DEMONSTRATIO MATHEMATICA
Vol. XXIV No 1-2 1991

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LATTICES WHOSE CONGRUENCE LATTICES SATISFY LEE'S IDENTITIES

1. Introduction

An elementary fact about congruence lattices of lattices is that they are distributive and pseudocomplemented, i.e. they can be investigated as distributive p-algebras. In G. Birkhoff's monography [2] the problem of characterizing lattices with Boolean congruence lattices was formulated. Three solutions of the problem were given in [20], [6] and [4]. Lattices with Stonean and relative Stonean (in our terminology, (L_1)- and relative (L_1)) congruence lattices were characterized in [13] and [9]. Semi-discrete lattices with (L_n)- and relative (L_n)- congruence lattices are for any $n\geq 1$ described in [10].

This paper is a continuation of [6], [13], [9] and a generalization of [10]. We characterize lattices with (L_n)- and relative (L_n)-congruence lattices for any $n\geq 1$ which gives a partial solution of the Problems III.5 and III.6 from G. Grätzer's monography [5]. In particular, we get slightly different descriptions (from those of [13] and [9]) of lattices with Stonean and relative Stonean congruence lattices. The results are presented in terms of the weak projectivity (Section 3). They can be simplified for *weak-modular* lattices and *semi-discrete* lattices. In Section 4 the distributive case is investigated.

2. Preliminaries

An algebra $(L;\vee,\wedge,^*,0,1)$ is called a (distributive) *p-algebra* or *pseudocomplemented* lattice (=*PCL*) if $(L;\vee,\wedge,0,1)$ is a bounded (distributive) lattice and * is a unary operation of pseudocomplement, i.e. $x\le a^*$ iff $a\wedge x=0$. The class $\mathbb{B}_\omega$ of all distributive p-algebras is equational. In [18] it is shown that the lattice of all equational subclasses of $\mathbb{B}_\omega$ is a chain

$$\mathbb{B}_{-1}\subset \mathbb{B}_0 \subset \mathbb{B}_1\subset \dots \subset \mathbb{B}_n\subset \dots\subset \mathbb{B}_\omega$$

of the type $\omega + 1$, where $\mathbb{B}_{-1}$, $\mathbb{B}_0$, $\mathbb{B}_1$ are the classes of all trivial p-algebras, Boolean algebras and Stonean algebras, respectively.

The elements of the subvariety $\mathbb{B}_n (n\ge 1)$ are called *(L_n)-lattices* as they are completely characterized by the identity

$$(L_n) \qquad l_n(x_1,\dots,x_n) = 1, \text{ where}$$

$$l_n(x_1,\dots x_n) = (x_1\wedge \dots \wedge x_n)^*\vee \bigvee_{i=1}^{n} (x_1\wedge \dots \wedge x_i^* \wedge \dots \wedge x_n)^*.$$

Distributive PCL's of which ev ry interval is an (L_n)-lattice are named *relative (L_n)-lattices*. The class of relative (L_n)-lattices can also be described as a subvariety of the variety of all Brouwerian lattices (see [10]). Recall that a Brouwerian lattice is an algebra $(L;\vee,\wedge,^*,1)$ where $(L;\vee,\wedge,1)$ is a lattice with unit and * is the binary operation of relative pseudocomplementation, i.e. $x\le y^*z$ iff $x\wedge y\le z$. A well-known fact about Brouwerian lattices is that they are distributive. In [10] relative (L_n)-lattices are described as Brouwerian lattices satisfying the identity

$$(L'_n) \qquad l'_n(x_1,\dots,x_n,y) = 1, \qquad \text{where}$$

$$l'_n(x_1,\dots,x_n,y)=(x_1\wedge\dots\wedge x_n)^*y\vee \bigvee_{i=1}^{n}(x_1\wedge\dots\wedge x_i^* y\wedge\dots\wedge x_n)^*y.$$

Brouwerian lattices with zero are called Heyting algebras. Distributive PCL's are obtained form Heyting algebras when we define $x^* = x^*0$. Congruence lattices of lattices form Heyting algebras. Therefore they can be investigated as relative (L_n)-lattices as well as (L_n)-lattices.

Also recall that a *semi-discrete* lattice is a lattice in which there exists a finite maximal chain between all comparable pairs of elements. Every finite lattice is obviously semi-discrete. Lattices in which all bounded chains are finite are called *discrete*.

We shall use this notation: Con(L) for the lattice of all congruence relations on a lattice L, 0 (I) for the smallest (largest) congruence relation, $a/b \longrightarrow c/d$ for the weak projectivity of quotients of a lattice in the sense of [13] or [6]. Further, we shall usually write $(x_1,\dots,\hat{x}_i,\dots,x_n)$ instead of $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$. All undefined terms as well as general lattice theoretic results may be found in [2], [5] or [19].

3. <u>The characterization theorems</u>

We begin with some definitions and lemmas.

