^\wedge$, ϕ is linear and surjective, therefore ker ϕ has codimension one in $\mathcal{A}$, and consequently $\ker\phi \in \text{Spec}\mathcal{A}$. And *vice versa*, let $M \in \text{Spec}\mathcal{A}$. Since $\mathcal{A}/M$ is one-dimensional, there is a unique isomorphism $\Psi: \mathcal{A}/M \longrightarrow \mathbb{K}$. If Π: $\mathcal{A} \longrightarrow \mathcal{A}/M$ is a canonical projection, one has $\phi = \Psi \circ \Pi \in \mathcal{A}^\wedge$ and $\ker \phi = M$.

The kernel of Gelfand representation $\ker(\rho^{\mathcal{A}})$ is called the *strict radical* of $\mathcal{A}$ and will be denoted by $Rad_{\mathcal{A}}(0)$. If $Rad_{\mathcal{A}}(0)=0$, $\mathcal{A}$ is said to be *strictly semi-simple*. In such a case the Gelfand representation of $\mathcal{A}$ is faithful. We can see that every strictly semi-simple algebra can be treated as an algebra of functions on a subset of $\mathcal{A}^\wedge$.

Now, we shall discuss topology on $\mathcal{A}^\wedge$. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. By taking the the Gelfand representation of $\mathcal{A}$ we can regard $\mathcal{A}$ as an algebra of $\mathbb{K}$-valued functions on $\mathcal{A}^\wedge$ (if the Gelfand representation is not faithful we can regard $\mathcal{A}/\mathrm{Rad}_{\mathcal{A}}(0)$ instead of $\mathcal{A}$). Now, there is an important correspondence. On the one hand, to each $\mathcal{B}\subseteq\mathcal{A}$ there corresponds the largest subset $V(\mathcal{B})$ of $\mathcal{A}^\wedge$ such that all functions of $\mathcal{B}$ vanish on $V(\mathcal{B})$ and, on the other hand, to every subset N of $\mathcal{A}^\wedge$ there corresponds the subset I(N) of all functions of $\mathcal{A}$ that identically vanish on N.

Subsets of $\mathcal{A}^\wedge$ of the form $V(\mathcal{B})$ are called *Z-closed;* they are closed sets for a T_1-topology called the *Zariski topology* (or *Z-topology*, for short) of $\mathcal{A}^\wedge$. It can be shown that subsets of $\mathcal{A}^\wedge$ of the form $\mathcal{A}^\wedge_x := \{\phi\in\mathcal{A}^\wedge: \rho^{\mathcal{A}}(x)(\phi) \neq 0\}$, $x\in\mathcal{A}$, called *basic open sets,* constitute a base for the Z-topology of $\mathcal{A}^\wedge$. The Z-topology of $\mathcal{A}^\wedge$ is the weakest topology of $\mathcal{A}^\wedge$ for which all mappings $\hat{x}: \mathcal{A}^\wedge \longrightarrow \mathbb{K}$, given by $\rho^{\mathcal{A}}(x)\in\mathbb{K}^{\mathcal{A}^\wedge}$, are continuous provided that $\mathbb{K}$ is equipped with its weakest T_1-topology. If $\mathbb{K}$ is a topological field, the weakest topology, in which every function $\hat{x}: \mathcal{A}^\wedge \longrightarrow \mathbb{K}$ is continuous, is called the W-topology of $\mathcal{A}^\wedge$. It is at least as strong as the Z-topology of $\mathcal{A}^\wedge$ (all these statements are proved in [17], sections 1.3-1.4; see also [19], p.20-21).

