DEMONSTRATIO MATHEMATICA
Vol. XXIV No 1-2 1991

Olga Nánásiová

MARTINGALES AND SUBMARTINGALES ON QUANTUM LOGIC

In this paper, we introduce the notion of submartingales and martingales in the state m on L, relative to sequence $\{a_n\}_{n=1}^{\infty}$ from L, such that $m(a_n)=1$, $R(x_n)\cup L_n\cup R(x_{n+1})$ is partially compatible with respect to a_n where x_n is a sequence of observables on L, $\{L_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of sublogic of L. This submartingales and martingales of an integrable functions on a probability space $(\Omega,\mathcal{S},\mu)$ and the case which is investigated in 1 is special case, when $a_n=a_{n+1}$ for all n. The main result is Theorem 3.3. It is analogous to martingale convergence theorem [9], [2], but this convergence is only in the state m. The author does not know how to prove this theorem for the convergence a.e. in a state m.

0. Preliminaries

Let L be a quantum logic, i.e. an orthomodular σ-lattice [13]. Explicitly: L is a quantum logic if the following axioms are fulfilled:

I) L is a non empty, partially ordered set with the relation "$\le$", with the maximum and minimum element (1 and 0 resp.) where $1\neq 0$;

II) For any sequence $\{a_n\}_{n=1}^{\infty}\subset L$ we have $\vee a_n\in L$ or $\wedge a_n\in L$, where $\vee,\wedge$ are lattices operations;

III) There is 1-1 mapping $\perp: L\longrightarrow L$ satisfying: a) for all $a\in L$ $(a^{\perp})^{\perp}=a$; b) for all $a\in L$ we have $a\vee a^{\perp}=1$, $a\wedge a^{\perp}=0$, for all $a\in L$; c) if $a,b\in L$, $a\le b$ then $b^{\perp}\le a^{\perp}$;

IV) Orthomodular law: For any $a,b\in L$, $a\le b$, it holds $b=a\vee(a^{\perp}\wedge b)$.

Let L be a quantum logic and $a,b\in L$. We shall say that a,b are orthogonal ($a\perp b$) if $a\le b^{\perp}$. The elements a,b will be called compatible ($a\leftrightarrow b$) if there are three mutually orthogonal elements $a_1, b_1, c\in L$ such that $a=a_1\vee c$, $b=b_1\vee c$.

It is know [13], that for all $a,b\in L$ $a\leftrightarrow b$ iff L is a Boolean algebra.

A subset $L_1\subset L$ will be called a sublogic of L if a) $0,1\in L_1$; b) $a^{\perp}\in L_1$ for each $a\in L_1$; c) $\vee a_n\in L_1$ for each sequence $\{a_n\}_{n=1}^{\infty}$ L_1.

A subset $B\subset L$ will be called a sub-σ-algebra of L, if B is a sublogic of L such that for any $a,b\in B$ $a\leftrightarrow b$.

A sublogic $L_1\subset L$ will be called separable if any sequence of pairwise orthogonal elements from L_1 is at most countable. In this case, of $\{a_\alpha\}_{\alpha\subset\mathcal{A}}\subset L$, then there is a countable sequence $\{b_n\}_{n=1}^{\infty}\subset L$, such that

$$\bigvee_{\alpha\in\mathcal{A}} a_\alpha = \bigvee_{n=1}^{\infty} b_n.$$

Let $(X,\mathcal{S})$ be a measurable space. A σ-homomorphism from S to L is any mapping h with the following properties: 1) $h(X)=1$; 2) If $A,B\in\mathcal{S}$ and $A\cap B=\emptyset$, then $h(A)\perp h(B)$; 3) If $\{A_n\}_{n=1}^{\infty}\subset\mathcal{S}$, then $h\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} h(A_n)$. If $X=R^1$ and $\mathcal{S}=\mathcal{B}(R^1)$ then the σ-homomorphism h will be called an observable on L.

In subsequent paragraphs we shall often use the following theorem.

Theorem 0.1. (Lommis - Sikorski [6],[12],[13]) Let B be a Boolean σ-algebra. Then there exist a measurable space $(X,\mathcal{S})$ and a σ-homomorphism h from $\mathcal{S}$ onto B.

