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GENERALIZED ORTHOMODULAR POSETS

Introduction

A weak generalized orthomodular poset (abbreviated as WGOMP) is defined as a poset, every interval [0,a] of which is equipped with a structure of orthomodular poset, satisfying some axioms. A structure of WGOMP can be defined on the set of all idempotent elements of any ring, and also on any *-ring satisfying the *-cancellation law [7].

Theorem 1 states that any WGOMP can be embedded in an orthomodular poset. This embedding preserves all existing infima, and the supremum of any two orthogonal elements.

Those WGOMPs for which the above embedding preserves all existing suprema of any two elements are characterized by a simple condition, and are called generalized orthomodular posets (abbreviated as GOMPs).

Generalized orthomodular lattices defined by Janowitz [4] are examples of GOMPs and we compare the embedding given in [4] with ours in this particular case.

An interesting class of GOMPs consists of all Rickart *-rings equipped with the *-order introduced by M.P.Drazin [3].

For notation and basic notions concerning orthomodular posets and orthomodular lattices, let us refer to [1] or [6]. For Rickart *-rings, the standard reference is [2].

1. Weak generalized orthomodular posets

Definition 1. Let $(A,\le)$ be a poset with the least element 0, such that every interval [0,a] of A is equipped with an unary operation $x \longrightarrow x^{\#_a}$. We shall say that A is a weak generalized orthomodular poset (abbreviated as WGOMP) if it satisfies the following conditions:

(G1) If $a \in A$ then $([0,a], \leq, \#_a)$ is an orthomodular poset.

(G2) If $a \leq b \leq c$ then $a^{\#_b} = a^{\#_c} \wedge b$.

(G3) If $a \leq c$, $b \leq c$ and $a \leq b^{\#_c}$ then the supremum $a \vee b$ exists in A.

We shall write $a \perp b$ if and only if $a \vee b$ exists in A and $a \leq b^{\#_{a \vee b}}$.

(G4) If $a,b,c \in A$ with $a \perp b$, $c \perp a$ and $c \perp b$ then $c \perp a \vee b$.

Remarks 1.

(a) It is easy to prove that $a \perp b$ if and only if there exists $c \in A$ with $a \leq c$, $b \leq c$ and $a \leq b^{\#_c}$.

(b) If $a \perp b$, $b \leq c$ and $a \leq c$ then $a \leq b^{\#_c}$.

(c) The poset of figure 1 is an example of poset satisfying Axioms (G1),(G2),(G3) but not Axiom (G4).

The poset of figure 2 satisfies Axioms (G1),(G2),(G4) but not Axiom (G3).

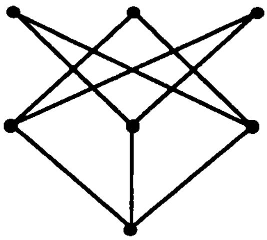

Fig. 1

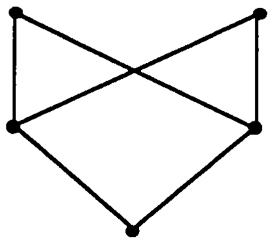

Fig. 2

Examples 1.

(a) Any orthomodular poset is a WGOMP.

(b) A generalized orthomodular lattice [4] is a WGOMP.

(c) The poset of figure 3 is a WGOMP.

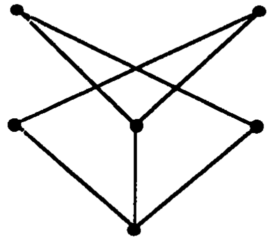

Fig. 3

The set of all idempotent elements of a ring also provides an example of WGOMP. Precisely, let R be a ring and A be the set of all its idempotent elements. Let us consider the binary relation $\leq$ on A defined by $x\leq y$ if and only if $x=xy=yx$. Clearly, the relation $\leq$ is an order relation. For any $a\in A$ and any $x\in[0,a]$ define $x^{\#_a}=a-x$.

Lemma 1. (i) The operation $x\longrightarrow x^{\#_a}$ is an orthocomplementation on $[0,a]$.

(ii) If $x,y\in A$ with $xy=yx$ the supremum $x\vee y$ and the infimum $x\wedge y$ of x,y exist in A. We have $x\vee y=x+y-xy$ and $x\wedge y=xy$.