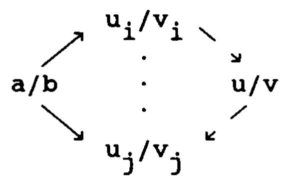
Definition 1. Let L be a lattice, $n \geq 1$ and a/b, u_1/v_1, ..., u_{n+1}/v_{n+1} nontrivial quotients of L. Then L is said to be *n-weakly modular* whenever

$$a/b \longrightarrow u_i/v_i,\ i = 1,\dots,n+1$$

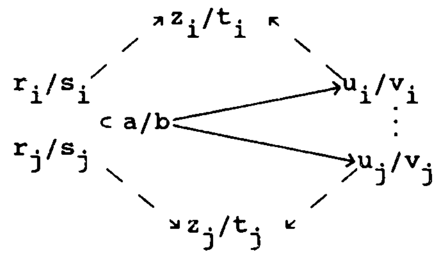
implies that one of the following conditions holds:

(i) there exist $i,j \in \{1,\dots,n+1\}$, $i \neq j$ and a quotient u/v with $u_i/v_i \longrightarrow u/v$ and $u_j/v_j \longrightarrow u/v$.

(ii) for all $i \in \{1,\dots,n+1\}$ there exist nontrivial proper subquotients $r_i/s_i \subset a/b$ and nontrivial quotients z_i/t_i such that $r_i/s_i \longrightarrow z_i/t_i$ and $u_i/v_i \longrightarrow z_i/t_i$ (see Figure 1).



(i)



(ii)

Figure 1

Definition 2. Let L be a lattice, $n\geq 1$ and $\theta_1,\dots,\theta_n$ nontrivial congruence relations on L. Then an (unordered) n- tuple $(\theta_1,\dots,\theta_n)$ is said to be n-weakly separable if for any $b<a$ in L there exists a chain $b=z_0<\dots<z_m=a$ such that for all $i\in\{0,\dots,m-1\}$ either

(i) $z_{i+1}/z_i \longrightarrow u/v$ and $u\equiv v(\theta_1\cap\dots\cap\theta_n)$ yields $u=v$ or

(ii) there exists $j\in\{1,\dots,n\}$ such that for every nontrivial proper subquotient $r/s \subset z_{i+1}/z_i$ the following holds: $r/s \longrightarrow u/v$, $u\neq v$ and $u\equiv v(\theta_1\cap\dots\cap\theta_{j-1} \cap I \cap\theta_{j+1}\cap\dots\cap\theta_n)$ imply the existence of a nontrivial quotient u'/v' with $u/v \longrightarrow u'/v'$ and $u'\equiv v'(\theta_j)$.

Remark 1. It is easy to verify that n-weakly modular lattice is also (n+1)-weakly modular for any $n\geq 1$. Similarly, (n+1)-tuple $(\theta_1,\dots,\theta_{n+1})$ of nontrivial congruence relations on L is (n+1)-weakly separable if some n-tuple $(\theta_1,\dots,\hat{\theta}_i,\dots,\theta_{n+1})$ is n-weakly separable, where $i\in\{1,\dots,n+1\}$.

Lemma 1. ([5; Theorem III.1.2]). Let L be a lattice, $a,b,c,d\in L$, $b\leq a, d\leq c$. Then $c\equiv d(\theta(a,b))$ if and only if there exists a chain $d=z_0\leq\dots\leq z_m=c$ such that

$$a/b \longrightarrow z_{i+1}/z_i \quad \text{for every } i=0,\dots,m-1.$$

Lemma 2. ([16; 1.4.]). Let L be a lattice and $\theta,\varphi \in \mathrm{Con}(L)$. Then the relative pseudocomplemented of θ with respect to φ is

$$\theta^{*}\varphi = \bigvee(\theta(u,v),\ (u,v)\in S),\ \text{where}$$

S is the set of all pairs of elements (u,v) $(u,v\in L)$ such that $u/v \longrightarrow z/t$ and $z\equiv t\ (\theta)$ implies $z\equiv t(\varphi)$ for all $z,t\in L$.

Now we can present the first result.

Theorem 1. Let L be a lattice. Then Con(L) is an (L_n)-lattice if and only if

(i) L is n-weakly-modular and

(ii) every n-tuple $(\theta_1,\ldots,\theta_n)$ of mutually distinct nontrivial congruence relations on L is n-weakly separable.

