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MAXIMAL ANTICHAINS AND DISSECTING IDEALS

1. Introduction

Let $(X,\le)$ be a poset. A subset I of X is called an ideal of $(X,\le)$ if whenever $y\in I$ and $x\in X$ are such that $x\le y$ then $x\in I$. A subset A of X is called an antichain if any two distinct elements of A are incomparable. Let A(X) be the set of all antichains of $(X,\le)$. An antichain A is called maximal if it is a maximal element of A(X) with respect to set-theoretic inclusion. Let MA(X) be the set of all maximal antichains of $(X,\le)$.

It is well known (see, for example, [2], Ch. III, Thm. 3) that the set of all ideals of $(X,\le)$ forms a distributive lattice under set-theoretic inclusion, and that every finite distributive lattice can be represented in this way. For any subset Y of X let Max(Y) and Min(Y) be the set of maximal, respectively minimal, elements of Y. On the set of all antichains of $(X,\le)$ a partial order can be defined as follows. If $A,B\in A(X)$ then $A\le B$ if for each $a\in A$ there exists $b\in B$ such that $a\le b$. Furthermore, if X is finite then the mapping $Y\mapsto \mathrm{Max}(Y)$ is an order-isomorphism between the set of ideals of $(X,\le)$ and the set $(A(X),\le)$. Thus $(A(X),\le)$ is a distributive lattice, and every finite distributive lattice arises in this way.

R.P. Dilworth [3] showed that if X is finite, then the set MSA(X) of all antichains of $(X,\le)$ of maximum size forms a distributive lattice under the order induced from $(A(X),\le)$, in fact, a sublattice of $(A(X),\le)$. It was shown by K.M. Koh [4] that every finite distributive lattice can be represented in this way.

In [1], the author showed that the set MA(X) of all maximal antichains of the finite poset $(X,\le)$ forms a lattice under the order induced from $(A(X),\le)$, and that furthermore every finite lattice can be represented in this way. In general, the lattice $(MA(X),\le)$ is only a subposet of $(A(X),\le)$ but not a sublattice. This suggests to consider the question of what can be said about the sublattice of $(A(X),\le)$ which is generated by MA(X). This question shall be dealt with in this paper.

2. The lattice of dissecting ideals

We call an ideal I of a poset $(X,\le)$ <u>dissecting</u> if whenever $x\in I$ and $y\in X\setminus I$ are such that x and y are incomparable then there exists $y'\in X$ with $y'<y$ such that x and y' are incomparable.

Theorem 2.1. Let $(X,\le)$ be a poset. Then the set of all dissecting ideals forms a complete sublattice of the lattice of all ideals of $(X,\le)$.

Proof. First note that X and the empty set both are dissecting. Let S be a non-empty set of dissecting ideals of $(X,\le)$ and let J and M be the set-theoretic union respectively intersection of all members of S. Let $x\in J$ and $y\in X\setminus J$ such that x and y are incomparable. There exists $I\in S$ such that $x\in I$. As $I\subseteq J$, we also have $y\in X\setminus I$, and thus there exists $y'\in X$ with $y'<y$ such that x and y' are incomparable. Therefore J is dissecting. Let $x\in M$ and $y\in X\setminus M$. There exists $I\in S$ such that $y\in X\setminus I$, and as $M\subseteq I$ we have $x\in I$. Hence there exists $y'\in X$ with $y'<y$ such that x and y' are incomparable, and M is dissecting.

A first connection between maximal antichains and dissecting ideals is given as follows.

Proposition 2.2. Let $(X,\le)$ be a poset, let $A\in MA(X)$, and let $I=\{x\in X \mid \text{there exists } a\in A \text{ such that } x\le a\}$. Then I ia a dissecting ideal.

Proof. Clearly I is an ideal. Let $x\in I$ and $y\in X\setminus I$ such that

x and y are incomparable. By definition of I, there exists $a \in A$ such that $x \leq a$. By maximality of A, there exists $b \in A$ such that y and b are comparable, and as $y \in X \setminus I$, we must have $y < b$. If $a = b$ then $x \leq y$, which is a contradiction, thus we have $a \neq b$. If $b \leq x$ then $b \leq a$ which is a contradiction to $a \neq b$ and A being an antichain. If $x \leq b$ then again we get $x \leq y$, giving a contradiction. Thus x and b are incomparable, and as $y \in X \setminus I$ and $b \in I$ we have $b < y$. Therefore I is dissecting.

Next we see that the lattice of dissecting ideals is, in fact, generated by the ideals which are generated by the maximal antichains.

Theorem 2.3. Let $(X, \leq)$ be a poset. Then the lattice of dissecting ideals of $(X, \leq)$ is the complete sublattice of the lattice of all ideals of $(X, \leq)$ which is generated by the empty set and the ideals $I(A) = \{x \in X \mid \text{there exists } a \in A \text{ such that } x \leq a\}$ for the maximal antichains A.

