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IMPLICATIVE ORTHOPOSETS

1. Introduction

In [2] the present authors in a cooperation with Professor Jules Varlet, introduced a notion of implication , as a partial operation, on an orthomodular poset P. This operation was supposed to reduce to the Hardegree's implication

$$a \longrightarrow b = a' \vee (a \wedge b) .$$

in the case P be an orthomodular lattice.

Surprisingly, it happens that in the case of implicative orthoposet P (see the definition 7 below)

$$a \longrightarrow b \text{ exists iff } a \wedge b \text{ exists in } P .$$

This statement is the main result of the present paper. It also has some nice consequences, which are discussed in section 4 below.

2. Basic definition and notions

Let us recall that an orthocomplemented poset, abbreviated orthoposet, (see [1]) is an algebraic structure

$$(P, \le, ', 0, 1),$$

where (P, ≤, 0, 1) is a poset with the least and the greatest element - denoted by 0 , 1 respectively, and $a \longmapsto a'$ is a unary operation on P satisfying the following conditions:

(OCP1) $a'' = a$

(OCP2) $a \le b \Longrightarrow b' \le a'$

(OCP3) $a \le b' \Longrightarrow a \vee b$ exists

(OCP4) $a \vee a' = 1$

An orthoposet P is said to be orthomodular if it satisfies:

(OCP5) $(a \le b \text{ and } a \vee b' = 1) \Longrightarrow a = b$.

If $a \le b'$, we say that a and b are orthogonal and we write $a \perp b$. We say that a commutes with b and we write aCb, if there exist mutually orthogonal elements a_1, b_1, c in P such that :

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c .$$

An ortoposet P is said to be a Boolean poset (see [4]) if it satisfies:

$$\text{(BP)} \quad a \wedge b = 0 \Longrightarrow a \perp b .$$

Every Boolean poset is orthomodular. In a Boolean poset, if for some a, b the l.u.b. $a \vee b$ exists, then aCb (see [4]) . We say that an orthomodular poset (P, ≤, ′, 0, 1) is a horizontal sum of a family $\{(P_t, \le, ', 0, 1), t \in T\}$ of orthomodular posets (see [1]) , if the following conditions hold :

H1) $t \ne s \Longrightarrow P_t \cap P_s = \{0, 1\}$,

H2) $P = \bigcup \{P_t, t \in T\}$,

H3) $a \le b$ in P iff there exists $t \in T$ such that $a, b \in P_t$, and $a \le b$ in P_t ,

H4) the unary operation ′ on P_t is a restriction of the operation ′ on P .

A partial field of sets (see [1]) is a nonempty family M of subsets of a set X satisfying for any $A, B \in M$ the following conditions:

(PFS1) $A \in M \Longrightarrow X \setminus A \in M$,

(PFS2) $[A, B \in M, A \cap B = \emptyset] \Longrightarrow A \cup B \in M$.

Observe that a partial field of sets M forms an orthomodular poset under inclusion as a partial order and natural operation

$$A' = X \setminus A ,$$

where $0 = \emptyset$, $1 = X$ and for any $A, B \in M$, $A \perp B$ iff $A \cap B = \emptyset$.

A typical, nontrivial example of a partial field of sets is a family M of even subsets of even (finite) set X . Before giving another example let us observe that if for any $x \in X$, $\{x\} \in M$ then M forms a Boolean poset .

Example 1. Let $X = \{1, 2, 3, 4, \dots\}$, $A = \{2, 4, 6, 8, \dots\}$, $B = \{3, 6, 9, 12, \dots\}$. We define a family M as follows :
$Z \in M$ iff
1) Z is finite or
2) $X \setminus Z$ is finite or
3) there exist finite subsets C , D of X such that
$Z = (A \setminus C) \cup D$ or
4) there exist finite subsets C, D of X such that
$Z = ((X \setminus A) \setminus C) \cup D$ or
5) there exist finite subsets C , D of X such that
$Z = (B \setminus C) \cup D$ or
6) there exist finite subsets C, D of X such that
$Z = ((X \setminus B) \setminus C) \cup D$.
Observe that this family is a partial field of sets and forms a Boolean poset (we denote it by K2). Moreover $A \wedge B$ does not exist in M. The idea of this construction is due to J. Klukowski ([4]) .

3. Implicative orthoposets

Definition 2. (see [2]). An orthoimplication on an orthoposet P is a partial binary operation $\longrightarrow$ defined as follows :

$c = a \longrightarrow b$ iff c is the smallest element in P such that:
1) $a' \leq c$
2) $(x \leq a \text{ and } x \leq b) \Longrightarrow x \leq c$.