Under the canonical bijection $\mathcal{A}^\wedge \longrightarrow \mathrm{Spec}\,\mathcal{A}$ subsets $V(\mathcal{B})$ of $\mathcal{A}^\wedge$ correspond to the subsets $V(\mathcal{B}) := \{\mathcal{M}\in\mathrm{Spec}\,\mathcal{A}: \mathcal{B}\subset\mathcal{M}\}$ of Spec $\mathcal{A}$. Let N be a subset of $\mathcal{A}^\wedge$. The sets of the form $I(N) := \bigcap \{\ker\phi: \phi\in N\}$ are called *strict radical ideals* of $\mathcal{A}$. There is a bijective correspondence between Z-closed subsets of $\mathcal{A}^\wedge$ and strict radical ideals of $\mathcal{A}$.

2. Ringed spaces

Crucial structures in our considerations will be ringed spaces. First, we define a *structural ring* (over $\mathbb{K}$) of a set M to be a subalgebra $\mathcal{A}$ of the algebra $\mathbb{K}^M$ of $\mathbb{K}$-valued functions on M which separate points in M. Then a *ringed space* (over $\mathbb{K}$) is a pair $(M,\mathcal{A})$ where M is any set and $\mathcal{A}$ is a structural ring of M.[1]

Let $(M,\mathcal{A})$ be a ringed space. The *Z-topology* on M is defined to be the topology of M induced from the Z-topology of $\mathcal{A}^\wedge$. The *basic open sets*, i.e. sets of the form $M_f := \{p\in M: f(p)\neq 0\}$, $f\in\mathcal{A}$, constitute a base for the Z-topology of M.

If $(M,\mathcal{A})$ is a ringed space and $N\subset M$, the pair $(N, \mathcal{A}|N)$, where $\mathcal{A}|N$ is the algebra of functions being restrictions to N of those of $\mathcal{A}$, is said to be a *ringed subspace* of $(M,\mathcal{A})$.

Let us notice that since the functions of the structural ring $\mathcal{A}$ separate points in M, the mapping $Ev: M \longrightarrow \mathcal{A}^\wedge$ is injective. This allows us to regard Ev as an identification of M with a subset of $\mathcal{A}^\wedge$. In these circumstances the Gelfand representation $\rho^{\mathcal{A}}$ of $\mathcal{A}$ is given by $f = \rho^{\mathcal{A}}(f)|M\subset\mathcal{A}^\wedge$, where $f\in\mathcal{A}$. If Ev is surjective, one has $M = \mathcal{A}^\wedge$ and the ringed space $(M,\mathcal{A})$ is called *complete*. A complete ringed space $(\bar{M},\bar{\mathcal{A}})$ such that M is Z-dense in $\bar{M}$ is called a *completion* of $(M,\mathcal{A})$. One can show that if $(M,\mathcal{A})$ is a ringed space then $(\mathcal{A}^\wedge,\mathcal{A})$ is its completion. It can be proved that the category of complete ringed spaces over $\mathbb{K}$ is isomorphic to the dual of the category of semi-simple algebras over $\mathbb{K}$.

Later on we shall need the following concepts. Let $(M,\mathcal{A})$ be a ringed space. A function $h: M \longrightarrow \mathbb{K}$ is said to be *regular* if it can be written in the form f/g, where $f,g\in\mathcal{A}$ and $g\neq 0$ everywhere. The ring of regular functions on M is denoted by $\mathcal{A}_{reg}$. A ringed space $(M,\mathcal{A})$ is called a *regular ringed space* if $\mathcal{A} = \mathcal{A}_{reg}$. The ringed space $(M,\mathcal{A}_{reg})$ is said to be the *regularization* of $(M, \mathcal{A})$.

One can easily see that a C^k-manifold can be regarded as a ringed space. Indeed, let $\mathbb{K} = \mathbb{R}$ and M be a C^k-(paracompact) manifold, where k is any non-negative integer, ∞, or ω. If $C^k(M)$ is the family of real C^k-functions on M, the manifold in question can be considered as a ringed space $(M,C^k(M))$ with $C^k(M)$ as its structural ring. It can be shown that the manifold topology of M is the W-topology of the ringed space

[1] In the literature the name *ringed space* is usually reserved for what is called by Palais *local ringed space* (see, footnote 3).