Proof. Assume that Con(L) is an (L_n)-lattice, i.e. it satisfies the equation (L_n). Take any nontrivial quotients a/b, u_1/v_1, ..., u_{n+1}/v_{n+1} in L with $a/b \longrightarrow u_i/v_i$, $i=1,\ldots,n+1$. Consider that there are no $i,j\in\{1,\ldots,n+1\}$, $i\neq j$ and a nontrivial quotient u/v such that $u_i/v_i \longrightarrow u/v$ and $u_j/v_j \longrightarrow u/v$. Set

$$\theta_1 = \theta(u_1,v_1),\ldots,\theta_{n+1} = \theta(u_{n+1},v_{n+1}).$$

First we shall prove that

$$(1)\qquad \theta_1{}^{*}\vee\ldots\vee\theta_{n+1}{}^{*}=I.$$

In order to show that $\theta_i \cap \theta_j = 0$ for all $i,j\{1,\ldots,n+1\}$, $i\neq j$, suppose on the contrary that there exist elements $u\neq v$ in L with $u\equiv v(\theta_i \cap \theta_j)$ (we can assume $u>v$). Then by Lemma 1 there exists a nontrivial subquotient $u'/v' \subseteq u/v$ with $u_i/v_i \longrightarrow u'/v'$ and $u_j/v_j \longrightarrow u'/v'$, a contradiction. Hence, $\theta_i \leq \theta_j^{*}$ for all $i,j\in\{1,\ldots,n+1\}$, $i\neq j$. In the case $n=1$ we have $\theta_1^{*} \vee \theta_1^{**} = I$ as Con(L) is Stonean. Since $\theta_2 \leq \theta_1^{*}$, i.e. $\theta_1^{**} \leq \theta_2^{*}$, it follows $I = \theta_1^{*} \vee \theta_1^{**} \leq \theta_1^{*} \vee \theta_2^{*}$, thus (1) holds. Now we assume $n\geq 2$. Set

$$\alpha_1 = \theta_2 \vee\ldots\vee \theta_n \vee \theta_{n+1},$$

$$\alpha_2 = \theta_1 \vee \theta_3 \vee\ldots\vee \theta_n \vee \theta_{n+1},$$

$$\cdots$$

$$\alpha_n = \theta_1 \vee\ldots\vee \theta_{n-1} \vee \theta_{n+1}.$$

We have $1_n(\alpha_1,\dots,\alpha_n) = I$ by the assumption. We shall prove that

(2) $$(\alpha_1\wedge\dots\wedge\alpha_n) = \theta_{n+1}$$

and $\alpha_1\wedge\dots\wedge\alpha_{i-1}\wedge\alpha_i^* \wedge \alpha_{i+1}\wedge\dots\wedge\alpha_n = \theta_i$, $i=1,\dots,n$.

Clearly, $\theta_{n+1} \le \alpha_1\wedge \dots \wedge\alpha_n$. Suppose on the contrary that there are $u,v\in L$, $u\ne v$ (we can assume $u>v$ again) with $u\equiv v(\alpha_1\wedge \dots \wedge\alpha_n)$ and $u\not\equiv v(\theta_{n+1})$. Then (as $u\equiv v(\alpha_1)$) there exist $i\in\{2,\dots,n\}$ and a nontrivial subquotient $u'/v' \subseteq u/v$ such that $u'\equiv v'(\theta_i)$, $u'\not\equiv v'(\theta_{n+1})$. Since $u'\equiv v'(\alpha_i)$, it follows than there exist $j\in\{1,\dots,n\}$, $j\ne i$ and a nontrivial subquotient $u''/v'' \subseteq u'/v'$ such that $u''\equiv v''(\theta_j)$. Then $u''\equiv v''(\theta_i \cap \theta_j)$, which contradicts $\theta_i \cap \theta_j = 0$. Thus $\alpha_1\wedge \dots \wedge\alpha_n = \theta_{n+1}$. Using distributivity and the fact that $\theta_i \le \theta_j^*$ for all $i\ne j$, the remaining equalities in (2) can be easily verified. Now, (1) directly follows from (2) and the assumption. Hence, $a\equiv b(\theta_1{}^*\vee\dots\vee\theta_{n+1}{}^*)$. Considering the existence of some $i\in\{1,\dots,n+1\}$ with $a\equiv b(\theta_i^*)$ would lead to $u_i\equiv v_i(\theta_i \wedge \theta_i^*)$, a contradiction. Thus for every $i\in\{1,\dots,n+1\}$ there exists a nontrivial proper subquotient $r_i/s_i \subseteq a/b$ with $r_i\not\equiv s_i(\theta_i^*)$. Now using Lemma 2 there exists a nontrivial quotient z'_i/t'_i with $r_i/s_i \longrightarrow z'_i/t'_i$ and $z'_i\equiv t'_i(\theta_i)$. By Lemma 1 this yields that for every $i\in\{1,\dots,n+1\}$ there exists a nontrivial subquotient $z_i/t_i \subseteq z'_i/t'_i$ with $u_i/v_i \longrightarrow z_i/t_i$. Hence, L is n-weakly modular.

Now, let $\theta_1,\dots,\theta_n \in \mathrm{Con}(L)$ be mutually distinct and nontrivial and $b<a$. According to the assumptions there exists a chain $b=z_0< \dots<z_m=a$ such that for every $i\in\{1,\dots,m-1\}$ either $z_{i+1}\equiv z_i((\theta_1\wedge \dots \wedge\theta_n)^*)$ or $z_{i+1}\equiv z_i((\theta_1\wedge\dots\wedge\theta_j^*\wedge\dots\wedge\theta_n)^*)$ for some $j\in\{1,\dots,n\}$. In the first case we immediately get the condition (i) from Definition 2. We shall show that in the second case the condition (ii) of Definition 2 is satisfied. Let $z_{i+1} \equiv z_i((\theta_1\wedge \dots \wedge\theta_j^* \wedge \dots \wedge\theta_n)^*)$ for some $j\in\{1,\dots,n\}$.