The above definition is a natural generalization of Hardegree's orthoimplication (see [3]).

Lemma 3. (see [2]). $a \longrightarrow b$ exists in P iff the least upper bound

$$c = \bigvee_{x \leq a,b} (a' \vee x)$$ exists. Then $a \longrightarrow b = c$.

Corollary 4. If $a \wedge b$ exists in P then $a \longrightarrow b$ exists and:

$$a \longrightarrow b = a \longrightarrow (a \wedge b) = a' \vee (a \wedge b) .$$

Example 5. Observe that in the orthoposet K2 (Example 2.1), $A \longrightarrow B$ does not exist.

It follows from Corollary 4 that in particular,

$$\text{if } a \leq b \text{ , then } a \longrightarrow b = 1 \text{ .}$$

The converse statement is not true in general, even if the orthoposet P is orthomodular.

Example 6. Let M be a partial field of even subsets of the set $X = \{ 1 , 2 , 3 , 4 , 5 , 6 \}$. Let $A = \{ 1 , 2 , 3 , 4 \}$, and $B = \{ 1 , 2 , 3 , 5 \}$. Then $A \longrightarrow B = 1$, and $A \not\leq B$.

Definition 7. (see [2]). An orthoposet P is said to be implicative if

$$a \longrightarrow b = 1 \quad \text{implies} \quad a \leq b \text{ , for all } a,b \in P \text{ .}$$

In [2] it is proved that every implicative orthoposet is orthomodular. Observe that a horizontal sum of a family F of orthomodular posets is implicative iff every member of F is implicative .

Lemma 8. Let P be an orthomodular poset. If $a \longrightarrow b$ exists in P then :

1) $a \wedge (a \longrightarrow b)$ exists
2) $[a \wedge (a \longrightarrow b)] \longrightarrow b = 1$
3) $[x \leq a \text{ and } x \leq b] \Longrightarrow x \leq a \wedge (a \longrightarrow b)$.

Proof. 1) Observe that $a \, C \, (a \longrightarrow b)$ because $a' \leq a \longrightarrow b$. Then the meet $c = a \wedge (a \longrightarrow b)$ exists .
2) We have to show that $c \longrightarrow b = 1$. Suppose that $d \in P$ is such that :

1) $c' \leq d$
2) $[x \leq c \text{ and } x \leq b] \Longrightarrow x \leq d$

or, equivalently :

1a) $a' \leq d$
1b) $(a \longrightarrow b)' \leq d$
2) $[x \leq c \text{ and } x \leq b] \Longrightarrow x \leq d$.

It follows from 1a) and 2) that $a \longrightarrow b \leq d$. Then from this fact and from 1b) we obtain $d = 1$.
3) If $x \leq a$ and $x \leq b$ then it follows from the definition of orthoimplication that $x \leq a \longrightarrow b$. Thus $x \leq a \wedge (a \longrightarrow b)$.

Now we can prove the converse to Corollary 4 .

Proposition 9. In an implicative orthoposet P, if $a \to b$ exists then $a \wedge b$ exists in P and $a \wedge b = a \wedge (a \to b)$.

Proof. Evidently $a \wedge (a \to b) \le a$. From Lemma 8.2) it follows that $a \wedge (a \to b) \le b$. This fact together with Lemma 8.3) completes the proof.

Corollary 10. Let P be an implicative orthoposet such that $a \to b$ exists for all $a,b \in P$. Then P is an orthomodular lattice.

Definition 11. (see [2]). A partial orthomodular lattice (abbreviated poml) is an orthomodular poset satisfying the condition:

$$a \wedge b \text{ exists} \implies a' \wedge b \text{ exists.}$$

Observe that any orthomodular lattice and any Boolean poset is a poml . Moreover a horizontal sum (as well as a product) of a family F of orthomodular posets is a poml iff every member of F is .

In [2] it is proved that every poml is implicative. In particulary any orthomodular lattice and any Boolean poset is implicative.