$(M, C^k(M))$, and if $k \neq \omega$ it is also the Z-topology of $(M, C^k(M))$. In general, the Z-topology is weaker than the W-topology (of the same ringed space).

If M is a compact manifold, then the ringed space $(M, C^k(M))$ is complete.

3. Differential spaces in the sense of Sikorski

The last example is crucial. One could reverse the reasoning and ask which condition should be imposed upon a ring of functions defined on a set M to change this set into a manifold or some its workable generalization. From the answer to this question various definitions of differential spaces arise. The essential idea consists in selecting a family of functions on M in such a way that it would be possible to treat it, *ex definitione*, as the family of smooth functions on M. If the family of such functions is a ring, almost everything else which is needed is, as we have seen above, nicely done by the algebraic properties of ringed spaces.

One of the possible ways was chosen by Sikorski [3], [4], [18]. He decided to work with a non-empty family C of real functions on a set M with the weakest topology τ_C in which functions of C are continuous. The family C was supposed to be (1) closed with respect to localization, and (2) closed with respect to superposition with smooth functions on the Euclidean space.

A function f, defined on $A \subset M$, is said to be a *local C-function* if, for every $p \in A$ there is a neighbourhood B in the topological space (A, τ_A), where τ_A is the topology induced in A by τ_C, and a function $g \in C$ such that $g|B = f|B$. The set of all local C-functions is denoted by C_A. One obviously has $C \subset C_M$. If $C = C_M$ the family C is said to be *closed with respect to localization*.

Let C be a family of real functions on M. It is said to be *closed with respect to superposition with smooth Euclidean functions* if for any $n \in \mathbb{N}$ and any function $\omega \in C^\infty(\mathbb{R}^n)$, $f_1, \ldots, f_n \in C$ implies $\omega \circ (f_1, \ldots, f_n) \in C$.

A family C of real functions on M satisfying conditions (1) and (2) is called a *differential structure* on M, and it is treated, *ex definitione*, as the family of smooth functions on M. A pair (M,C), where C is a differential structure on M, is called a *differential space (in the sense of Sikorski)*. Let us notice that on the strength of condition (2) C is a (linear) ring, and consequently a differential space (M,C) is also a ringed space.

It turns out that the notion of differential space is both a vast generalization of the ordinary manifold concept[2] and a workable tool: large parts of differential geometry can be done entirely in terms of it (see [18]). Let us briefly consider the two conditions defining this concept.

The closure with respect to localization is a natural condition. It ensures the global meaning to all local constructions. This can be seen from the following. Let (M,C) be a ringed space, where M is *a priori* given a certain topology. Let $\{U_i\}$, $i \in I$, be an open covering of M in this topology. By restricting C to each U_i one obtains the presheaf of rings on M (for details see below, section 4). To connect rings over all open sets U_i into a global structure, i.e. to change the presheaf into a sheaf, one must assume that C is closed with respect to localization.

Moreover, condition (1) is satisfied by smooth functions on manifolds. Without having it satisfied the differential space concept could not be a generalization of that of smooth manifold.

The closure with respect to superposition with smooth Euclidean functions is also a useful postulate. It makes the topology τ_C of a differential space (M,C) identical with the Zariski topology which, in the case when (M,C) is a manifold, will coincide with the manifold topology. Moreover, as we have seen, condition (2) guarantees that a family C of real functions is a ring. This, however, could be achieved in many

[2] By postulating that a differential space should be locally diffeomorphic to an open subset of a Euclidean space, the differential space changes into a smooth manifold.

different ways. For instance, in place of demanding the closure with respect to superposition with smooth Euclidean functions one could postulate the closure with respect to superposition with "Euclidean polynomials". In such a case the topology would in general fail to be a Zariski topology, but the way seems to be worthwhile to try since it clearly drifts towards the algebraic geometry.