Further, let $r/s \subset z_{i+1}/z_i$ be a nontrivial proper subquotient and let $r/s \to u/v$, $u \neq v$ and $u \equiv v(\theta_1 \cap \dots \cap \hat{\theta}_j \cap \dots \cap \theta_n)$. Suppose on the contrary that $u/v \to u'/v'$, $u' \geq v'$ and $u' \equiv v'(\theta_j)$ imply $u'=v'$. Then $u \equiv v\ (\theta_j^*)$ by Lemma 2, hence we have $u \equiv v(\theta_1 \cap \dots \cap \theta_{j-1} \cap \theta_j^* \cap \theta_{j+1} \cap \dots \cap \theta_n)$. Since also $u \equiv v((\theta_1 \cap \dots \cap \theta_j^* \cap \dots \cap \theta_n)^*)$, we get $u=v$, a contradiction. Thus there exist elements $u' > v'$ with $u/v \to u'/v'$ and $u' \equiv v'(\theta_j)$. This proves that every (unordered) n-tuple $(\theta_1,\dots,\theta_n)$ of mutually distinct nontrivial congruence relations on L is n- weakly separable.

Conversely, suppose that the conditions (i), (ii) are satisfied. Let $\theta_1,\dots,\theta_n \in \mathrm{Con}(L)$ be nontrivial. In order to prove that Con (L) is an (L_n)-lattice, it is sufficient to show that for any $b<a$ in L $a \equiv b(1_n(\theta_1,\dots,\theta_n))$. The nontrivial case is when $\theta_1,\dots,\theta_n$ are mutually distinct. Take $b<a$ in L. Since the n-tuple $(\theta_1,\dots,\theta_n)$ is n-weakly separable, there exists a chain $b=c_0<\dots<c_m=a$ such that for every $i \in \{0,\dots,m-1\}$ either the condition (i) or that of (ii) from Definition 1 holds. In the first case we immediately get $c_{i+1} \equiv c_i((\theta_1 \wedge \dots \wedge \theta_n)^*)$. Now assume that (i) doesn't hold, i.e. there exists a nontrivial quotient u_{n+1} / v_{n+1} such that $c_{i+1}/c_i \to u_{n+1}/v_{n+1}$, $u_{n+1} \equiv v_{n+1}(\theta_1 \wedge \dots \wedge \theta_n)$ and simultaneously the condition (ii) holds. Two cases can occur:

1. there exists $j \in \{1,\dots,n\}$ such that $c_{i+1}/c_1 \to u/v$ and $u \equiv v(\theta_1 \wedge \dots \wedge \theta_j^* \wedge \dots \wedge \theta_n)$ imply $u=v$. Then by Lemma 2 $c_{i+1} \equiv c_i((\theta_1 \wedge \dots \wedge \theta_j^* \dots \wedge \theta_n)^*)$.

2. for every $j \in \{1,\dots,n\}$there exists a nontrivial quotient u_j/v_j such that $c_{i+1}/c_i \to u_j/v_j$ and $u_j \equiv v_j(\theta_1 \wedge \dots \wedge \theta_j^* \wedge \dots \wedge \theta_n)$. According to the assumptions at least one of the conditions (i), (ii) from Definition 1 holds for the quotients c_{i+1}/c_i, u_j/v_j $j=1,\dots,n+1$. Clearly,

condition (i) is not satisfied. Hence (ii) holds, thus for every $j\in\{1,\dots,n+1\}$ there exists a nontrivial proper subquotient $r_j/s_j \subset c_{i+1}/c_i$ and a nontrivial quotient z_j/t_j such that $r_j/s_j \longrightarrow z_j/t_j$ and $u_j/v_j \longrightarrow z_j/t_j$. Therefore, $z_j \equiv t_j(\theta_1 \wedge \dots \theta_j^* \wedge \dots \wedge\theta_n)$ for every $j\in\{1,\dots,n\}$. Since we are considering the case (ii) from Definition 2, it follows that for some $j\in\{1,\dots n\}$ there exists a nontrivial quotient z/t such that $z_j/t_j \longrightarrow z/t$ and $z\equiv t(\theta_j)$. Since $z_j \equiv t_j(\theta_j^*)$, we get $z\equiv t(\theta_j \wedge \theta_j^*)$, so $z=t$, a contradiction. Thus, the case 2. is impossible. Hence, for every $i\in\{1,\dots,m-1\}$ either $c_{i+1}\equiv c_i((\theta_1\wedge \dots\wedge \theta_n)^*)$ or $c_{i+1}\equiv c_i((\theta_1\wedge \dots\wedge\theta_j^*\wedge \dots \wedge\theta_n)^*)$for some $j\in\{1,\dots,n\}$. This yields that $a\equiv b(1_n(\theta_1,\dots,\theta_n))$. The proof of Theorem 1 is complete.