Proof. Let D be the set of all dissecting ideals and L the complete lattice generated by $\emptyset$ and all ideals $I(A)$ for the maximal antichains A. By Theorem 2.1 and Proposition 2.2 it is clear that $L \subseteq D$. For $x \in X$ define $K(x)$ to be the intersection of all ideals $I(A)$ where A is a maximal antichain containing x. Obviously $K(x) \in L$. Let I be a non-empty dissecting ideal. Let $x \in I$ and $y \in X \setminus I$. If $x < y$ then clearly $y \notin K(x)$. Suppose x and y are incomparable. Then there exists $y' < y$ such that x and y' are incomparable, and there exists $A \in MA(X)$ with $\{x, y'\} \subseteq A$. It then follows that $y \notin I(A)$, and thus $y \notin K(x)$. Hence we have $K(x) \subseteq I$. As $x \in K(x)$, it follows that I is the union of all ideals $K(x)$ for $x \in I$, and therefore $I \in L$, which concludes the proof of the theorem.

Note that the lattice of dissecting ideals of $(X, \leq)$ contains an atom (which is then unique) if and only if $(X, \leq)$ has at least one minimal element (and then the atom is the set of all minimal elements). Thus the complete lattice generated by the ideals $I(A)$ for the maximal chains A is the set of all dissecting ideals whenever $(X, \leq)$ has no minimal element, or otherwise it is the set of non-empty dissecting ideals.

3. Dissecting ideals in finite posets

If $(X,\leq)$ is a finite poset then by Theorem 2.1 its dissecting ideals form a finite distributive lattice, which in turn is isomorphic to the lattice of all ideals of a finite poset $(Y,\leq)$. We shall now investigate in which way $(Y,\leq)$ can be constructed from $(X,\leq)$.

Lemma 3.1. Let $(X,\leq)$ be a finite poset and $x\in X$. Let $K(x)$ be defined as in the proof of Theorem 2.3, let $A(x)=Max(K(x))$, and let $B(x)=\{y\in X \mid x \text{ and } y \text{ are incomparable}\}$. Then $A(x)=\{x\}\cup Min(B(x))\in MA(X)$.

Proof. Clearly $\{x\}\cup Min(B(x))$ is an antichain. Let $y\in X$. Then either y is comparable with x or $y\in B(x)$. But if $y\in B(x)$, as X is finite there exists $y'\in Min(B(x))$ with $y'\leq y$. Thus $\{x\}\cup Min(B(x))\in MA(X)$, and $K(x)\subseteq I(\{x\}\cup Min(B(x)))$. But on the other hand, if $A\in MA(X)$ with $x\in A$ then $A\subseteq\{x\}\cup B(x)$ and thus $\{x\}\cup Min(B(x))\subseteq I(A)$, hence $I(\{x\}\cup Min(B(x)))\subseteq K(x)$. Therefore $A(x)=\{x\}\cup Min(B(x))$.

Lemma 3.2. Let $(X,\leq)$ be a finite poset, and let $x,y\in X$. Then $A(x)=A(y)$ if and only if $\{z\in X \mid z<x\} = \{z\in X \mid z<y\}$.

Proof. Without loss of generality, we can assume that $x\neq y$. Let $A(x)=A(y)$, and let $z\in X$ with $z<x$. If $y\leq z$ then $y<x$ and thus $y\in A(y)\setminus A(x)$, which is a contradiction. If y, z are incomparable then we get a contradiction from the fact that $x\in A(y)$, thus $x\in Min(B(y))$. By symmetry, we then get $\{z\in X \mid z<x\} = \{z\in X \mid z<y\}$.

Conversely, suppose that $\{z\in X \mid z<x\} = \{z\in X \mid z<y\}$, and let $z\in A(x)$. Note that x and y must be incomparable. First assume that $z=x$. By the equality above, there can be no $z'\in X$ incomparable with y such that $z'<x$. Thus $x\in Min(B(y))\subseteq A(y)$. Now assume that $z\in Min(B(x))$. Clearly we can not have $z<y$. If $z>y$ then we get a contradiction to minimality of z in $B(x)$. Hence either $z=y\in A(y)$ or z and y are incomparable. If z, y are incomparable then suppose there exists $z'<z$ such that $z'\in B(y)$. We can not have $z'<x$ (and clearly nor can we have

$x \le z'$), therefore $z' \in B(x)$ in contradiction to the minimality of z. Thus $z \in Min(B(y)) \subseteq A(y)$. We have shown that $A(x) \subseteq A(y)$, and equality holds by symmetry.

Lemma 3.3. Let $(X,\le)$ be a finite poset, and let I be a dissecting ideal. Then I is join-irreducible in the lattice of dissecting ideals if and only if there exists $x \in X$ such that $I=K(x)$.

Proof. Let $x \in X$ and let I, J be dissecting ideals such that $K(x)=I \cup J$. As $x \in K(x)$, it follows that $x \in I$ or $x \in J$. But from the proof of Theorem 2.3 it follows that $K(x) \subseteq I$ or $K(x) \subseteq J$, and thus $K(x)=I$ or $K(x)=J$. Therefore $K(x)$ is join-irreducible. Let I be a join-irreducible dissecting ideal. Again, the proof of Theorem 2.3 shows that I is the join of all ideals $K(x)$ for $x \in I$, and thus there exists $x \in I$ such that $I=K(x)$.

