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# ORTHOSYMMETRY AND MODULARITY IN ORTHOLATTICES

Orthosymmetric orthomodular lattices were introduced by R. Mayet [6] who also proved their basic properties and indicated their significance within the axiomatics of quantum theories. In this paper we offer solutions to a few question suggested by the cited paper [6]. We first prove that the orthosymmetric structure on Hilbert lattices is unique. This result may find application in the foundation of algebraic and measure theoretic quantum mechanics. Then we take up some questions motivated by [6]. As the main results we show that modular lattices need not admit an orthosymmetric structure and that orthosymmetric orthomodular lattices need not have a strong set of states.

## 1. Introduction and basic notions

Motivated by quantum axiomatics, one of the main lines of investigation in orthomodular lattices (abbr. OML) is the effort so determine Hilbert lattices among general orthomodular lattices (see e.g. [10], [13], etc.). Following this line, R. Mayet introduced orthosymmetric orthomodular lattices (abbr. OSOML) and showed, roughly, that OSOML's form a useful subclass of OML's which contains all Hilbert lattices. The orthosymmetry seems also interesting in its own algebraic right. In this paper we carry on the investigation originated in [6] and resolve a few questions regarding the size of the class of OSOML's.

For the basic notions concerning orthomodular lattices, let us refer to [3] and [4]. Throughout this paper, let us denote by L and OML.

Recall that elements $a,b\in L$ are called <u>compatible</u> (abbr. aCb) if they are contained in a Boolean subalgebra of L. A maximal Boolean subalgebra of L is called a <u>block</u>. Further, by the <u>centre</u> of L (denoted by C(L)) we call the intersection of all blocks of L. Thus, C(L) is a Boolean subalgebra of L consisting of all "absolutely compatible" elements in L.

A nonnegative function s on L is called a <u>state</u> if it satisfies the following conditions:

(1) $s(1)=1$,

(2) $s\left(\vee_{i\in N} a_i\right) = \sum_{i\in N} s(a_i)$

for every mutually orthogonal sequence $\{a_i\}_{i\in N}\subset L$.

An <u>automorphism</u> of L is a bijection $P:L\to L$ satisfying the following conditions $(x,y\in L)$:

$$P(x') = (P(x))'$$

$$P(x\wedge y) = P(x)\wedge P(y).$$

If, moreover, $P\circ P=1_L$ (the identity on L), we call P a <u>symmetry</u>. We denote by Aut(L) the group of all automorphisms of L and by Sym(L) the set of all symmetries of L.

For $a\in L$, let the mapping $\phi_a$ denote <u>Sasaki projection</u> corresponding to a (i.e., $\phi_a(x)=a\wedge(a'\vee x)$ for all $x\in L$). (In a Hilbert lattice - the lattice of closed subspaces of a Hilbert space - the Sasaki projection is the ordinary orthogonal projection onto a.)

**1.1 Definition.** An <u>orthosymmetric orthomodular lattice</u> (abbr. OSOML) is an OML equipped with a mapping $S:a\to S_a$ from L to Sym(L) (called an orthosymmetry) such that all $a,b\in L$ satisfy the following conditions:

(1) $S_a\circ S_b\circ S_a = S_{S_a(b)}$,

(2) $b\vee S_a(b) = b\vee\phi_a(b)$,

(3) $a\perp b \Rightarrow S_a\circ S_b = S_{a\vee b}$.

We shall need the following properties of the orthosymmetry.

**1.2. Proposition** [6]. Let L be an OSOML and $a,b\in L$. Then $S_a=S_b$ if and only if

(R) $aCb$ and $(a\wedge b)\vee(a'\wedge b')\in C(L)$.
In particular, $S_a=S_{a'}$ for all $a\in L$.

**1.3. Proposition [6].** Let L be an OSOML and $a,b\in L$. Then $S_a(b)=b$ if and only if $aCb$.

## 2. Orthosymmetric structure of the lattice of closed subspaces in an inner product space

Throughout this section we shall investigate an orthosymmetric structure of the lattices of closed subspaces. Especially, we shall study the orthosymmetry on the lattice of closed subspaces in a Hilbert space.