(iii) If $x,y\in[0,a]$ with $x\leq y^{\#_a}$ then $x\vee y$ exists.

(iv) If $x,y\in[0,a]$ with $x\leq y$ then $y=x\vee(x^{\#_a}\wedge y)$ holds.

Proof. Statements (i) and (ii) can be proved by a straightforward calculation.

(iii) If $x\leq y^{\#_a}$ then $xy=yx=0$, and we can apply (ii).

(iv) If $x\leq y$ we have $x=xy=yx$ hence $x^{\#_a}y=yx^{\#_a}$ and, by (ii), $x^{\#_a}\wedge y$ exists. As $(x^{\#_a}\wedge y)x=x(x^{\#_a}\wedge y)=0$, $x\vee(x^{\#_a}\wedge y)$ exists and $x\vee(x^{\#_a}\wedge y)=x+(a-x)y=x+y-xy=y$. □

Note, by Lemma 1, that $[0,a]$ is an orthomodular poset and thus, A is an orthomodular poset if R has a unit.

Lemma 2. For elements a and b of A, $a\perp b$ if and only if $ab=ba=0$ (where the relation $\perp$ is defined as in Definition 1).

Proof. Clearly, if $ab=ba=0$ then $a\perp b$. Conversely, if $a\vee b$

exists and $a \leq b^{\#_{a \vee b}}$ we have $a=a(a \vee b-b)=(a \vee b-b)a$, thus $a=a-ab=a-ba$, and $0=ab=ba$. □

Proposition 1. The set of all idempotent elements of a ring is a WGOMP.

Proof. Axiom (G1) follows from Lemma 1. Using Lemma 2 we show easily that (G2), (G3) and (G4) are satisfied. □

An important example of WGOMP is given by *-rings equipped with the *-order. We will study this example in the following section.

2. The example of *-order in *-rings

In this section A will denote a *-ring. That means that on A there exists an involution $x \longrightarrow x^*$ satisfying $(x+y)^*=x^*+y^*$ and $(xy)^*=y^*x^*$. We assume further that A fulfills the so-called *-cancellation law:

$$a^*a=0 \text{ implies } a=0.$$

In such a ring the *-order defined as follows:

$$a \leq b \text{ if and only if } a^*a=a^*b \text{ and } aa^*=ba^* \text{ hold,}$$

is an order relation [3].

In a *-ring one can also define an orthogonality relation [7] by:

$$a \perp b \text{ if and only if } a^*b=ab^*=0.$$

We shall use results that can be found in [5]. Among them are:

(R_1) If $x \perp y$ then $x \wedge y$, $x \vee y$ exist and $x \wedge y=0$, $x \vee y=x+y$.

(R_2) Every interval $[0,x]$ of A is an orthomodular poset and for $a \in [0,x]$ the orthocomplement of a is $a^{\#_x}=x-a$.

The following lemma proves that, for *-rings, the orthogonality relation of [7] agrees with the one introduced in Section 1.

Lemma 3. For elements a and b of A, $a^*b=ab^*=0$ if and only if there exists $c \in A$ such that $a \leq c$, $b \leq c$ and $a \leq b^{\#_c}$.

Proof. Assume $a^*b=ab^*=0$, then by (R1) we have $a \vee b=a+b$ and so $a \leq a \vee b-b=b^{\#_{a \vee b}}$. Conversely, if there exists $c \in A$ such that $a \leq c$, $b \leq c$ and $a \leq b^{\#_c}$ then, as $a \leq c-b$, we have $a^*a=$

$=a^*(c-b)=a^*c-a^*b$. Now, $a\le c$ implies $a^*a=a^*c$ and therefore, $a^*b=0$. Similarly, $ba^*=0$ holds. □

Proposition 2. Every *-ring satisfying the *-cancellation law is a WGOMP.

Proof. Axiom (G1). It is the result (R_2).

Axiom (G2). Let $a,b,c\in A$ and assume $a\le b\le c$. As $[0,c]$ is an orthomodular poset, $a\le b$ implies $b=a\vee(a^{\#c}\wedge b)$. By $a\perp a^{\#c}\wedge b$, we infer $a\vee(a^{\#c}\wedge b)=a+(a^{\#c}\wedge b)$ and so, $a^{\#b}=b-a^{\#c}\wedge b$.