The set $R(x)=\{x(E) \mid E\in\mathcal{B}(R^1)\}$ is said to be the range of the observable x. It is clear that $R(x)$ is a Boolean sub-σ-algebra of L and if f is any Borel function then $R(f\circ x)\subset R(x)$.

Definition 0.1. Let L be a quantum logic and x, y be observables on L. We shall say that x, y are compatible ($x\leftrightarrow y$) if for any $b\in R(x)$, $a\in R(y)$ it holds $a\leftrightarrow b$.

S.V. Varadarajan [13] has proved the following properties: If B is a countably generated Boolean sub-σ-algebra of L then there exists such an observable z on L that $R(z)=B$. If x,y are any observables and $x\leftrightarrow y$ then there are an observable z on L and Borel functions f,g satisfying $x=f\circ z$, $y=g\circ z$.

Put $L_{[0,a]}=\{b\in L \mid b\le a\}$, $a\in L$, $a\neq 0$. Then $L_{[0,a]}$ is a quantum logic with the maximum element a and orthocomplement "*", which is defined as follows:

$$b^{*} = b^{\perp}\wedge a, \text{ for } b\in L_{[0,a]}.$$

Now we are going to define a measure on the quantum logic L. A function $m:L\to[0,\infty)$ will be called a measure on L if it holds: a) $m(0)=0$; b) If $\{a_n\}_{n=1}^{\infty}\subset L$, $a_n\perp a_t$ for $n\neq t$, then $m(\bigvee_n a_n) = \sum_n m(a_n)$. We shall consider nontrivial measures only $m(1)\neq 0$. If $m(1)=1$, then m will be called a state on L.

If x is an observable on L and m is a state, then the function $m_x:\mathcal{B}(R^1)\to[0,1]$ where $m_x(E)=m(x(E))$, $E\in\mathcal{B}(R^1)$ said to be a probability distribution of the observable x in the state m.

An expectation of an observable x, in the state m is the number

$$m(x) = \int x\, dm = \int \lambda m_x(d\lambda),$$

if the integral on the right side exists.

1. Partial Compatibility

Definition 1.1. [10] Let L be a quantum logic and $M\subset L$, $a\in L$, $a\neq 0$. We shall say that M is partially compatible with respect to a (abb. as M is p.c.[a]) if the following is true: 1) For all $b\in M$ we have $b\leftrightarrow a$ ($M\leftrightarrow a$); 2) For all $b,c\in M$ we have $b\wedge a\leftrightarrow c\wedge a$ is a compatible set, ($M\wedge a=\{b\wedge a \mid b\in M\}$).

Let $a\in L$, $a\neq 0$. Then the subset $M\wedge a$ is compatible in L iff it is compatible in $L_{[0,a]}$.

For $F=\{a_1,\dots,a_n\}\subset L$ put

$$\text{com}(F)=\bigvee_{d\in D^n} a_1^{d_1}\wedge\dots\wedge a_n^{d_n},$$ where $D=\{0,1\}$, $d=\{d_1,\dots,d_n\}$, $a^0=a^{\perp}$, $a^1=a$.

The set M is p.c.[com(F)] [11].

The cardinality of a set G will be denote by G.

Definition 1.2. [11] Let $B\subset L$. Put

$$com(B)=\{com(F)\ |\ F\subset B,\ |F|<\infty\}.$$

The element $com(B)\in L$, if it exists, is said to be a commutator of the set B.

Note that B is p.c.[com(B)] [11].

Theorem 1.1. [11]. Let $B\subset L$ such that com(B) exists. Then the set B is p.c.[a] iff $a\leftrightarrow B$ and $a\le com(B)$ where $a\in L$, $a\ne 0$.

Let M be a set of states on L. The pair (L,M) is a quite full system (abbr. q.f.s.) if $\{m\in M\ |\ m(a)=1\}\subset\{m\in M\ |\ m(b)=1\}$ implies $a\le b$.

Let (L,M) be q.f.s. We say that L has the property U if $m(x)=m(y)$ for all $m\in M$ implies $x=y$, where x,y are bounded observables on L. We say that L has property E if for any pair x,y of bounded observables there is a unique bounded observable z such that $m(z)=m(x)+m(y)$ for any $m\in M$. The observable z is called the sum of observables x,y and we write $z=x+y$. A pair (L,M) is called a sum logic if it is q.f.s. and L has the properties U and E, (see [4], [3]).

