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GENERALIZATION OF THE PYTHAGOREAN THEOREM

1. Introduction

In this note a generalized form of the Pythagorean theorem is shown and proved. Because the Pythagorean theorem is being often formulated in terms of a unitary space, for the purpose of this work a special version of the theorem is used.

Let V be a unitary space over field $\mathbb{R}$ and let $<.,.>$ be an inner product in V.

If $x,y\in V$ and $<x,y>=0$, then according to the Pythagorean theorem we have

$$\|x+y\|^2=\|x\|^2+\|y\|^2 \tag{1}$$

where $\|x\|^2=<x,x>$.

The geometrical interpretation of the formula (1) may be shown in the following way:
Let H be the Hilbert space and let $<.,.>$ be the inner product in H.

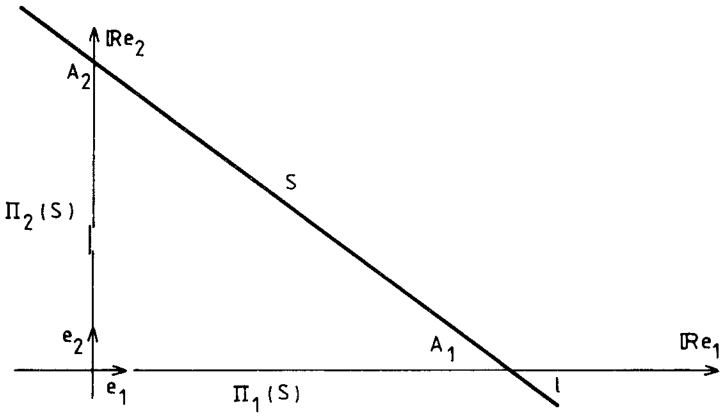
Let A_i be a point of the axis $\mathbb{R}e_i$ (for $i=1,2$) of the space given by two orthonormal vectors $e_1,e_2\in H$.

Denoting by l the affine line passing through the points A_1, A_2 and S the segment of l whose ends are A_1, A_2 we receive that

$$\|\Pi_1(S)\|^2+\|\Pi_2(S)\|^2=\|S\|^2$$

where Π_i is an orthogonal projection onto axis $\mathbb{R}e_i$ (for $i=1,2$).

Fig 1. shows this situation.



Now we are ready to generalize the Pythagorean theorem.

2. Theorem

Let H be a Hilbert space with an inner product $<.,.>$. Choose an orthonormal system $\{v_1,\dots,v_n\}$ of n vectors of H. Now let us fix two natural numbers k,m such that $1 \le k < n$, $k \le m < n$. Let L denote a k-dimensional affine subspace of $\mathrm{span}\{v_1,\dots,v_n\}$ and S any measurable subset of L. Defining $\Pi_{\alpha_1 \dots \alpha_m}$ as an orthogonal projection nto $\mathrm{span}\{v_{\alpha_1},\dots,v_{\alpha_m}\}$ (where $1 \le \alpha_i \le n$ for $i=1,\dots,m$) and denoting $|S|$ as k-dimensional measure of set S in space L, $|\Pi_{\alpha_1 \dots \alpha_m}(S)|$ as k-dimensional measure of $\Pi_{\alpha_1 \dots \alpha_m}(S)$ in $\Pi_{\alpha_1 \dots \alpha_m}(L)$ we can express the theorem in the following generalized version

$$\binom{n-k}{m-k} |S|^2 = \sum_{\substack{\alpha_1,\dots,\alpha_m=1 \\ \alpha_1<\dots<\alpha_m}}^{n} |\Pi_{\alpha_1,\dots,\alpha_m}(S)|^2. \tag{2}$$

3. Proof of the theorem

We shall discuss two cases.

Suppose first that dimension of S is smaller than k. Hence, the image of any projection of the set S is of the dimension less than k. Therefore, every k-dimensional measure of S and $\Pi_{\alpha_1 \dots \alpha_m}(S)$ vanishes and the theorem is true.

Assuming now that dimension of S is equal to k and $g:\mathbb{R}^k \xrightarrow{\ominus} L$ is an affine parametrization of the subspace L we receive

$$g(t_1,\dots,t_k)=(x_1,\dots,x_n)$$

and

(3) $\qquad x_i=a_0^i+a_1^it_1+\dots+a_k^it_k$, where $a_j^i\in\mathbb{R}$ for $\begin{matrix} i=1,\dots,n \\ j=1,\dots,k \end{matrix}$.