This result can be simplified if the lattice L is weak-modular or semi-discrete. By a *weak-modular* lattice L we mean the lattice L in which $a/b \longrightarrow c/d (a\geq b,\ c\geq d,\ a,b,c,d,\in L)$ yields the existence of a subquotient $a_1/b_1 \leq a/b$ such that $c/d \longrightarrow a_1/b_1$. The class of weak-modular lattices includes all modular lattices (cf.[5]).

Omitting (i) and modifying (ii) in Theorem 1 we get the following result:

Corollary 1. Let L be a weak-modular lattice. Then Con(L) is an (L_n)-lattice $(n\geq 1)$ if and only if for every (unordered) n-tuple $(\theta_1,\dots\theta_n)$ of mutually distinct nontrivial congruence relations and every $b<a$ in L there exists a chain $b=z_0< \dots<z_m=a$ such that for each $i\in\{1,\dots,m-1\}$ either

(i) $u\equiv v(\theta_1 \cap \dots \cap\theta_n)$ for any subinterval $[u,v]\leq[z_i,z_{i+1}]$ implies $u=v$ or

(ii) there exits $j\in\{1,\dots,n\}$ such that for every nontrivial proper subinterval $[u,v]\subset[z_i,z_{i+1}]$ with $u\equiv v(\theta_1 \cap \dots \cap\hat{\theta}_j \cap \dots \cap\theta_n)$ there exists a nontrivial subinterval $[u',v']\leq[u,v]$ with $u'\equiv v'(\theta_j)$.

Corollary 2. ([10, Corollary 4]). Let L be a semi-discrete lattice. Then Con(L) is an (L_n)-lattice ($n\geq 1$) if and only if for any prime quotients $p, q_1, \ldots, q_{n+1}$ of L satisfying

$$p \longrightarrow q_k,\ k=1,\ldots,n+1$$

there exists a prime quotient s of L such that

$$q_i \longrightarrow s \text{ and } q_j \longrightarrow s$$

for some $i,j\in\{1,\ldots n+1\}, i\neq j$ (see Figure 2).

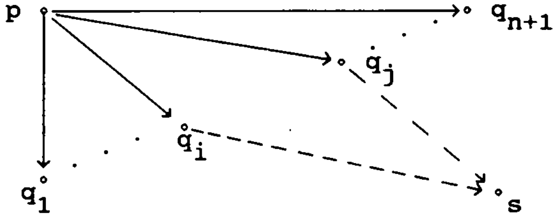


Figure 2

Proof. Obviously, in a semi-discrete lattice L the condition (ii) from Definition 1 doesn't hold for any prime quotients. Every n-tuple congruences on L is evidently n-weakly separable.

One can easily verify the following statement (see also [10; Theorem 1]):

Lemma 3. Let L be a distributive lattice with 1. Then L be a relative (L_n)-lattice if and only if for every $a\in L$, $[a,1]$ is an (L_n)-lattice.

Corollary 3. Let L be a lattice. Then Con(L) is a relative (L_n)-lattice ($n\geq 1$) if and only if for every $\Pi\in Con(L)$

(i) the factor lattice L/Π is n-weakly modular and

(ii) every (unordered) n-tuple $(\theta_1/\Pi, \ldots, \theta_n/\Pi)$ of mutually distinct nontrivial congruence relations on L/Π is n-weakly separable.

Proof. By Lemma 3, Con(L) is a relative (L_n)-lattice if and only if $Con(L/\Pi)$ is an (L_n)-lattice for every $\Pi\in Con(L)$.

Remark 2. For n=1 we get slightly different descriptions (from those of [13] or [9]) of lattices with Stonean and relative Stonean congruence lattices, respectively. More precisely, if L is Π-almost weakly modular (cf.[9]) then L/Π is 1-weakly modular for any $\Pi\in Con(L)$. On the other hand, if for some $\theta,\Pi \in Con(L)$, $\theta>\Pi$, the congruence θ/Π on the factor lattice L/Π is 1-weakly separable, then θ is also Π-weakly separable in the sense of [9].

Finally, we are able to prove much simpler result for semi-discrete lattices (see also [10]).

Theorem 2. Let L be a semi-discrete lattice. Then Con(L) is a relative (L_n)-lattice $(n\geq 1)$ if and only if for any prime quotients $p,q_1,\dots,q_{n+1}$ of L the relations $p \longrightarrow q_k$, $k=1,\dots,n+1$ imply $q_i \longrightarrow q_j$ or $q_j \longrightarrow q_i$ for some $i,j\in\{1,\dots,n+1\}$, $i\neq j$ (see Figure 3).