Let x,y be such observables, for which the sum x+y exists. In what follows we shall suppose that the following condition is fulfilled:

α) If $a\in L-\{0\}$ and $R(x)\cup R(y)\leftrightarrow a$ then $x+y\leftrightarrow a$ and $x+y\wedge a= =x\wedge a+y\wedge a$, where $z\wedge a$ is an observable on $L_{[0,a]}$, which is defined by $z\wedge a(E)=z(E)\wedge a$, for $E\in\mathcal{B}(R^1)$. For example, the quantum logic on the Hilbert space satisfies the condition α).

Definition 1.3. Let L be a quantum logic and $\{a_n\}_{n=1}^{\infty}\subset L$. We shall say that the sequence $\{a_n\}_{n=1}^{\infty}$ has a limit equal to a

$$\lim_{n\to\infty} a_n=a \quad\text{if}\quad \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty} a_k = \bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty} a_k = a.$$

Lemma 1.2. Let L be a quantum logic and $\{a_n\}_{n=1}^{\infty}\subset L$. If there exists $\lim_{n\to\infty} a_n=a$ and there is a state m with the property $m(a_n)=1$ for all n, then $m(a)=1$.

The proof is obvious.

Let $\{M_n\}_{n=1}^{\infty}$ be such system of subset of L that $M_n\subset M_{n+1}$ and M_n be p.c.[a] for all n, a be any element $a\in L-\{0\}$. Then it is clear, that $\bigcup_{n=1}^{\infty} M_n$ is p.c.[a]. In addition, if $com(M_n)$ exist for all n, then $com \bigcup_{n=1}^{\infty} M_n = \lim_{n\to\infty} comM_n = \bigwedge_{n=1}^{\infty} com(M_n)$.

2. Conditional Expectation

A relative conditional expectation is defined and analyzed in [7], [8]. At this place we introduce only the definition and some fundamental properties.

Throughout we shall assume that (L,M) be a summable quantum logic fulfilling the condition α).

Definition 2.1. Let $L_0 \subset L$ be a sublogic. Let x be an observable on L, $m \in M$, $a \in L-\{0\}$. We shall suppose that

a) $R(x) \cup L_0$ is p.c.[a];

b) $m(a)=1$;

c) There exists $m(x)$.

Then by a version of conditional expectation of the observable x in the state m, for L_0, relativized by a notation: $E_m(x/L_0,a)$ we understand any observable z with the properties:

a) $z \leftrightarrow a$;

b) $R(z) \wedge a \subset L_0 \wedge a$;

c) For any $b \in L_0$ $\int_b x\, dm = \int_b z\, dm$, where $\int_b x\, dm = \int m(x(d\alpha) \wedge b)$ if the integral on the right side exists.

Definition 2.2. Let x,y be observables. We shall say, that x,y are equal "modulo" a state m $(x \simeq y[m])$ if for any $E \in \mathcal{B}(R^1)$ $m(x(E) \Delta y(E))=0$, where $a \Delta b = (a \wedge b^{\perp}) \vee (a^{\perp} \wedge b)$.

The relation "$\simeq[m]$" is reflexive and symmetric. Moreover, if $R(x) \cup R(y) \cup R(z)$ is p.c.[a], $m(a)=1$, then $x \simeq y[m]$, $y \simeq z[m]$ implies $x \simeq z[\,]$ [7].

Theorem 2.1. [7]. Let x,y be such observables that $R(x) \cup R(y)$ be p.c.[a], where $m(a)=1$ and L_0 be a sublogic of L. Then $E_m(x+y/L_0,m) \simeq E_m(x/L_0,a) + E_m(y/L_0,a)$ [m].

3. Generalized Martingales and Submartingales

In this part we introduce a definition of generalized martingales and submartingales. As before we shall assume that (L,M) is a summable quantum logic with the property α).