On a poset $(X,\le)$ we define an equivalence relation ~ by $x \sim y$ if and only if $\{z \in X \mid z<x\} = \{z \in X \mid z<y\}$. Let $E(X)$ be the set of equivalence classes of this relation. Note that each element of $E(X)$ is an antichain of $(X,\le)$. For $C,D \in E(X)$ let $C \le D$ if and only if $\{z \in X \mid z<c\} \subseteq \{z \in X \mid z<d\}$ whenever $c \in C$ and $d \in D$. It is easy to see that this relation is a partial order.

Lemma 3.4. Let $(X,\le)$ be a finite poset, let $x,y \in X$, and let $C(x)$ and $C(y)$ be the elements of $E(X)$ containing x respectively y. Then $A(x) \le A(y)$ if and only if $C(x) \le C(y)$.

Proof. Let $A(x) \le A(y)$. Then there exists $y' \in A(y)$ such that $x \le y'$. Let $z \in X$ with $z<x$. If $y \le z$ then $y<x$ which is a contradiction to $A(x) \le A(y)$. If y and z are incomparable, then we have $z \in B(y)$ but $z<y'$, which again is a contradiction. Thus $z<y$, and we have $C(x) \le C(y)$. Conversely, suppose that $C(x) \le$ $\le C(y)$. Then $x \le y$ (and hence also $A(x) \le A(y)$) or $x \in Min(B(y))$. Let $x' \in A(x)$, and suppose there exists $y' \in A(y)$ such that $y'<x'$. Thus $y'<x$ but not $y'<y$, which is a contradiction. Therefore, as $A(y)$ is a maximal antichain, there exists $y'' \in A(y)$ such that $x' \le y''$, and hence $A(x) \le A(y)$.

We now can give the main result of this section.

Theorem 3.5. Let $(X,\le)$ be a finite poset. Then the lattice of dissecting ideals of $(X,\le)$ is isomorphic to the lattice of all ideals of $(E(X),\le)$.

Proof. By the preceding Lemmas, the poset of join-irreducible dissecting ideals is isomorphic to $(E(X),\le)$. The rest follows from [2], Ch. III, Thm. 3.

4. <u>A representation theorem</u>

We shall finally give a characterization of those finite lattices which are isomorphic to the lattice of dissecting ideals of a partially ordered set. In view of Theorem 3.5 this mainly reduces to the problem of characterizing the posets $(E(X),\le)$.

Lemma 4.1. Let $(X,\le)$ be a finite poset with a unique minimal element. Then there exists a finite poset $(Y,\le)$ such that $(X,\le)$ is isomorphic to $(E(Y),\le)$.

Proof. Let x_0 be the minimal element of X. Define $Y=X\times X$, and for $(x_1,x_2),(x_1',x_2')\in Y$ let $(x_1,x_2)\le(x_1',x_2')$ if and only if $(x_1,x_2)=(x_1',x_2')$ or $x_1<x_1'$ and $x_2\le x_1'$. It is clear that this is a partial order on Y. Now let $(x,y_1),(x,y_2)\in Y$. If $(z_1,z_2)\in Y$ is such that $(z_1,z_2)<(x,y_1)$ then $z_1<x$ and $z_2\le x$. Thus we also have $(z_1,z_2)<(x,y_2)$. By symmetry, we get $(x,y_1)\sim(x,y_2)$. On the other hand, let $(x_1,y_1),(x_2,y_2)\in Y$ such that $x_1\ne x_2$. If $x_1<x_2$ then $(x_1,x_1)<(x_2,y_2)$ and $(x_1,x_2)<(x_2,y_2)$, but at least one of (x_1,x_1) and (x_1,x_2) is incomparable with (x_1,y_1). If x_1 and x_2 are incomparable, then both are distinct from x_0, and we have $(x_0,x_1)<(x_1,y_1)$ but we do not have $(x_0,x_1)<(x_2,y_2)$. Thus we have seen that $E(Y) = \{\{(x,y) \mid y\in X\} \mid x\in X\}$, and it is easy to see in a similar way that the mapping $x\mapsto\{(x,y) \mid y\in X\}$ gives the desired isomorphism.

Theorem 4.2. Let $(L,\le)$ be a finite distributive lattice with more than one element. Then $(L,\le)$ is isomorphic to the lattice of dissecting ideals of a finite poset if and only if $(L,\le)$ contains a unique atom.

Proof. It is clear that a finite distributive lattice contains a unique atom if and only if the poset of its join-irreducible elements contains a unique minimal element. By Lemma 4.1 and Theorem 3.5 every finite distributive lattice with a unique atom is isomorphic to the lattice of dissecting ideals of a finite poset. The converse follows from Theorem 2.1 and the remark after Theorem 2.3.

As a sequence we get that every finite distributive lattice $(L,\leq)$ is isomorphic to the lattice generated by the maximal antichains of some finite poset $(X,\leq)$ (which is the lattice of non-empty dissecting ideals of $(X,\leq)$).

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