Axiom (G3). Obvious, by making use of (R1) and Lemma 3.

Axiom (G4). Let $a,b,c\in A$ such that $a\perp b$, $c\perp a$ and $c\perp b$. We have $a\vee b=a+b$, $c^*a=ca^*=0$, $c^*b=cb^*=0$. Thus $c^*(a\vee b)=c^*a+c^*b=0$ and $c(a\vee b)^*=ca^*+cb^*=0$. Therefore $c\perp a\vee b$ and (G4) holds. □

Examples 2. (a) A commutative ring A has a canonical structure of *-ring with $x^*=x$. Such a ring satisfies the *-cancellation law if and only if it has no non-zero nilpotent element and, in this case, the *-order is defined by $a^2=ab$. Recall that commutative rings without a non-zero nilpotent element are all subrings of direct products of fields.

(b) Every Rickart *-ring fulfills the *-cancellation law and, equipped with the *-order, is a WGOMP. Note that *-order extends the classical order on the set of all projections.

(c) In a C^*-algebra, the norm satisfies the identity $\|x^*x\|=\|x\|^2$. Hence, $x^*x=0$ implies $x=0$ and a C^*-algebra, when endowed with the *-order, is a WGOMP.

3. The embedding

This section, in which A denotes a WGOMP, is devoted to the proof of the following result:

Theorem 1. Every WGOMP A can be embedded as an order ideal in an orthomodular poset $\hat{A}$ so that, for any $x\in\hat{A}$, $x\in A$ or $x^\perp\in A$.

Let $A^{\#}$ be a set disjoint from A with the same cardinality. Consider a bijection $a\longrightarrow a^{\#}$ from A onto $A^{\#}$ and let us denote the set $A\cup A^{\#}$ by $\hat{A}$.

Lemma 4. Consider the relation $\le^*$ on $\hat{A}$ defined by the following conditions:

(i) For $a\in A$, $b\in A$, $a\leq^{*}b$ holds if and only if $a\leq b$.

(ii) For $a\in A$, $b\in A^{\#}$ with $b=b_1^{\#}$, $a\leq^{*}b$ holds if and only if $a\perp b_1$.

(iii) For $a\in A^{\#}$, $b\in A^{\#}$ with $a=a_1^{\#}$, $b=b_1^{\#}$, $a\leq^{*}b$ holds if and only if $b_1\leq a_1$.

The relation $\leq^{*}$ is an order relation on $\hat{A}$.

Proof. The reflexive and antisymmetric laws are obvious. In order to prove the transitive law, let $a,b,c\in\hat{A}$ such that $a\leq^{*}b$ and $b\leq^{*}c$.

If $a,b,c\in A$ or $a,b,c\in A^{\#}$ it is obvious that $a\leq^{*}c$.

If $a,b\in A$ and $c\in A^{\#}$ with $c=c_1^{\#}$, $a\leq^{*}b$ and $b\leq^{*}c$ mean $a\leq b$ and $c_1\perp b$.

Then there exists $m\in A$ with $c_1\leq m$, $b\leq m$ and $b\leq c_1^{\#_m}$. We have $a\leq m$, $c_1\leq m$ and $a\leq c_1^{\#_m}$, therefore (by Remark 1 (a)), $a\perp c_1$ and $a\leq^{*}c$.

If $a\in A$ and $b,c\in A^{\#}$ the proof is similar. □

Note that $(\hat{A},\leq^{*})$ is a bounded poset with 0 as smallest element and $0^{\#}$ as largest element.

We define $a^{\perp}$, for any a in $\hat{A}$, by:

$$a^{\perp}=a^{\#} \quad \text{if} \quad a\in A,$$
$$a^{\perp}=a_1 \quad \text{if} \quad a\in A^{\#},\ a=a_1^{\#}.$$

Lemma 5. The map $a\longrightarrow a^{\perp}$ is an orthocomplementation on $\hat{A}$.