Definition 3.1. Let $\{L_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of sublogic of L, $\{x_n\}_{n=1}^{\infty}$ be a sequence of observables on L, $\{a_n\}_{n=1}^{\infty}\subset L$. Let there be a state m with $m(a_n)=1$ for all n. Then the triple (x_n,L_n,a_n) will be called a submartingale in the state m if it is holds:

1) $L_n\subset L_{n+1}$ for all n;
2) $R(x_n)\cup R(x_{n+1})\cup L_n$ is p.c.$[a_n]$ and moreover $R(x_n)\wedge a_n\subset \subset L_n\wedge a_n$.
3) For all $b\in L_n$, $\int_b x_n\, dm \le \int_b E_m(x_{n+1}/L_n,a_n)\, dm$.

A submartingale will be called a martingale in the state m if for any $b\in L_n$

$\int_b x_n\, dm = \int_b E_m(x_{n+1}/L_n,a_n)\, dm$.

Definition 3.2. [5]. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of observables on L. We shall say that

a) x_n converges to x in $L_p(m)$ (denote $x_n\xrightarrow{p}x$), if $m(|x_n-x|^p)\longrightarrow 0$;

b) x_n converges to x in the measur $m(x_n\longrightarrow x[m])$, if for any $\varepsilon>0$ $\lim_n m((x_n-x)[-\varepsilon,\varepsilon])=1$.

c) x_n converges to x almost everywhere with respect to m ($x_n\longrightarrow x$ a.e.[m]) if for any $\varepsilon>0$ $m\Big(\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}(x_n-x)[-\varepsilon,\varepsilon]\Big)=1$.

Lemma 3.1. Let (x_n,L_n,a_n) be a submartingale resp. a martingale in a state m. Let $\{z_n\}_{n=1}^{\infty}$ be any sequence of observables with the following properties:

1) $R(z_n)\cup R(z_{n+1})\cup L_n$ is p.c.$[a_n]$;
2) $R(z_n)\wedge a_n\subset L_n\wedge a_n$;
3) $x_n\simeq z_n[m]$ for all n.

Then (z_n,L_n,a_n) is a submartingale resp. a martingale in a state m.

Proof. Since $x_n\simeq z_n[m]$, we have $\int|x_n-z_n|\, dm=0$. Let $b\in L_n$. Then

$$0 = \int |x_n - z_n| \; dm = \int_{a_n} |x_n \wedge a_n - z_n \wedge a_n| \; dm = \int_{b \wedge a_n} |x_n \wedge a_n - z_n \wedge a_n| \; dm +$$

$$+ \int_{b^{\perp} \wedge a_n} |x_n \wedge a_n - z_n \wedge a_n| \; dm = \int_{b \wedge a_n} |x_n \wedge a_n - z_n \wedge a_n| \; dm = \int_b |x_n - z_n| \; dm \geq$$

$$\geq \left| \int_b x_n - z_n \; dm \right| \geq 0.$$

It means that $\int_b x_n - z_n \; dm = 0$ for all $b \in L_n$. This implies $\int_b x_n dm = \int_b z_n \; dm$. We conclude that (z_n, L_n, a_n) is the submartingale resp. the martingale in the state m. (Q.E.D.)

Let us denote by L(T) the smallest sublogic which contained T $(T \subset L)$.

Lemma 3.2. Let (x_n, L_n, a_n) be a submartingale resp. a martingale in a state m. Let $\{z_n\}_{n=1}^{\infty}$ be any sequence of observables with the following properties:

1) $R(z_n) \wedge a_n \subset L_n \wedge a_n$;
2) $x_n \approx z_n [m]$ for all n.

Moreover let be $a_n \in L_n$ and there exist $\lim_{n \to \infty} a_n = a$. If x is such an observable that $x \leftrightarrow a$ and $R(x) \wedge a \subset L\left(\bigcup_{n=1}^{\infty} L_n\right) \wedge a$, then

a) $x_n \xrightarrow{P} x$ iff $z_n \xrightarrow{P} x$;
b) $x_n \longrightarrow x[m]$ iff $z_n \longrightarrow x[m]$.