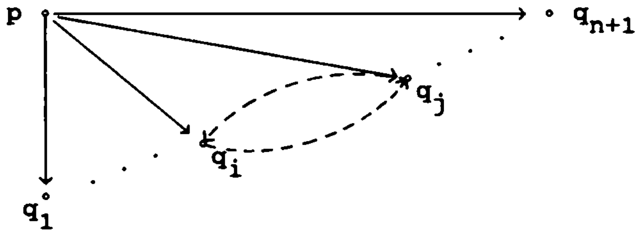


Figure 3

Proof. Let Con(L) be a relative (L_n)-lattice and a/b, $u_1/v_1,\dots,u_{n+1}/v_{n+1}$ prime quotients such that $a/b \longrightarrow u_i/v_i$, $i=1,\dots,n+1$. Denote

$$\theta_i = \theta(u_i,v_i), \quad i=1,\dots,n+1.$$

Further, set

$$\alpha_j = \theta_1\vee\dots\vee\theta_{j-1}\vee\theta_{j+1}\vee\dots\vee\theta_{n+1}, \ j=1,\dots,n,$$

$$\Pi = \bigvee_{\substack{i,j=1\\ i<j}}^{n} (\theta_i\wedge\theta_j).$$

We have $a\equiv b(1'_n(\alpha_1,\dots,\alpha_n))$ by the assumption. Since a/b is prime, we conclude either

$$a\equiv b(\alpha_1\wedge\dots\wedge\alpha_n)^*\Pi \text{ or } a\equiv b(\alpha_1\wedge\dots\wedge\alpha_{i-1}\wedge\alpha_i^*\Pi\wedge\alpha_{i+1}\wedge\dots\wedge\alpha_n)^*\Pi$$

for some $i\in\{1,\dots n\}$. In the first case we get $u_{n+1}\equiv v_{n+1}(\Pi)$ by Lemma 2 since $u_{n+1}\equiv v_{n+1}(\bigcap_{i=1}^{n}\alpha_i)$. This yields $u_{n+1}\equiv v_{n+1}(\theta_i\wedge\theta_j)$

for some $1 \le i < j \le n+1$ as u_{n+1}/v_{n+1} is prime. Then by Lemma 1 $u_i/v_i \longrightarrow u_{n+1}/v_{n+1}$. Now assume that $a \equiv b(\alpha_1 \wedge \ldots \wedge \alpha_i^* \Pi \wedge \ldots \wedge \alpha_n)^* \Pi$. If we show that $u_i \equiv v_i(\alpha_1 \wedge \ldots \wedge \alpha_i^* \Pi \wedge \ldots \wedge \alpha_n)$ then $u \equiv v(\Pi)$ will hold by Lemma 2. But clearly $u_i \equiv v_i(\alpha_1 \wedge \ldots \hat{\alpha}_i \wedge \ldots \wedge \alpha_n)$. In order to show that $u_i \equiv v_i(\alpha_i^* \Pi)$ by Lemma 2 it suffices verify that $u_i/v_i \longrightarrow u/v$ and $u \equiv v(\alpha_i)$ implies $u \equiv v(\Pi)$ for any $u, v \in L$, $u \ge v$. But this is really true, since $u_i \equiv v_i(\theta_i)$ and $\alpha_i \cap \theta_i \le \Pi$. Hence, we have $u_i \equiv v_i(\Pi)$. This means $u_i \equiv v_i(\theta_k \cap \theta_m)$ for some $1 \le k < m \le n+1$ as u_i/v_i is prime, so $u_k/v_k \longrightarrow u_i/v_i$ for some $1 \le k \le n+1$.

Conversely, suppose that the identity (L_n') is not satisfied in Con(L). Hence there exist mutually different congruences $\theta_1, \ldots, \theta_n, \Pi$ on L and a prime quotient a/b such that $a \not\equiv b(1_n'(\theta_1, \ldots, \theta_n, \Pi))$. Thus $a \not\equiv b(\theta_1 \wedge \ldots \wedge \theta_n)^* \Pi$ and $a \not\equiv b(\theta_1 \wedge \ldots \wedge \theta_{i-1} \wedge \theta_i^* \Pi \wedge \theta_{i+1} \wedge \ldots \wedge \theta_n)^* \Pi$ for all $i \in \{1, \ldots, n\}$. This yields by Lemma 2 that there exist prime quotients u_{n+1}/v_{n+1}, $u_1/v_1, \ldots, u_n/v_n$ such that $a/b \longrightarrow u_{n+1}/v_{n+1}$, $u_{n+1} \equiv v_{n+1}(\theta_1 \cap \ldots \cap \theta_n)$ and $u_{n+1} \not\equiv v_{n+1}(\Pi)$, and for all $i = 1, \ldots, n$ $a/b \longrightarrow u_i/v_i$, $u_i \equiv v_i(\theta_1 \cap \ldots \cap \theta_{i-1} \cap \theta_i^* \Pi \cap \theta_{i+1} \cap \ldots \cap \theta_n)$ and $u_i \not\equiv v_i(\Pi)$.