Proof. Clearly, $(a^{\perp})^{\perp}=a$, and $a\leq^{*}b$ implies $b^{\perp}\leq^{*}a^{\perp}$ follows easily from definitions. In order to prove that $a\wedge a^{\perp}=0$ holds for all $a\in\hat{A}$, we consider two cases.

If $a\in A$ and if m is a lower bound of a and $a^{\perp}=a^{\#}$ then $m\in A$ and we have in A, $m\leq a$ and $m\leq a^{\#_a}$. Therefore $m=0$ and $a\wedge a^{\perp}=0$.

If $a\in A^{\#}$ with $a=a_1^{\#}$, then $a^{\perp}=a_1$ and $a_1\wedge a_1^{\#}=0$ by the same proof. □

Lemma 6. If $a,b\in\hat{A}$ with $a\leq^{*}b^{\perp}$ then $a\vee b$ exists in $\hat{A}$.

Proof. We consider different cases.

If $a,b\in A$, $a\leq^{*}b^{\perp}$ means $a\leq^{*}b^{\#}$. Thus $a\perp b$ holds and $a\vee b$ exists in A. If $m^{\#}\in A^{\#}$ is an upper bound of a and b in $\hat{A}$ then we have

$a\perp m$ and $b\perp m$. It follows from (G4) that $m\perp a\vee b$ and $a\vee b\leq^{*} m^{\#}$. Hence, the supremum of a and b in A is the supremum of a and b in $\hat{A}$.

If $a\in A$, $b\in A^{\#}$ with $b=b_1^{\#}$, and $a\leq^{*} b^{\perp}$ then, as $a\leq b_1$, we have $a^{\#_{b_1}}\in[0,b_1]$ and $b_1^{\#}\leq^{*}\left(a^{\#_{b_1}}\right)^{\#}$ thus $b\leq^{*}\left(a^{\#_{b_1}}\right)^{\#}$. Moreover, as $a\perp a^{\#_{b_1}}$, $a\leq^{*}\left(a^{\#_{b_1}}\right)^{\#}$ holds. Let $c\in\hat{A}$ with $c=c_1^{\#}$, $c_1\in A$ and assume that $a\leq^{*} c$ and $b\leq^{*} c$. As $a\leq b_1$, $a\perp c_1$ and $c_1\leq b_1$ it follows from Remark 1(b) that $a\leq c_1^{\#_{b_1}}$. Therefore, $c_1\leq a^{\#_{b_1}}$, $\left(a^{\#_{b_1}}\right)^{\#}\leq^{*} c$ and thus the supremum of a and b exists in $\hat{A}$ and $a\vee b=\left(a^{\#_{b_1}}\right)^{\#}$.

If $a\in A^{\#}$ and $b\in A$ the the proof is similar. □

Remark 2. The second part of the above proof shows that if $a,b_1\in A$ and $a\leq b_1$ then the supremum of a and $b_1^{\#}$ exists in $\hat{A}$ and $a\vee b_1^{\#}=\left(a^{\#_{b_1}}\right)^{\#}$.

Lemma 7. For two elements a and b of $\hat{A}$, $a\leq^{*} b$ implies $b=a\vee(a^{\perp}\wedge b)$.

Proof. If $a,b\in A$, with $a\leq^{*} b$, then $a^{\#_b}\leq^{*} b$ and $a^{\#_b}\leq^{*} a^{\#}$ as $a^{\#_b}\perp a$. If $c\in\hat{A}$ with $c\leq^{*} b$ and $c\leq^{*} a^{\#}$ we have $c\in A$ and $c\leq b$, $c\perp a$. As $a\leq b$, it follows by Remark 1(b) that $c\leq a^{\#_b}$. Hence $a^{\#}\wedge b$ exists and $a^{\#}\wedge b=a^{\perp}\wedge b=a^{\#_b}$. Since $a^{\#_b}\perp a$, $a\vee a^{\#_b}$ exists and $a\vee a^{\#_b}=b$.