Proof. As L_n is p.c.$[a_n]$ and $\{L_n\}_{n=1}^{\infty}$ is nondecreasing subsets of L we have $L\left(\bigcup_{n=1}^{\infty} L_n\right)$ is p.c.[a]. Because $\{a \wedge a_n\} \cup \cup R(x) \wedge a \subset L\left(\bigcup_{n=1}^{\infty} L_n\right) \wedge a$ we have $x \wedge a \vee x \wedge a^{\perp} = x \leftrightarrow a \wedge a_n$. Now we put $x'_n = (x_n \wedge a_n) \vee (x_0 \wedge a_n^{\perp})$, $z'_n = (z_n \wedge a_n) \vee (x_0 \wedge a_n^{\perp})$, where x_0 is such an observable, that $x_0(\{0\}) = 1$, $x_0(\{1\}) = 0$. From this we have $x'_n \leftrightarrow z'_n$ for all n. Moreover $x'_n \approx z'_n [m]$, $x'_n \approx x_n [m]$, $z'_n \approx z_n [m]$. As we can write $x_n = x_n \wedge a_n \vee x_n \wedge a_n^{\perp}$ and $z_n = z_n \wedge a_n \vee z_n \wedge a_n^{\perp}$ then $\{x_n, z_n\} \leftrightarrow a \wedge a_n$ for all n. Then $m((x_n - x)(E)) = m((x_n \wedge a_n \wedge a - x \wedge a_n \wedge a)(E)) = m((x'_n \wedge a_n \wedge a - x \wedge a_n \wedge a)(E)) = m((x'_n - x)(E))$, for all $E \in \mathcal{B}(R^1)$. And $m((z_n - x)(E)) = m((z'_n - x)(E))$ for all n and for all

$E\in\mathcal{B}(R^1)$. Moreover the set $(R(x_n')\cup R(z_n')\cup R(x))\wedge a_n\wedge a\subset L(\bigcup_{n=1}^{\infty}L_n)\wedge a$ and $a\wedge z_n'\wedge a_n\leftrightarrow x_n\wedge a_n\wedge a[m]$. Thus

$$m((x_n'-x)(E)) = m((x_n'\wedge a_n\wedge a-x\wedge a_n\wedge a)(E)) = m((z_n'\wedge a_n\wedge a-x\wedge a_n\wedge a)(E))= = ((z_n'-x)(E)).$$

Now we get

$$m((x_n-x)(E)) = m((z_n-x)(E)) \text{ for any } E\in\mathcal{B}(R^1) \text{ and for all } n.$$

a) Let $x_n\xrightarrow{p}x$. It means $0=\lim_{n\to\infty}\int|x_n-x|^p\,dm$.

But

$$\int|x_n-x|^p\,dm = \int|t|^p m((x_n-x)(dt)) = \int|t|^p m((z_n-x)(dt)) = =\int|z_n-x|^p\,dm.$$

It means that $x_n\xrightarrow{p}x$ iff $z_n\xrightarrow{p}x$.

b) If $x_n\longrightarrow x[m]$ then for all $\varepsilon>0$

$1 = \lim_{n\to\infty} m((x_n-x)[-\varepsilon,\varepsilon]) = \lim_{n\to\infty} m((z_n-x)[-\varepsilon,\varepsilon])$. It means that $x_n\longrightarrow x[m]$ iff $z_n\longrightarrow x[m]$. (Q.E.D.)

Theorem 3.3. Let (L,M) be a summable logic. Let (x_n,L_n,a_n) be a submartingale in the state m, $a_j\in L_n$ and there be a $\lim_{n\to\infty} a_n=a$. Let $\sup_n(|x_n|)<\infty$. Then there exists an observable x with the properties:

$$R(x)\wedge a\subset(\bigcup_{n=1}^{\infty}L_n) \quad\text{and}\quad x_n\longrightarrow x[m].$$