First we introduce some more notations. Let V be a vector space over a division ring K(card K>2), endowed with a hermitean form <., .>. A subspace X of V is called closed if $X=(X^{\perp})^{\perp}$ (we use the notation X for the space $X^{\perp}=\{v\in V|<x,v>=0$ for each $x\in X\}$). We denote by L(V) the lattice of all closed subspaces of V (ordering is given by the set inclusion). A space V is said to be an orthomodular space if $V=X+X^{\perp}$ for each $X\in L(V)$ (or equivalently, if $(L(V),{}^{\perp})$ forms an orthomodular lattice - see [2]). For each $X\in L(V)$ we define a projection $P_X$ putting

$$P_X(x_1+x_2) = x_1 \quad \text{for any} \quad x_1\in X,\ x_2\in X^{\perp}.$$

**2.1. Proposition [6].** Let V be an orthomodular space and let $S_X:L(V)\longrightarrow L(V)$ $(X\in L(V))$ be the mapping defined by the formula

$S_X(Y) = (2P_X-I)(Y)$ for any $Y\in L(V)$, where I is the identity mapping. Then the mapping $S:X\longrightarrow S_X$ $(X\in L(V))$ is an orthosymmetry on L(V).

If V is a Hilbert space H, then the mapping S from the foregoing proposition is a usual geometrical symmetry on L(H) (i.e. $S_X(Y)$ is the closed subspace of H symmetrical to Y with respect to X). It should be observed that the lattice L(H) is an OSOML with a strong set of states, i.e., for each $X,Y\in L(H)$ $(X\neq Y)$, there is a state s such that $s(X)\neq s(Y)$.

Since all other examples dealt with in [6] also have this property, the question arises of whether it has to be the case in general. Let us point out that it does not have to be so. Indeed, according to [5] there is an orthomodular space V such that L(V) does not have a strong set of states.

Let us concentrate to a Hilbert lattice L(H). Let us show that this lattice admits exactly one orthosymmetry (which, of course, has to be the orthosymmetry from Proposition 2.1).

**2.2 Theorem.** Let H be a Hilbert space with $\dim H \geq 3$. Then there is exactly one orthosymmetry on L(H).

**Proof.** Let S be an orthosymmetry on L(H) and let S′ be the usual orthosymmetry (i.e., $S'_X(Y)=(2P_X-I)(Y)$, $X,Y\in L(H)$). Take a nonzero element $X\in L(H)$. Making use of Wigner theorem [11], [12] we see that $S_X(Y)=U_X(Y)$ $(Y\in L(H))$, where $U_X$ is a suitable unitary or antiunitary mapping on H. Since $S_X$ is an identity on X and $X^{\perp}$ (see Proposition 1.2. and Proposition 1.3.), $U_X$ has to be a unitary mapping. According to the equality $S_X\circ S_X=1_{L(H)}$, we see that $U_X$ can be taken so that $U_X^2=I$. Put $Q=\frac{U_X+I}{2}$. Then Q is a projection and we can find $Y\in L(H)$ such that $Q=P_Y$. According to Proposition 1.2., it suffices to show that $X=Y$ or $X=Y^{\perp}$.

Since $S_X(Y)=S'_Y(Y)=Y$, we see that XCY. Let us now consider the case of $X\wedge Y\neq 0$. If $X-X\wedge Y\neq 0$, then there exists a closed subspace $R\leq X$ such that R is not compatible with $X\wedge Y$. Thus, R is not compatible with Y and therefore $S_Y(R)=R$. But $S_Y(R)\neq S_X(R)=R$, which is a contradiction. We obtain that $X\leq Y$. Applying the same argument to X and $Y\wedge X^{\perp}$, we see that X=Y. Indeed, if $X\neq Y$, we can find a closed subspace $K\leq Y$ such that K is not compatible with X. Then $S_X(K)\neq K$, but $S_X(K)=S'_Y(K)=K$ - a contradiction.

Finally, let $X\wedge Y=0$. Then $X\leq Y^{\perp}$. If $X\neq Y^{\perp}$, then there is a closed subspace $E\leq Y^{\perp}$ such that E is not compatible with X. Therefore $S_X(E)\neq E$ and $S_X(E)=S'_Y(E)=S'_{Y^{\perp}}(E)=E$ - again a contradiction. So we have $X=Y^{\perp}$. The proof is complete.

## 3. Orthosymmetry of modular ortholattices

It is proved in [6] that every Boolean algebra admits a unique orthosymmetry (an easy consequence of Proposition 1.3.). Both Boolean algebras and finite-dimensional Hilbert lattices are examples of modular OSOML's. Infinite-dimensional Hilbert lattices are OSOML's which are not modular. In [6] there are examples of finite nonmodular OSOML's, so orthosymmetry does not imply modularity. The question is whether modularity implies orthosymmetry.

**3.1. Proposition [6].** Every finite modular ortholattice admits an orthosymmetry.

Thus, it remains the case of infinite modular lattices. Here the answer may be in the negative as we see from the following result.