It can be easily seen that g is a diffeomorphism. In this case a k-dimensional measure of set S takes the form

$$|S| = \int_{g^{-1}(S)} \sqrt{D(1,\dots,n)(t)}\, dt_1\dots dt_k,\ t=(t_1,\dots,t_k)$$

where

$$D(1,\dots,n)(t) = \det\left[\left(\frac{\partial g}{\partial t_i}(t)\,\middle|\,\frac{\partial g}{\partial t_j}(t)\right) \right]_{i,j=1}^{k} =$$

$$= \det\left[\sum_{l=1}^{n} \frac{\partial x_l}{\partial t_i}(t)\frac{\partial x_l}{\partial t_j}(t) \right]_{i,j=1}^{k} = \det\left[\sum_{l=1}^{n} a_i^l a_j^l \right]_{i,j=1}^{k}.$$

But the Jacobian $D(1,\dots,n)(t)$ does not depend on variable t, and from now on we shall write $D(1,\dots,n)$.

For every system of numbers $\alpha_1,\dots,\alpha_m$ ($\alpha_i=1,\dots,n$; $i=1,\dots,m$) we can perform an affine parametrization of the subspace $\Pi_{\alpha_1\dots\alpha_m}(L)$ of the mapping $g_{\alpha_1\dots\alpha_m}:\mathbb{R}^k \xrightarrow{\ominus} \Pi_{\alpha_1\dots\alpha_m}(L)$ in such a way that

$$g_{\alpha_1\dots\alpha_m}(t_1,\dots,t_k) = (x_{\alpha_1},\dots,x_{\alpha_m})$$

where x_{α_i} is defined by (3) for $1\le\alpha_i\le n$.

It is clear that

$$|\Pi_{\alpha_1\dots\alpha_m}(S)| = \int_{g^{-1}_{\alpha_1\dots\alpha_m}(\Pi_{\alpha_1\dots\alpha_m}(S))} \sqrt{D(\alpha_1,\dots,\alpha_m)(t)}\, dt_1\dots dt_k$$

and

$$D(\alpha_1,\dots,\alpha_m)(t) = D(\alpha_1,\dots,\alpha_m) =$$

$$= \det\left[\left(\frac{\partial g_{\alpha_1\dots\alpha_m}}{\partial t_i}(t) \,\middle|\, \frac{\partial g_{\alpha_1\dots\alpha_m}}{\partial t_j}(t) \right) \right]_{i,j=1}^{k} =$$

$$= \det\Big[\sum_{l=1}^{m} a_i^{\alpha_l}\, a_j^{\alpha_l}\Big]_{i,j=1}^{k}.$$

Since any determinant of the matrix is a n-linear form of the rows of this matrix we get

$$D(1,\dots,n) = \sum_{l_1,\dots,l_k=1}^{n} \det\Big[a_i^{l_j} a_j^{l_j}\Big]_{i,j=1}^{n} =$$

$$= \sum_{l_1,\dots,l_k=1}^{n} a_1^{l_1}\cdot a_2^{l_2}\cdot\ldots\cdot a_k^{l_k} \det\Big[a_i^{l_j}\Big]_{i,j=1}^{n}$$

Note that if for some $s,r=1,\dots,k$ l_s is equal to l_r then two rows of the matrix $\Big[a_i^{l_j} a_j^{l_j}\Big]_{i,j=1}^{n}$ are proportional and the determinant of the matrix vanishes. Finally

$$D(1,\dots,n) = \sum_{\substack{l_1,\dots,l_k=1\\ l_1<\dots<l_k}}^{n} a_1^{l_1}\cdot a_2^{l_2}\cdot\ldots\cdot a_k^{l_k} \det\Big[a_i^{l_j}\Big]_{i,j=1}^{n}.$$

Similarly

$$D(\alpha_1,\dots,\alpha_m) = \sum_{\substack{l_1,\dots,l_k=1\\ l_1<\dots<l_k}}^{n} a_1^{\alpha_{l_1}}\cdot a_2^{\alpha_{l_2}}\cdot\ldots\cdot a_k^{\alpha_{l_k}} \det\Big[a_i^{\alpha_{l_j}}\Big]_{i,j=1}^{n}.$$

Consider now the sum $\sum\limits_{\substack{\alpha_1,\dots,\alpha_m=1\\ \alpha_1<\dots<\alpha_m}}^{n} D(\alpha_1,\dots,\alpha_m)$.