Proof. Put $y_n=(x_n\wedge a_n)\vee(x_0\wedge a_n^{\perp})$. Then (y_n,L_n,a_n) is a submartingale in the state m (Lemma 3.1). From Lemma 3.2 it follows that $x_n\longrightarrow x[m]$ iff $y_n\longrightarrow x[m]$. We know, that $L(\bigcup_{n=1}^{\infty}L_n)$ is p.c.[a]. But $R(y_n)\subset L(\bigcup_{n=1}^{\infty}L_n)\wedge a$ and $L(\bigcup_{n=1}^{\infty}L_n)\wedge a$ is a Boolean-σ-algebra. If we use the Loomis-Sikorski theorem we get $(X,\mathcal{S})$, h, $\{f_n\}_{n=1}^{\infty}$ measurable space, σ-homomorphism, $\mathcal{S}$-measurable function resp. such that $f_n\circ h=y_n\wedge a$. Put $\mathcal{S}_n=\{E\in\mathcal{S} \mid h(E)\in L_n\wedge a\}$. If $C\in\mathcal{B}(R^1)$ then $h(f_n^{-1}(C))\in L_n\wedge a$. It means that $f_n^{-1}(C)\in\mathcal{S}_n$. Hence f_n is the measurable function for any n. Now we have for $E\in\mathcal{S}_n$ $\int_E f_n(t)m_h(dt) = \int_{h(E)} f_n\circ h\ dm = \int_{b\wedge a} y_n\wedge a\ dm =$

$= \int_b y_n dm = \int_b x_n dm \le \int_b x_{n+1} dm = \int_E f_{n+1}(t)m_h(dt)$.

Therefore, $(f_n, \mathcal{S}_n)$ is a submartingale on the probability space $(X,\mathcal{S},m_h)$. Because $m(x_n) = \int f_n(t)m_h(dt)$, $\sup_n m_h(|f_n|)<\infty$.

From this it follows that the assumption for the convergence theorem [9], [2] are fulfilled on some probability space. Thus, there exists a $\sigma(\bigcup_{n=1}^{\infty}\mathcal{S}_n)$-measurable function f (where $(\bigcup_{n=1}^{\infty}\mathcal{S}_n)$ is the smallest σ-algebra which contains $\bigcup_{n=1}^{\infty}\mathcal{S}_n)$ with the property: $f_n\longrightarrow f$ a.e.$[m_h]$. It means that $y_n\wedge a\longrightarrow f\circ h$ a.e.$[m_h]$.

Now we put $x=f\circ h\vee x_0\wedge a^{\perp}$. Then $R(x)\wedge a\subset L(\bigcup_{n=1}^{\infty}L_n)\wedge a$ and $y_n\longrightarrow x[m]$. Finally from Lemma 3.2 we have $x_n\longrightarrow x[m]$. (Q.E.D.)

REFERENCES

[1] **J. Bán:** Martingale convergence theorem in quantum logic, Math. Slovaca 37 (1987) 313-322.

[2] **J.L. Doob:** Stochastic processes. New York, London. Chapman § hall, (1953).

[3] **A. Dvurečenskij, S. Pulmannová:** On the sum of observables in a logic. Math. Slovaca 30 (1980) 393-399.

[4] **S.P. Gudder:** Uniqueness and existence properties of bounded observables. Pac. J. Math. 15(1966) 81-92, 588-589.

[5] **S.P. Gudder, H.G. Mullikin:** Measure theoretic convergences of observables and operators. J. Math. Phys. 14(1973) 234-242.

[6] **S. Loomis:** On the representation of σ-complete Boolean algebras. Bull. AMS 53(1947) 757-760.

[7] **O. Nánásiova, S. Pulmannová:** Relative conditional expectationson a quantum logic. Aplik. Matematiky

30(1985) 332-350.

[8] **O. Nánásiova:** Ordering observables and characterization of conditional expectations on a logic. Math. Slovaca 37(1987) 323-340.

[9] **T. Neubrun, B. Riečan:** Miera a integral. Veda, Bratislava (1981).

[10] **S. Pulmannová:** Compatibility and partial compatibility in a quantum logic. Ann. Inst. H.Poincaré 43 (1981) 391-403.

[11] **S. Pulmannová:** Commutator in orthomodular lattices. Demonstratio Math. 18(1985) 187-208.

[12] **R. Sikorski:** Boolean algebras. Berlin (1964).

[13] **S.V. Varadarajan:** Geometry of quantum theory. Princeton I. Van Nonstrand (1968).

SLOVENSKÁ TECHNICKÁ UNIVERZITA, STAVEBNÁ FAKULTA
KAT. MATEMATIKY A DESK. GEOM.,
RADLINSKÉHO 11, 813 68 BRATISLAVA, CZECHOSLOVAKIA

Received June 15, 1989.