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A NOTE ON DISTRIBUTIVITY IN ORTHOPOSETS

It is well-known that distributive ortholattices are Boolean (i.e., they fulfil the condition $a \wedge b = 0 \Rightarrow a'$). In this note we formulate some distributivity-like condition valid in Boolean orthoposets and prove that Boolean ω-orthocomplete poset has to be orthomodular. Our results generalize results of Klukowski [2,3] and, also, might find application in the axiomatics of quantum theories (see [1,5]).

Notions and results

Let us first review basic notions as we shall use them throughout the paper.

1. Definition. An *orthoposet* is a triple $(P,\le,')$ such that
(1) $(P,\le)$ is a partially ordered set with a least element, 0, and greatest element 1,
(2) $' : P \longrightarrow P$ is a *orthocomplementation*, i.e., (i) $a''=a$, (ii) $a\le b \Rightarrow b'\le a'$, (iii) $a\wedge a' = 0$ for every $a,b \in P$.

An orthoposet $(P,\le,')$ is called an *ω-orthocomplete poset* if $a\vee b$ exists in P for every $S \subset P$ such that $s_1\le s_2'$ for any pair $s_1, s_2 \in S$.

Further, an ω-orthocomplete poset is called *orthomodular* if $b = a\vee(b\wedge a')$ for every $a,b \in P$ such that $a\le b$.

An orthoposet $(P,\le,')$ is called *Boolean* ([2]) if the condition $a\wedge b = 0$ implies $b\le a'$.

For any orthoposet $(P,\le,')$, let us write $[a,b] = \{c\in P;\ a\le c\le b\}$ for every $a,b \in P$, $S_1\le S_2$ $(S_1, S_2 \subset P)$ if $s_1<s_2$ for every $s_1 \in S_1$ and for every $s_1\in S_2$, $s\le S$ $(s\in P,\ S\subset P)$ if

$\{s\} \le S_1 \wedge \ldots \wedge S_n = \{s_1 \wedge \ldots \wedge s_n;\ s_1 \in S_1, \ldots, s_n \in S_n\} (S_1, \ldots, S_n \subset P)$.

2. Lemma. Suppose that $(P,\le,')$ is a Boolean orthoposet and that $S_1, \cup \ldots \cup S_n \subset P$. Let us write

$$L = \bigcup \{[0,s_1] \cap \ldots \cap [0,s_n];\ (s_1,\ldots,s_n) \in S_1 \times \ldots \times S_n\},$$

$$U = \bigcup_{k=1}^{n} \bigcap_{s_k \in S_k} [s_k, 1].$$

Then $L \le U$ and $l \le u$ for every $l \le U$ and for every $u \ge L$.

Proof. The inequality $L \le U$ is evident. Suppose that $l \not\le u$. Then there is an $a \in P \setminus \{0\}$ such that $a \le l, u'$. Since $l \le \bigcap_{s_1 \in S_1} [s_1, 1]$, we obtain $a' \notin \bigcap_{s_1 \in S_1} [s_1, 1]$. It means that $s_1 \not\le a'$ for some $s_1 \in S_1$. Hence, there is an $a_1 \in P \setminus \{0\}$ such that $a_1 \le s_1, a, u'$. Proceeding by induction, we obtain an $a_n \in P \setminus \{0\}$ such that $a_n \le s_1, \ldots, s_n, u'$ for some $s_1 \in S_1, \ldots, s_n \in S_n$. Therefore we have $a_n \in L \le u$ and $a_n \le u \wedge u' = 0$, which is a contradiction.

3. Theorem. Suppose that $(P,\le,')$ is a Boolean orthoposet and that $S_1 \cup \ldots \cup S_n \subset P$ such that $S_1 \wedge \ldots \wedge S_n$, $\vee S_1, \ldots, \vee S_n$ exist in $(P,\le,')$. Then

$$\vee(S_1 \wedge \ldots \wedge S_n) = (\vee S_1) \wedge \ldots \wedge (\vee S_n)$$

if at least one side of this equality exists.

Proof. The left side of this equality exists if and only if $\vee L$ exists (L taken from Lemma 2) and both expressions are equal. The right side of this equality exists if and only if $\wedge U$ exists (U taken from Lemma 2) and both expressions are equal. According to Lemma 2, $\vee L$ exists if and only if $\wedge U$ exists and then $\vee L = \wedge U$.

4. Corollary. Every Boolean ω-orthocomplete poset is orthomodular.

5. Corollary. Every Boolean ortholattice is a Boolean algebra.

Let us recall that an orthoposet $(P,\leq,')$ is called *atomic* if for any $b\in P\backslash\{0\}$ there is an $a\in P\backslash\{0\}$ such that $[0,a] = \{0,a\}$ (i.e., a is an *atom*) and $a\leq b$.

6. **Theorem.** Every atomic Boolean orthocomplete poset is a Boolean algebra.

Proof. It follows from Corollary 4 and from [2], Theorem 2.

Let us note that Boolean orthoposets are *concrete* (i.e., they are set-representable in such a manner that the supremum of a finite number of mutually disjoint sets (if it exists) is the set-theoretic union, see[4] for Boolean orthomodular posets - the orthomodularity was not used in the proof). As the following simple example shows, not every Boolean orthoposet is orthomodular.

7. **Example.** Let X be a four-element set and let $(P,\leq,')$ be a triple such that P consists of $\emptyset$, one-element subsets of X and sets-complements of these sets, $\leq$ means the inclusion in X and $'$ the set-theoretic complementation. Then $(P,\leq,')$ is a Boolean orthoposet which is not orthomodular.

Finally, let us state for comparison results analogous to Theorem 3. We shall need the following definition.

8. **Definition.** Let $(P,\leq,')$ be an orthoposet. Then elements $a,b\in P$ are called *compatible* (denoted by $(a\leftrightarrow b)$ if there are $a_1,b_1,c\in P$ such that $a = a_1\vee c$, $b = b_1\vee c$ and $a_1\leq b_1'$, $a_1\leq c'$, $b_1\leq c'$.

9. **Proposition.** Suppose that $(P,\leq,')$ is an orthomodular lattice and that $S_1\cup\ldots\cup S_n\subset P$ such that $S_1\wedge\ldots\wedge S_n$, $\vee S_1,\ldots,\vee S_n$ exists in $(P,\leq,')$ and such that $s_i\leftrightarrow s_j$ for every pair $s_i\in S_i$, $s_j\in S_j$, $i,j\in\{1,\ldots,n\}$ with $i\neq i$. Then

$$\vee(S_1\wedge\ldots\wedge S_n) = (\vee S_1)\wedge\ldots\wedge(\vee S_n)$$

if the right side of this equality exists.

Proof. It follows from [5], Proposition 1.3.10, if we proceed by the induction.

10. Proposition. Suppose that $(P,\leq,')$ is an orthomodular poset and that $\{s_1\} \cup S_2 \subset P$ such that $\{s_1\} \wedge S_2$, $\vee\{s_1\}$, $\vee S_2$ exists in $(P,\leq,')$ and such that $s_1 \leftrightarrow s_2$ for every $s_2 \in S_2$. Then

$$\vee(\{s_1\} \wedge S_2) = (\vee\{s_1\} \wedge (\vee S_2))$$

if the left side of this equality exists.

Proof. See [1], Lemma 3.7.

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