Now consider $a\in A$, $b=b_1^{\#}\in A^{\#}$ and suppose $a\ll^{*} b$. As $a\perp b_1$, $a\vee b_1$ exists and we have $a\leq a\vee b_1$ and $b_1\leq a\vee b_1$. Therefore $(a\vee b_1)^{\#}\leq a^{\#}$ and $(a\vee b_1)^{\#}\leq b_1^{\#}$. Consider a lower bound c of $a^{\#}$ and $b_1^{\#}$. If $c\in A$, then $c\perp a$ and $c\perp b_1$. Hence, by (G4), $c\perp a\vee b_1$ and $c\leq^{*}(a\vee b_1)^{\#}$. Now, if $c=c_1^{\#}\in A^{\#}$ then $a\leq c_1$ and $b_1\leq c_1$. Thus $a\vee b_1\leq c_1$ and $c\leq(a\vee b_1)^{\#}$. Hence $a^{\perp}\wedge b$ exists and $a^{\perp}\wedge b=(a\vee b_1)^{\#}$.

As $a\leq a\vee b_1$, it follows from Remark 2 that $a\vee(a\vee b_1)^{\#}=$ $=(a^{\#_{a\vee b_1}})^{\#}$.

We have $b_1\leq a^{\#_{a\vee b_1}}$ and, by using the orthomodular law in

$[0, a\vee b_1]$, we infer that:

$$a^{\#_{a\vee b_1}} = b_1\vee(b_1^{\#_{a\vee b_1}}\wedge a^{\#_{a\vee b_1}}) = b_1\vee 0 = b_1.$$

Therefore, $a\vee(a\vee b_1)^{\#}=b_1^{\bullet}=b$ and the result follows.

If $a=a_1^{\#}\in A^{\#}$, $b=b_1^{\#}\in A^{\#}$ and $a\leq^{\bullet} b$ then $b_1\leq a_1$ and we have $b_1^{\#}\wedge a_1=b\wedge a^{\perp}=b_1^{\#_{a_1}}$. But $b_1^{\#_{a_1}}\vee a_1^{\#}=((b_1^{\#_{a_1}})^{\#_{a_1}})^{\#}$ by Remark 2 and thus, $a\vee(a^{\perp}\wedge b)=b$. □

As an immediate consequence of Lemmas 4,5,6 and 7 we have:

Proposition 2. If A is a WGOMP then A is an orthomodular poset.

Examples 3. (a) Let R be a ring without unit and A be the WGOMP of all its idempotent elements. The ring R can be embedded in a ring with unit $R^{\bullet}$. It is easy to show that the set $I(R^{\bullet})$ of all its idempotent elements is the set $A\cup\{1-a \mid a\in A\}$. Obviously, $I(R^{\bullet})$ is an orthomodular poset canonically isomorphic to the set $\hat{A}$ obtained from A by the previous construction.

(b) If A is the WGOMP of fig.3 then $\hat{A}$ is the orthomodular lattice G_{12} of figure 4.

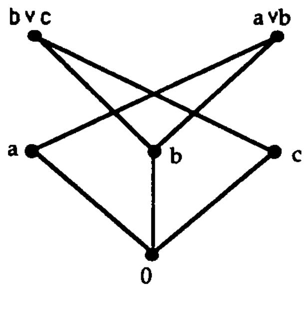

Fig. 3

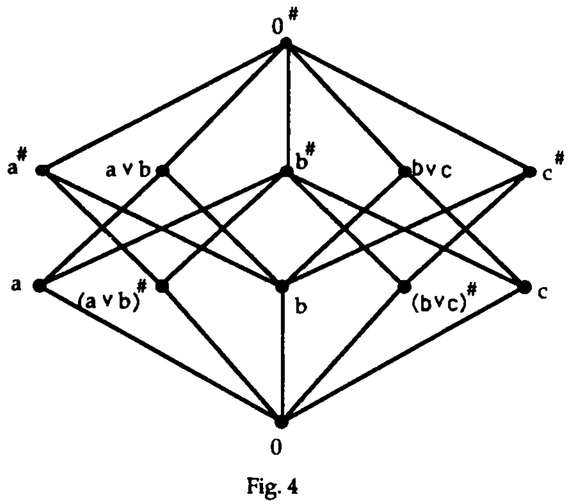

Fig. 4

(c) If A is already an orthomodular poset then $\hat{A}$ is isomorphic to the direct product $A\times\{0,1\}$.