We shall show that neither $u_i/v_i \longrightarrow u_j/v_j$ nor $u_j/v_j \longrightarrow u_i/v_i$ hold for any $1 \le i < j \le n+1$. First, assume that $u_{n+1}/v_{n+1} \longrightarrow u_i/v_i$ for some $i \in \{1, \ldots, n\}$. Then $u_i \equiv v_i(\theta_i \cap \theta_i^* \Pi)$ which implies $u_i \equiv v_i(\Pi)$, a contradiction. Now let $u_i/v_i \longrightarrow u_j/v_j$ for some $1 \le i < j \le n+1$. Then we again get $u_j \equiv v_j(\theta_i^* \Pi \cap \theta_i)$, so $u_j \equiv v_j(\Pi)$, a contradiction. The proof is complete.

4. Congruence lattices of distributive lattices

Several interesting results were obtained for congruence lattices of distributive lattices.

Lemma 4 ([6],[8]). Let L be a distributive lattice. Then Con(L) is a Boolean algebra if and only if L is discrete.

Lemma 5 ([9; Theorem 7]). Let L be a distributive lattice. Then Con(L) is a relative Stone lattice if and only if L is discrete, i.e. Con(L) is a Boolean algebra.

Now one can ask the following question. What do characterizations of the lattices with (L_n)-and relative (L_n)-lattices look like in the case of a distributive lattice L ?.

In the following Theorem we give an answer to this question in the special case when the lattice L is a chain.

Theorem 3. Let L be a chain. Then Con(L) is an (L_n)-lattice ($n \geq 1$) if and only if L is discrete, i.e. Con(L) is a Boolean algebra.

Proof. Let Con(L) be an (L_n)-lattice. Assume on the contrary that L is not discrete. Then there exist elements $a<b$ in L such that the interval $[a,b]$ contains an infinite sequence

$$a=a_{0,n+1}<a_{11}<a_{12}< \dots <a_{1n}<a_{1,n+1}<a_{21}<a_{22}< \dots <a_{2,n+1}<a_{32}< \dots$$

of elements of L. Define congruences $\theta_1,\dots,\theta_n$ on the lattice L as follows:

$$\theta_j = \theta(a,a_{1j}) \vee \bigvee_{i=1}^{\infty} (\theta(a_{i,j+1},a_{i+1,j})), \quad j=1,\dots,n$$

(i.e. so that $a_{i,n+1} \equiv a_{i+1,1}(\bigcap_{k=1}^{n}\theta_k)$, $i=0,1, \dots$ and

$a_{ij} \equiv a_{i,j+1}(\bigcap_{\substack{k=1 \\ k\neq j}}^{n}\theta_k)$, $i=1,2,\dots$, $j=1,\dots,n$ - see Figure 4a).