We should notice that if we assume $\mathbb{K} = \mathbb{C}$, the Sikorski construction in general does not work. The reason is the following. Let H be a complex analytic manifold. We might try to construct a ringed space by considering the pair $(H, \mathfrak{C})$ where $\mathfrak{C}$ is the family of holomorphic complex valued functions on H. As it is well known, if $f \in \mathfrak{C}$, and f reaches the maximum at $p \in \text{int } H$, then $f = \text{const}$ on H ([20], p. 307). Therefore, if H is compact, the family $\mathfrak{C}$ consists only of constant functions which of course ruins the idea. To circumvent this difficulty one must take into considerations sheaves of algebras over a given set. This will be done in the next section.

4. Structured spaces and Mostow differential spaces

Let M be a topological space and $\mathfrak{A}$ the category of commutative $\mathbb{K}$-algebras with identity. By topM we shall denote the category whose objects are open subsets of M and morphisms inclusion maps between them. Therefore, for any, $U, V \in$ topM, one has

(a) $\text{Hom}(U,V) = \emptyset$ iff $V \subset U$,

(b) $\text{Hom}(U,V) = \{i^U_V\}$ iff $V \subset U$,

where $i^U_V: V \hookrightarrow U$ is the inclusion map. By a *presheaf* of algebras on M we mean a system $S = \{S(U), \rho^U_V: U,V \in \text{topM}, V \subset U\}$, where $S(U) \in \mathfrak{A}$ and $\rho^U_V \in \text{Hom}_{\mathfrak{A}}(S(U),S(V))$ satisfying the following conditions:

(P1) $S(\emptyset) = 0$, where $\emptyset$ is the empty set,

(P2) for any $U \in$ topM, ρ^U_U is the identity of $S(U)$, (P3) for any U, V, W $\in$ topM, if $W \subset V \subset U$, then $\rho^U_W = \rho^V_W \rho^U_V$.

A presheaf S can be equivalently defined as a contravariant functor $\mathcal{S}$ from the category topM to the category $\mathfrak{A}$ which assigns the zero algebra to the empty set. In such a case one has $S(U) = \mathcal{S}(U)$ and $\rho^U_V = \mathcal{S}(i^U_V)$, for U, V $\in$ topM, $V \subset U$.

A presheaf $S = \{S(U), \rho^U_V: U,V \in \text{topM}, V \subset U\}$ of algebras on M is a *sheaf* if it satisfies the following additional conditions:

(S1) suppose that $\{V_t\}$ is an open covering of U; if $a,b \in S(U)$ and $\rho^U_{V_t}(a) = \rho^U_{V_t}(b)$, for all t, then a = b,

(S2) suppose that $\{V_t\}$ is an open covering of U; if there exist elements a_t such that $a_t \in S(V_t)$ for each t, and $\rho^{V_t}_{V_t \cap V_s}(a_t) = \rho^{V_s}_{V_t \cap V_s}(a_s)$ for each t, s, then there is an element $a \in S(U)$ such that $\rho^U_{V_t}(a) = a_t$ (notice that condition (S1) implies that a is unique).

Now, we define the *sheaf of germs of* $\mathbb{K}$*-valued functions on* M, $\Phi_M = \{\Phi_M(U), r^U_V: U, V \in \text{topM}, V \subset U\}$ as the following sheaf of algebras. For any non-empty U $\in$ topM, $\Phi_M(U)$ is the algebra of all $\mathbb{K}$-valued functions on U (under the pointwise algebra operations). If U, V $\in$ topM, $V \subset U$ and $V \neq \emptyset$, then the homomorphism $r^U_V: \Phi_M(U) \longrightarrow \Phi_M(V)$ is the restriction map $\alpha \longmapsto \alpha|V$. It can be easily seen that the above assumptions define the sheaf Φ_M uniquely.