Indeed, consider the map $\varphi:\hat{A}\longrightarrow A\times\{0,1\}$ defined as follows:

if $x\in A$, $\varphi(x)=(x,0)$,

if $x^{\#}\in A^{\#}$, $\varphi(x^{\#})=(x^{\perp_A},1)$ where $x^{\perp_A}$ denotes the orthocomplement of x in A.

Then, it is easy to show that φ is an isomorphism.

4. Generalized orthomodular poset

The embedding of a WGOMP A in an orthomodular poset $\hat{A}$ preserves the infimum but not generally the supremum whenever they exist in A. If $a,b\in A$ with $a\perp b$, then, by (G4), the supremum of a and b in A is also the supremum of a and b in $\hat{A}$. But if we consider the WGOMP A of figure 5, suggested by M.Roddy, we have $z\perp x$ and $z\perp y$ (thus $x\leq^{\bullet} z^{\#}$ and $y\leq^{\bullet} z^{\#}$), and $x\vee y$ is not the supremum of x and y in $\hat{A}$ because we have not $x\vee y\perp z$.

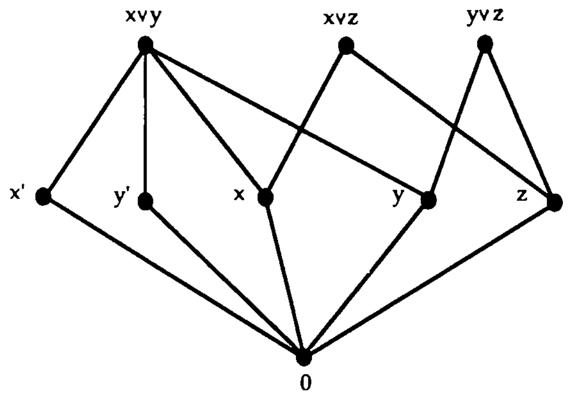

Fig. 5

Notice that for a WGOMP A the following conditions are equivalent:

(i) The embedding preserves all existing supremum of two elements,

(ii) If $a,b,c\in A$ with $c\perp a$ and $c\perp b$ and if $a\vee b$ exists in A then $c\perp a\vee b$.

This remark leads us to introduce a new definition.

Definition 2. A generalized orthomodular poset (abbreviated as GOMP) is a poset $(A,\le)$ with 0 satisfying Axioms (G1), (G2), (G3) and
(G4)′ If $a,b,c\in A$ are such that $c\perp a$, $c\perp b$ and $a\vee b$ exists then $c\perp a\vee b$.

Theorem 2. Every GOMP A can be embedded as an order ideal in an orthomodular poset $\hat{A}$ so that, for any $x\in\hat{A}$, $x\in A$ or $x^{\perp}\in A$. Moreover, the embedding preserves all existing supremum of two elements.

Every generalized orthomodular lattice (in particular, every ideal of an orthomodular lattice) is a GOMP. Now, we will compare the present embedding with the one of Janowitz [4].

Recall the embedding of Janowitz. Let L be a generalized orthomodular lattice. For a subset M of L define $M^{\perp}=\{x\in L \mid x\perp y \text{ for all } y\in M\}$ and let $J_x=[0,x]$. The set $\mathcal{L}$ of all ideals of L of the form J_x or $J_x^{\perp}$, for some element x of L, is an orthomodular lattice and the mapping $x\longrightarrow J_x$ is an embedding of L into $\mathcal{L}$.

Let $\hat{L}$ be the orthomodular poset obtained from L by the embedding of Theorem 2.

Proposition 3. If L is not an orthomodular lattice then $\hat{L}$ is canonically isomorphic to $\mathcal{L}$.