The above formula yields that $\sum\limits_{\substack{\alpha_1,\dots,\alpha_m=1\\ \alpha_1<\dots<\alpha_m}}^{n} D(\alpha_1,\dots,\alpha_m) =$

$$= \sum_{\substack{\alpha_1,\ldots,\alpha_m=1 \\ \alpha_1<\ldots<\alpha_m}}^{n} \sum_{\substack{l_1,\ldots,l_k=1 \\ l_1<\ldots<l_k}}^{n} a_1^{\alpha_{l_1}} \cdot a_2^{\alpha_{l_2}} \cdot \ldots \cdot a_k^{\alpha_{l_k}} \det\left[a_i^{\alpha_{l_j}}\right]_{i,j=1}^{n}.$$

Notice that every member of the form

$a_1^{l_1} \cdot a_2^{l_2} \cdot \ldots \cdot a_k^{l_k} \det\left[a_i^{l_j}\right]_{i,j=1}^{n}$ appears $\binom{n-k}{m-k}$ times because

$\alpha_{l_1}=l_1, \alpha_{l_2}=l_2, \ldots, \alpha_{l_k}=l_k$ and another m-k values α_j can be arbitrarily chosen from the remaining n-k numbers. Hence

$$\sum_{\substack{\alpha_1,\ldots,\alpha_m=1 \\ \alpha_1<\ldots<\alpha_m}}^{n} D(\alpha_1,\ldots,\alpha_m) =$$

$$= \binom{n-k}{m-k} \sum_{\substack{l_1,\ldots,l_k=1 \\ l_1<\ldots<l_k}}^{n} a_1^{\alpha_{l_1}} \cdot a_2^{\alpha_{l_2}} \cdot \ldots \cdot a_k^{\alpha_{l_k}} \det\left[a_i^{\alpha_{l_j}}\right]_{i,j=1}^{n} =$$

$$= \binom{n-k}{m-k} D(1,..,n).$$

If $\dim \Pi_{\alpha_1\ldots\alpha_m}(S) = k$ then the inverse images satisfy $g^{-1}(S) = g^{-1}_{\alpha_1,\ldots,\alpha_m}(\Pi_{\alpha_1\ldots\alpha_m}(S))$. This is because there exist k independent coordinates (of the parametrization g) such that each of the remaining coordinates of the projections can be regarded as a (unique) linear combination of the others. If, to the contrary, $\dim \Pi_{\alpha_1\ldots\alpha_m}(S) < k$, then the k-dimensional measure of $\Pi_{\alpha_1\ldots\alpha_m}(S)$ is equal to zero.

This allows us to set the following equalities:

$$\binom{n-k}{m-k} |S|^2 = \binom{n-k}{m-k} \left[\int_{g^{-1}(S)} \sqrt{D(1,\ldots,n)}\, dt_1\ldots dt_k \right]^2 =$$

$$= \binom{n-k}{m-k} D(1,\ldots,n) \left[\int_{g^{-1}(S)} dt_1\ldots dt_k \right]^2 =$$

$$= \sum_{(\alpha_1,\dots,\alpha_m)\in J}^{n} D(\alpha_1,\dots,\alpha_m)'(t)\left[\int_{g^{-1}_{\alpha_1..\alpha_m}(\Pi_{\alpha_1\dots\alpha_m}(S))} dt_1\dots dt_k\right]^2 =$$

$$= \sum_{\substack{\alpha_1,\dots,\alpha_m=1\\ \alpha_1<\dots<\alpha_m}}^{n} D(\alpha_1,\dots,\alpha_m)\left[\int_{g^{-1}_{\alpha_1..\alpha_m}(\Pi_{\alpha_1\dots\alpha_m}(S))} dt_1\dots dt_k\right]^2 =$$

$$= \sum_{\substack{\alpha_1,\dots,\alpha_m=1\\ \alpha_1<\dots<\alpha_m}}^{n} \left|\Pi_{\alpha_1\dots\alpha_m}(S)\right|^2 .$$

Therefore the proof of the theorem is completed.

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