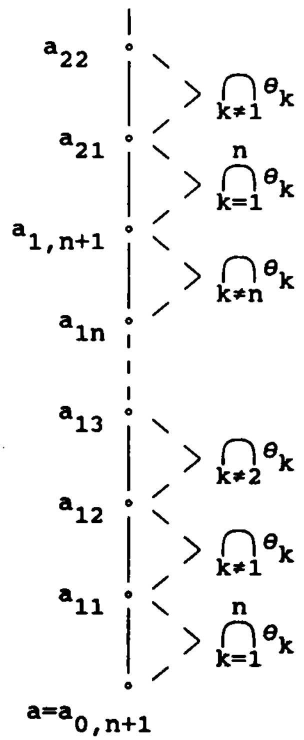


Figure 4a

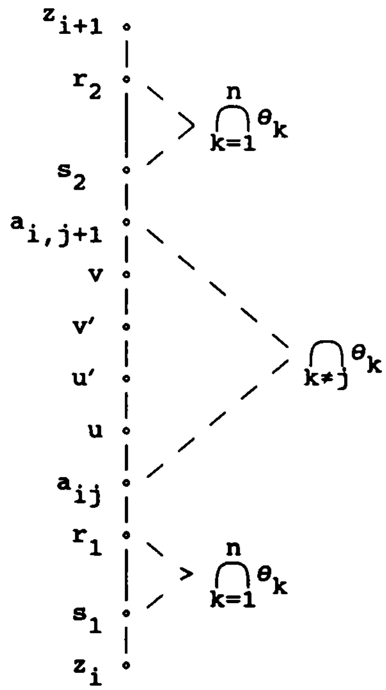


Figure 4b

Evidently, there are infinitely many nontrivial factor classes (i.e. containing more than one element) related to the congruence $\bigcap_{k=1}^{n}\theta_k$ in the interval [a,b]. The n-tuple of congruences $(\theta_1,\dots,\theta_n)$ is n-separable by the assumption and Theorem 1. Thus there exists a chain $a=z_0<\dots z_m=b$ such that for each $i\in\{1,\dots,m-1\}$ either (i) or (ii) from Corollary 1 is satisfied. If (i) holds, then $z_i\equiv z_{i+1}((\theta_1\cap\dots\cap\theta_n)^*)$ using Lemma 2, i.e. there is no nontrivial factor class related to the congruence $\bigcap_{k=1}^{n}\theta_k$ in the interval $[z_i,z_{i+1}]$. If (i) doesn't hold, then (as L is a chain) there exists a subinterval $[s,r]\subseteq[z_i,z_{i+1}]$ such that $r\equiv s(\bigcap_{k=1}^{n}\theta_k)$, and the condition (ii) from Corollary 1 holds. We shall prove that in the interval $[z_i, z_{i+1}]$ there is at most one nontrivial factor class related to the congruence $\bigcap_{k=1}^{n} \theta_k$ in this case. Suppose on the contrary that there are elements r_1, s_1, r_2, s_2 such that $z_i \leq s_1 < r_1 < s_2 < r_2 \leq z_{i+1}$, $r_i \equiv s_i (\theta_1 \cap \ldots \cap \theta_n)$, $i=1,2$ and $r_1 \not\equiv s_2 (\theta_1 \cap \ldots \cap \theta_n)$. Let $j \in \{1, \ldots n\}$ be the indices from (ii) of Corollary 1. By the definition of the congruences θ_i, there exists a nontrivial proper subinterval $[u,v] \subset [r_1, s_2]$ such that $u \equiv v (\theta_i \cap \ldots \cap \hat{\theta}_j \cap \ldots \cap \theta_n)$ (see Figure 4b). (Obviously, $[u,v] \subseteq \subseteq [a_{ij}, a_{ij+1}]$ for some i.) By (ii) from Corollary 1, there exists a nontrivial subinterval $[u', v'] \subseteq [u,v]$ such that $u' \equiv v' (\theta_j)$. But this evidently contradicts the definition of the congruences θ_i, $i=1, \ldots, n$.

Hence, there are only finitely many nontrivial factor classes related to the congruence $(\bigcap_{k=1}^{n} \theta_k)$ in the interval $[a,b]$, a contradiction. Therefore L must be discrete.

The converse statement is trivial.

REFERENCES

[1] **R. Beazer:** Lattices whose ideal lattice is Stone, Proc. Edinburgh Math. Soc. 26(1983), 107-112.

[2] **G. Birkhoff:** Lattice Theory. Third. Ed., Amer. Math. Soc. Colloq. Publ., vol. 25,1967.

[3] **O. Frink:** Pseudo-complements in semi-lattices, Duke Math.J. 29(1962), 505-514.

[4] **P. Craweley:** Lattices whose congruences form a Boolean algebra, Pacif. J. Math. 10(1960), 787-795.

[5] **G. Grätzer:** General Lattice Theory. Birkhäuser Verlag, Basel 1978.

[6] G. Grätzer and E.T. Schmidt: Ideals and congruence relations in lattices, Acta Math. Acad. Sci. Hungar. 9(1958), 137-175.

[7] G. Grätzer and E.T. Schmidt: On congruence lattices of lattices, Acta Math. Acad. Sci. Hungar. 13(1962), 179-185.

[8] J. Hashimoto: Ideal theory for lattices, Math. Japonicae 2 (1952), 149-186.

[9] M. Haviar and T. Katriňák: Lattices whose congruence lattice is relative Stone, Acta Sci. Math.(Szeged) 51(1987), 81-91.

[10] M. Haviar and T. Katriňák: Semi-discrete lattices with (L_n)-congruence lattices, Contributions to General Algebra 7 (to appear).

[11] J. Jakubík: Relácie kongruentnosti a slabá projektivnost vo zväzoch, Časop. pěst. mat. 80(1955), 206-216 (in Slovak).

[12] T. Katriňák: Notes on Stone lattices I, Mat. fyz. Časop. 16(1966), 128-142 (in Russian).

[13] T. Katriňák: Notes on Stone lattices II, Mat. fyz. Časop. 17(1967), 20-37 (in Russian).

[14] T. Katriňák: M-Polaren in halbgeordneten Mengen, Časop. pest. mat. 95(1970), 416-419.

[15] T. Katriňák: Die Kennzeichnung der distributiven pseudokomplwmentaren Halbverbande, J.Reine Angew. Math. 241(1970), 160-179.

[16] T. Katriňák: Eine Characterisierung der fast schwach modularen Verbande, Math. Z. 114(1970), 49-58.

[17] T. Katriňák and S. El-Assar: Algebras with Boolean and Stonean congruence lattices, Acta Math. Acad. Sci. Hungar. 48(1986), 301-316.

[18] K. B. Lee: Equational classes of distributive pseudocomplemented lattices, Canad. J. Math. 22(1970), 881-891.

[19] E. T. Schmidt: A Survey on Congruence Lattice Representations, Teubner Texte zur. Math., Leipzig, Band 42, 1982.

[20] T. Tanaka: Canonical subdirect factorization of lattices, J. Sci. Hiroshima Univ., Ser. A, 16(1952), 239-246.

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Received December 18, 1990.