Proof. Let us define a map $f:\hat{L}\longrightarrow\mathcal{L}$ by:

$$f(x)=J_x \text{ if } x\in L$$
$$f(x^{\#})=J_x^{\perp} \text{ if } x^{\#}\in L^{\#}.$$

It is easy to see that f is a bijection from $\hat{L}$ into $\mathcal{L}$ such that $f(x^{\perp})=(f(x))^{\perp}$. In order to prove that $x\le^{*}y$ is equivalent to $f(x)\subset f(y)$, the difficulties arise in the two following cases:

i) $x\in L$ and $y=y_1^{\#}\in L^{\#}$. If $x\le^{*}y$ then $x\perp y_1$ and let $a\in J_x$, $t\in J_{y_1}$. Since $a\perp y_1$ we have $t\perp a$ and $a\in J_{y_1}^{\perp}$. Thus $J_x\subset J_{y_1}^{\perp}$ and $f(x)\subset f(y)$. The converse is obvious.

ii) $x=x_1^{\#}\in L^{\#}$ and $y\in L$. We need only prove that $J_{x_1}^{\perp}\subset J_y$ does not hold. If $J_{x_1}{}^{\perp} \subset J_y$ then $x_1 \vee y$ is the largest element of L. Indeed, suppose the contrary. Then there exists $b \in L$ with $x_1 \vee y < b$. Thus, there exists an element u of L with $u \perp x_1 \vee y$ and $u \vee x_1 \vee y = b$. It follows that $x_1 \perp u$ and that $u \leq y$ does not hold, which contradicts the fact that $J_{x_1}{}^{\perp} \subset J_y$. Since L has a largest element, L is an orthomodular lattice and so we have a contradiction.□

5. The example of the *-order in a Rickart *-ring

In this section A is a Rickart *-ring and Proj(A) denotes the orthomodular lattice of all projections of A. For $a \in A$, a' is the projection which generates the right-annihilator of {a} and let $C(a)=\{e \in \mathrm{Proj}(A) \mid ae=ea\}$. Note that, if $a=a^*$, then C(a) is the projection lattice of the Rickart *-ring of all elements of A which commute with a.

Reading carefully Janowitz's paper [5] leads to the following result:

Proposition 4. ([5]) Let a be an element of a Rickart *-ring A equipped with the *-order. The mapping $\varphi_a : x \longrightarrow x''$ is an orthoisomorphism of [0,a] into $[0,a''] \cap C(a^*a)$.

Therefore, every interval [0,a] of a Rickart *-ring is an orthomodular lattice isomorphic to an interval of a subalgebra of Proj(A). The key of the proof of Proposition 4 is the following lemma:

Lemma 8. ([5]) For a Rickart *-ring A the following conditions are equivalent:

(i) $x \leq a$,

(ii) $x=ax''$ and $x'' \in C(a^*a)$,

(iii) $x''=a''(a-x)'$ and $x^*a=a^*x$.

Lemma 9. Let a and b be two elements of a Rickart *-ring.

(i) $a \perp b$ is equivalent to $a'' \perp b''$ and $a^{*}{}'' \perp b^{*}{}''$.

(ii) If $a \vee b$ exists then $(a \vee b)'' = a'' \vee b''$.

Proof. (i) If $a \perp b$ then $a \vee b$ exists, and by using the isomorphism of Proposition 4, we have $a'' \perp b''$ and $a^{*}{}'' \perp b^{*}{}''$ as, clearly, $a \perp b$ is equivalent to $a^* \perp b^*$. Conversely, if $a'' \perp b''$ then $a''b''=0$ and thus, $0=aa''b''=ab''$. Hence, $0=ab''b^*=ab^*$ and, by using $a^{*''} \perp b^{*''}$, $0=a^*b$ also holds. Therefore, we have $a \perp b$.

(ii) Obvious, by using the isomorphism $\varphi_{a \vee b}$. □

Proposition 5. Every Rickart *-ring is a GOMP.

If A is a Rickart *-ring then, by Proposition 2, A is a WGOMP. In order to prove (G$)′, let a,b,c be elements of A such that $c \perp a$, $c \perp b$ and $a \vee b$ exists. It follows from Lemma 9 that $c'' \perp a''$, $c'' \perp b''$, $c^{*''} \perp a^{*''}$ and $c^{*''} \perp b^{*''}$ hold. We have, in the orthomodular lattice Proj(A), $c'' \perp a'' \vee b''$ and $c^{*''} \perp a^{*''} \vee b^{*''}$. By using part (ii) of Lemma 9, $c'' \perp (a \vee b)''$ and $c^{*''} \perp (a^* \vee b^*)''$ and part (i) implies $c \perp a \vee b$. □

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