**3.2. Proposition.** There is a modular ortholattice admitting no orthosymmetry.

**Proof.** We take a finite nonmodular OML, $L_0$, with exactly one state s. We may assume (see [7], [9]) that s attains the value 1/3 on each atom. Obviously for this situation, each block of $L_0$ has three atoms.

We now apply to $L_0$ the process of "modularization" as introduced by Godowski [1]: For $i \in N$, let us define OML's $L_i$ by induction. Denote by $D_{i-1}$ the set of all pairs (a,b) of atoms in $L_{i-1}$ whose supremum is 1. (The existence of such pairs of atoms implies the circumstance that the OML in question is not modular.) For each $(a,b) \in D_{i-1}$ we add $L_{i-1}$ three atoms c, d, e and two blocks whose sets of atoms are {a,c,e} and {b,d,e}. Having done this for all $(a,b) \in D_{i-1}$, we obtain an OML $L_i$. (Obviously, in $L_i$ we have $a \vee b = e'$). We obtain $L_{i-1} \subset L_i$ and the identity mapping of $L_{i-1}$ into $L_i$ preserves the orthocomplements, the ordering and the orthogonal suprema. Let us endow the union $L = \bigcup_{i \in N} L_i$ with the orthocomplementation and the ordering inherited from $L_i$ ($i \in N$). Then L becomes an OML (the limit of OML's $L_i$, $i \in N$, see [8, Th. 4.11]). It remains to prove that L is the required example.

First, let us verify the modularity. Observe that L contains only 0, 1, atoms and coatoms (orthocomplements of atoms). The supremum of each two distinct atoms is a coatom and their infimum is 0. By de Morgan laws, the infimum of each two distinct coatoms is an atom and their supremum is 1. This ensures that L is modular.

By the construction, each state on $L_i$ can be extended to a state on $L_j$ for j>i and, also, to a state on L. For each atom $b \in L_i - L_{i-1}$ ($i \in N$) there is a state on $L_i$ (and also a state on L) attaining a value different from 1/3 at b. On the other hand, all states on L attain the value 1/3 at all atoms of $L_0$. As automorphisms preserve states, each automorphism of L maps $L_0$ onto $L_0$. But L is generated by $L_0$, hence each automorphism of L is uniquely determined by its values on $L_0$. We obtain that Aut(L) is isomorphic to Aut(M), which is a finite group.

Suppose that S is an orthosymmetry of L. Let us now apply Proposition 1.2. to L. As C(L)={0,1}, the relation (R) is equivalent to the condition $b \in \{a,a'\}$. For $b \notin \{a,a'\}$ we have $S_a \neq S_b$. We obtain an infinite family $\{S_a$ : a is an atom of L$\}$ of distinct automorphisms of L. This is a contradiction. Therefore L admits no orthosymmetry and the proof is complete.

## REFERENCES


[1] **R. Godowski:** Partial Greechie diagrams for modular ortholattices, Demonstratio Math. 20 (1987), 291-297.

[2] **H. Gross, A.M. Künzi:** On a class of orthomodular quadratic spaces, L'Enseignement Mathematique 31 (1985), 276-283.

[3] **S.P. Gudder:** Quantum probability, Academic Press, Inc. (1988).

[4] **G. Kalmbach:** Orthomodular Lattices, Academic Press, London, 1983.

[5] **H.A. Keller:** Measures on orthomodular vector space lattices, Stud. Math. 88 (1988), 183-195.

[6] **R. Mayet:** Orthosymmetreic ortholattices, Proc. Amer. Math. Soc., 1991.

[7] **M. Navara, V. Rogalewicz:** Construction of orthomodular

lattices with given state spaces, Demonstratio Math. 21 (1988), 481-493.

[8] **M. Navara, V. Rogalewicz:** The pasting construction for orthomodular posets, to appear in Math. Nachrichten.

[9] **M. Navara, V. Rogalewicz:** On constructions of orthomodular posets and hypergraphs, Proc. 1st Winter School on Measure Theory, Liptovský Ján, Czechoslovakia (1988), 93-98.

[10] **C. Piron:** Axiomatique quantique, Helv. Phys. Acta 37 (1964), 439-468.

[11] **K. Ylinen:** Vector Gleason measures and their Fourier transforms, Lecture Notes in Math. 1132 (1983), 589-594.

[12] **V. Varadarajan:** Geometry of Quantum Theory I, Van Nostrand, Princeton, New Jersey, 1968.

[13] **W.J. Wilbur:** On characterizing the standard quantum logics, Trans. Amer. Math. Soc. 233 (1977), 265-282.

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