Let $S = \{S(U), \rho^U_V: U,V \in \text{topM}, V \subset U\}$ be a sheaf of algebras on M. We say that a sheaf $S' = \{S'(U), \rho'^U_V: U,V \in \text{topM}, V \subset U\}$ of algebras on M is a *subsheaf* of S if, for any U, V $\in$ topM, $V \subset U$, the algebra $S'(U)$ is a subalgebra of $S(U)$, $\rho^U_V(S'(U)) \subset S'(V)$ and the homomorphism ρ'^U_V is the restriction of ρ^U_V. By a *sheaf of function algebras (functional structure)* on M we mean a subsheaf $\mathcal{F}$ of the sheaf

Φ_M such that $\mathcal{F}(U)$, for each non-empty $U \in \text{top}M$, is a non-zero algebra, i.e. $\mathcal{F}(U)$ contains all constant $\mathbb{K}$-valued functions on U.

Let $\mathcal{F}$ be a sheaf of function algebras on M. For any $p \in M$, we define the *stalk* $\mathcal{F}_p$ of $\mathcal{F}$ at p in the following way. Let us consider the family $\mathcal{F}(.,p) = \bigcup \{\mathcal{F}(U): U \in \text{top}M, \; p \in M\}$. If $\alpha \in \mathcal{F}(.,p)$, the domain of α will be denoted by $\text{dom}(\alpha)$. We define the relation $\simeq_p$ in $\mathcal{F}(.,p)$ by: $\alpha \simeq_p \beta$ if there is an open neighbourhood U of p in M such that $U \subset \text{dom}(\alpha) \cap \text{dom}(\beta)$ and $\alpha|_U = \beta|U$. Evidently, $\simeq_p$ is an equivalence relation. For any $\alpha \in \mathcal{F}(.,p)$, the *germ* of α at p is defined as the equivalence class α_p of α with respect to the relation $\simeq_p$. The *stalk* $\mathcal{F}_p$ is defined to be the set $\{\alpha_p: \alpha \in \mathcal{F}(.,p)\}$. Moreover, $\mathcal{F}_p$ can be regarded as an algebra under the following operations

$$\alpha_p + \beta_p := (\alpha' + \beta')_p,$$

$$\alpha_p\beta_p := (\alpha' \beta')_p \quad \text{and} \quad \lambda(\alpha_p) := (\lambda\alpha)_p,$$

where $\alpha, \beta \in \mathcal{F}(.,p)$, $\lambda \in \mathbb{K}$ and $\alpha', \beta' \in \mathcal{F}(.,p)$ are chosen in such a way that $\alpha'_p = \alpha_p$, $\beta'_p = \beta_p$ with $\text{dom}(\alpha') = \text{dom}(\beta')$. Obviously, every $\mathcal{F}_p$ is a non-zero algebra. More generally, one can define the stalk S_p, $p \in M$, for any sheaf S of algebras on M; in such a case S_p is also an algebra (not necessarily non-zero).

Following Hochschild [23], the pair $(M,\mathcal{O}_M)$, where M is a topological space and $\mathcal{O}_M$ a sheaf of function algebras (a functional structure) on M will be called the *(functional) structured space* (and for brevity it will also be denoted by M).[3] Let M and N be structured spaces. A continuous function $f: M \longrightarrow N$ is said to be a *structured map* from M to N if, for any non-empty $U \in \text{top}N$ and $\alpha \in \mathcal{O}_N(U)$, one has $\alpha \circ f \in \mathcal{O}_M(f^{-1}U)$. Clearly, structured spaces (as objects) together with structural maps (as morphisms) form the category called the *category of structured spaces*.

Let $(M,\mathcal{A})$ be a ringed space. By equipping M with the

[3] Such a notion is called by Palais a *local ringed space* [17].

Z-topology and, for each Z-open subset U of M, defining $\mathcal{O}_M(U) := \{f \in \mathbb{K}^U : \forall\, p \in U,\ \exists\, V \in \mathrm{top}_Z M,\ \exists\, g \in \mathcal{A},\ p \in V \subset U,\ f|V = g|V\}$ one obtains the structured space $(M, \mathcal{O}_M)$ called the *associated structured space*.