Proposition 12. (see [2]). Let P be an implicative orthoposet. Then the following conditions are satistied:

(1) $a \to 0 = a'$

$(a \to 0) \to 0 = a$

(2) $0 \to a = 1$

(3) $(a \to b = b \to a = 1) \implies a = b$

(5) if $a \to b,\ a \to c$ exist then

$(b \to c = 1) \implies (a \to b) \to (a \to c) = 1$

(6) $(a \to b = 1) \implies b \to a = (a \to 0) \to (b \to 0)$

(7) if $c \to a$ exists then :

$a \to (b \to 0) = b \to c = 1 \implies (c \to a) \to (b \to a) = 1$

(8) if $a \to b$ exists then

$(a \to b) \to (a \to 0) = (b \to a) \to (b \to 0)$

4. Partial orthoimplicative algebras

In [2] an orthoimplicative algebra has been introduced and investigated. It is, by definition an algebra $(P, \cdot, 0, 1)$ of type $(2, 0, 0)$ for which the basic properties (1) - (8) of $\longrightarrow$ serve as the set of axioms. It was proved in [2] that if we define an ordering relation in P by

$$a \leq b \quad \text{iff} \quad ab = 1$$

and an orthocomplementation operation by

$$a' = a0$$

then the associated system $(P, \leq, ', 0, 1)$ is an implicative orthoposet and $a \longrightarrow b \leq ab$ whenever $a \longrightarrow b$ exists.

In this section we define a partial orthoimplicative algebra as a partial algebra $(P, \cdot, 0, 1)$ with two constants 0 and 1 and the partial binary operation $\cdot$ fullfils some axioms. The set of axioms is divided into 3 parts.

The first part (axioms 1-3) contains existential axioms. Here $p(x_1,..,x_n)$ denotes any term function over $\cdot$. The axioms say when $p(x_1,..,x_n)$ must exist.

The second part (axiom 4) contain axioms of the form:

$$p(x) = q(x)$$

where the existence of p and q follows immediately from the axioms of first part.

The third part (axioms 5-11) contains axioms of the form: if $p_1(x_1,..,x_n)$, .., $p_k(x_1,..,x_n)$ exist and $\Phi(p_1,..,p_n)$ holds then $q_1(x_1,..,x_n)$, .., $q_1(x_1,..,x_n)$ exist and $\Psi(p_1,..,p_n, q_1,..,q_1)$ holds, where Φ and Ψ are conjuctions of equalities of terms p_i and q_i .

Definition 13. Let $(P, \cdot, 0, 1)$ be a partial algebra with two constants 0 and 1 and a partial binary operation $\cdot$. We say that P is a partial orthoimplicative algebra if the following hold (where T denotes the set of all term function over $\cdot$):

A1. $p(0, a, 1)$ exists for any $a \in P$ and any $p \in T$,

A2. ab exists $\Longrightarrow$ ba exists for any $a, b \in P$, A3. ab exists and ab = 1 $\Longrightarrow$ p(0, a, b, 1) exists for any a, b$\in$P and any p$\in$T,

A4. 0a = 1 for any a$\in$P ,

A5. if ab exists and ab = ba = 1 then a = b ,

A6. if ab exists and ab = 1 then :

1. ba = a0·b0
2. b0·a0 = 1
3. b0·a = b
4. a0·b = b

A7. if ab exists then a0·ab exists and a0·ab = 1 ,

A8. if ab, bc exist and ab = bc = 1 then ac exists and ac = 1 ,

A9. if a·b0, bc, ac exist and a·b0 = bc = 1 then ca·ba exists and ca·ba = 1 ,

A10. if ab exists then ab·a0 = ba·b0 ,

A11. if ab, bc, ac exist and bc = 1 then ab·ac exists and ab·ac = 1.

The main result of this section is the following

Theorem 14. Let (P, ·, 0, 1) be a partial orthoimplicative algebra. We define:

$$a \le b \quad \text{iff} \quad ab \text{ exists and } ab = 1$$

$$a' = a0.$$

Then (P, ≤, ′, 0, 1) is an orthomodular poset.
We will call it shortly an associated orthomodular poset.
We precede the proof with some lemmas.

Lemma 15. Let (P, ·, 0, 1) be a partial orthoimplicative algebra. Then the following hold:

(W1) 00 = 1 ,

(W2) x0·0 = x ,

(W3) 1x = x ,

(W4) x1 = 1 ,

(W5) xx = 1 ,

(W6) if x·y0 exists and x·y0 = 1, then y·x0 = 1 and x0·y = y0·x,

Proof.

(W1): put $a = 0$ in A4 ,
(W2): put $a = 0$, $b = x$ in A6.3 ,
(W3): put $a = 0$, $b = x$ in A6.4 and apply (W1) ,
(W4): put $a = 0$, $b = x0$ in A6.2 and apply (W1) and (W2) ,
(W5): put $b = 0$, $a = x0$ in A7 and apply (W2) ,
(W6): put $a = x$, $b = y0$ in A6.2 and apply (W2) ,
put $a = x$, $b = y0$ in A6.1 and apply (W2) .

Lemma 16. Let $(P, \cdot, 0, 1)$ be a partial orthoimplicative algebra and $\leq$ be the relation defined in Theorem 14. Then $(P, \leq, 0, 1)$ is a bounded poset.

Proof. This is a consequence of axioms A4, A5, A8 and the above Lemma 15 (W4, W5).

Lemma 17. Let $(P, \cdot, 0, 1)$ be a partial orthoimplicative algebra and $\leq$ be the partial order defined in Theorem 14. Suppose that for some $x, y \in P$, $x \leq y0$. Then $y0 \cdot x$ is the least upper bound of x and y, i.e.:

$$x \leq y0 \Longrightarrow y0 \cdot x = x \vee y.$$

Proof. First we prove that $x , y \leq y0 \cdot x$. Put $a = x0$, $b = y$ in A7 we obtain $(x0 \cdot 0) \leq (x0 \cdot y)$, i.e. $x \leq x0 \cdot y$. Observe that it follows from Lemma 15. (W6) that $y \leq x0$ and $y0 \cdot x = x0 \cdot y$. Then $y \leq y0 \cdot x = x0 \cdot y$.

Now suppose that for some $z \in P$ we have $x , y \leq z$. We will show that $y0 \cdot x \leq z$. Put $a = x$, $b = z0$, $c = y0$ in A9. We obtain: $(y0 \cdot x) \leq (z0 \cdot x)$. Put $a = x$, $b = z$ in A6.3. We obtain $z0 \cdot x = z$. Thus $y0 \cdot x \leq z$.

Corollary 18. If $x \leq y$ then $y \cdot x = x \vee y0$.

Lemma 19. Let $(P, \cdot, 0, 1)$ be a partial orthoimplicative algebra and let $x, y, z \in P$ be such that xy exists and $z \leq x, y$. Then $z \leq xy$.

Proof. Put $a = x$, $b = z$, $c = y$ in A11, we obtain $xz \leq xy$. But $xz = z \vee x0 \geq z$. Then $z \leq xy$.

Proof of Theorem. It was proved (Lemma 16) that $(P, \le, 0, 1)$ is a bounded poset. The property $x'' = x$ follows from Lemma 15 (W2). The implication $x \le y \Longrightarrow y' \le x'$ follows from A6.2. It is proved in Lemma 17 that $x \le y0 \Longrightarrow x \vee y = x0 \cdot y = y0 \cdot x$. Hence (putting $y = x0$) we obtain $x \vee x' = x0 \cdot x0 = 1$. Finally suppose that $x \le y$ and $x \vee y' = 1$. We have to prove that $x=y$. Actually, if $x \le y$ then (Corollary 18) $x \vee y0 = yx$, thus $yx=1$, i.e. $y \le x$. Therefore $x=y$. This completes the proof of our Theorem.

Lemma 20. Let $(P, \cdot, 0, 1)$ be a partial orthoimplicative algebra with associated orthomodular poset $(P, \le, ', 0, 1)$. Let $a, b \in P$ be such that ab exists in P. Then $a \wedge b$ exists and

$$a \wedge b = a \wedge ab = b \wedge ba .$$

Proof. (A7) says that $a' \le ab$. Then (Corollary 18) $ab \cdot a' = a' \vee (ab)' = (a \wedge ab)'$. But (A10) $ab \cdot a' = ba \cdot b'$. Hence $a \wedge ab = b \wedge ba$. Therefore $a \wedge ab \le a, b$. On the other side, if $c \le a, b$ then (Lemma 19) $c \le a \wedge ab$.

Corollary 21. If $\cdot$ is a total operation on P then the associated orthomodular poset $(P, \le, ', 0, 1)$ forms an orthomodular lattice

Theorem 22. If ab exists in a partial orthoimplicative algebra then $a \longrightarrow b$ exists in the associated orthomodular poset and $a \longrightarrow b = ab$.

Proof. If ab exists then (Lemma 20) $a \wedge b$ exists and $a \longrightarrow b$ exists. Moreover $a \longrightarrow b = a' \vee (a \wedge b) = a' \vee (a \wedge ab)$. But $a' \le ab$. Then $a' \vee (a \wedge ab) = ab$. Therefore $a \longrightarrow b = ab$.

Example 2. Let $(P, \le, ', 0, 1)$ be an orthomodular poset. Define a partial operation $\cdot$ on P as follows:

$$a \cdot b \text{ exists iff } a \, C \, b \text{ ; in this case } a \cdot b = a' \vee b .$$

Then $(P, \cdot, 0, 1)$ forms a partial orthoimplicative algebra such that $a \le b$ iff $ab = 1$.

This example shows that, in general, the orthomodular poset associated with a partial orthoimplicative algebra can be non-implicative.

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