The structured space notion is a common generalization of such concepts as: smooth manifold, real analytic manifold, complex manifold, algebraic variety, differential space. For instance, every differential space M can be regarded as a structured space $(M, \mathcal{O}_M)$ with the functional structure given by the sheaf $\mathcal{O}_M$ of germs of smooth real functions on M. Moreover, the categories of smooth manifolds, real analytic manifolds, etc. are full subcategories of the category of structural spaces.

In particular, by using structured spaces instead of ringed spaces we can overcome difficulties met in our attempt to construct a complex analytic differential space (end of section 3). Indeed, let H be a complex analytic manifold and $\mathcal{O}_H(U)$ the algebra of holomorphic functions on U, where U is an open subset of H. Evidently, the assignment $U \longrightarrow \mathcal{O}_H(U)$ defines a sheaf $\mathcal{O}_H$ of function algebras on H. Therefore, any complex analytic manifold can be represented as a structured space $(H, \mathcal{O}_H)$. Basing on this fact, one can construct a complex differential space in an analogous manner to what has been done in the preceding section in the case of real differential spaces. To this end we could adapt a construction proposed by Mostow (1979). A *differential space in the sense of Mostow* is a topological space M together with a sheaf C_M of germs of continuous (in the assumed topology) $\mathbb{K}$-valued functions on M (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), called *smooth functions*, satisfying the following closure condition: for any $n \in \mathbb{N}$, if $f_1, \ldots, f_n \in C_M(U)$, where U is an open subset of M, and g is a smooth Euclidean function on $\mathbb{R}^n$ (resp. a holomorphic function on $\mathbb{C}^n$), then $g \circ (f_1, \ldots, f_n) \in C_M(U)$ (Mostow originally considered only the real case).

Let us notice that we do not need to assume the closure with respect to localization since it is contained in axiom (S2) of the sheaf definition. Therefore, with every differential space in the sense of Sikorski is associated a (real) differential space in the sense of Mostow. Indeed, a differential space in the sense of Sikorski (M,C) is a ringed space; by taking its associated structured space we obtain the differential space in the sense of Mostow. Relationships between Sikorski and Mostow differential spaces were studied in [10].

5. Concluding remarks

Since the rapid development of algebraic geometry there is no doubt that geometry can be given a sound algebraic foundations. From the work of Palais (1981) it clearly follows that it is equally true as far as differential geometry is concerned. By taking any algebra and, if necessary, representing it with the help of Gelfand representation as a functional algebra, we can construct algebraic structures such as ringed spaces or structured spaces which in many respects behave like geometric structures (one can algebraically define vector tangent and cotangent spaces to them, etc.). Moreover, many authentic differentially geometric objects such as real and complex manifolds turn out to be special instances of such structures.

This suggests a nice way of generalizing the standard differential geometry. One should take a suitable algebraic structure (ringed space or structured space), enrich it with the help of axioms which would make this structure more "geometrically flexible", and try to implement known differentially geometric procedures. As we have seen, Sikorski's theory of differential spaces naturally arises according to this method when one starts from the geometric concept of ringed space, and Mostow's theory of differential spaces when one begins with structured spaces. In fact, both authors worked with no help of algebraic considerations, and they should be praised for their correct intuitions.

Sikorski's theory is simpler than that of Mostow and proves to be sufficient as far as a generalization of the real manifold concept is concerned. However, it is inadequate to deal with complex analytic manifolds. In section 4 we have shown that it is Mostow's theory that could be naturally adapted to this case.

A nice thing with Sikorski's theory is that one can work entirely with the corresponding ring of functions, (almost entirely) forgetting about the set on which these functions are defined. This follows from the fact that if (M,C) is a differential space in the sense of Sikorski, M can be regarded as a subset of C^, the algebraic dual of C, and that there is a canonical bijection between C^ and Spec C, the set of strictly maximal ideals of C.

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VATICAN ASTRONOMICAL OBSERVATORY,

V-00120 VATICAN CITY STATE

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