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SUBVARIETIES OF THE VARIETY DEFINED BY EXTERNALLY COMPATIBLE IDENTITIES OF DISTRIBUTIVE LATTICES

We consider algebras of type $\tau=(2,2)$ with two fundamental operation symbols $+$, $\cdot$.

An identity $\phi=\psi$ of type τ is called externally compatible iff it is of the form $x=x$ or $\phi_1+\phi_2=\psi_1+\psi_2$ or $\phi_1\cdot\phi_2=\psi_1\cdot\psi_2$, where x is a variable and ϕ_1, ϕ_2, ψ_1, ψ_2 are terms of type τ (see[1]). An identity $\phi=\psi$ of type τ is called non-trivializing iff it is of the form $x=x$ or ϕ and ψ are different from a variable (see [2], [4]).

Let D be the variety of all distributive lattices and T the trivial variety of type τ. If K is a variety of type τ, then we denote by Id(K) the set of all identities of type τ satisfied in K, by Ex(K) the set of all externally compatible identities of type τ satisfied in K, and by N(K) the set of all non-trivializing identities of type τ satisfied in K. Obviously the sets Ex(K), N(K) are equational theories.

If Σ is a set of identities of type τ, then $V(\Sigma)$ denotes the variety of all algebras of type τ defined by Σ.

We denote by ϕ^* the canonical form of a term ϕ in D.

In this paper we describe the lattice of all subvarieties of the variety V(Ex(D)).

We have obviously

Lemma 1. (a) The identities

$$x\cdot y=u\cdot v$$
$$x+y=u+v$$

form an equational base of V(Ex(T)).

(b) The identity

$$x\cdot y=u+v$$

forms an equational base of $V(N(T))$.

Lemma 2. The lattice $L(V(Ex(T)))$ of all subvarieties of the variety $V(Ex(T))$ has the following diagram,

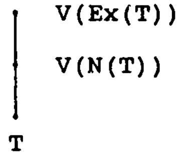

Figure 1.

i.e. $L(V(Ex(T)))$ is isomorphic to a three element chain.

Proof. By Lemma 1 it is easy to see that T, $V(N(T))$, $V(Ex(T))$ are different subvarieties of $V(Ex(T))$ and

$$T \subset V(N(T)) \subset V(Ex(T)).$$

It is known (see [2]) that the following identities

$$x+(y+z)=(x+y)+z \qquad x\cdot(y\cdot z)=(x\cdot y)\cdot z$$
$$x+y=y+x \qquad x\cdot y=y\cdot x$$
$$x+xy=x+x \qquad x\cdot(x+y)=x+x$$
$$x\cdot(y+z)=x\cdot y+x\cdot z$$
$$x+x=x\cdot x$$

form an equational base of $V(N(D))$.

Lemma 3. The lattice $L(V(N(D)))$ of all subvarieties of the variety $V(N(D))$ has the following diagram

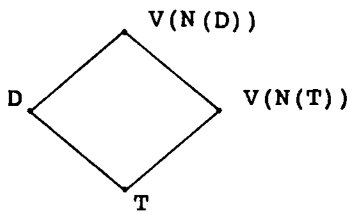

Figure 2.

Proof. An arbitrary subvariety K of $V(N(D))$ different from $V(N(D))$ satisfies at least one of the following identities:

(a) $x=\phi$ and $(x=\phi)\in Id(D)$,

(b) $\phi=\psi$ and $(\phi=\psi)\notin Id(D)$,

(c) $x=\phi$ and $(x=\phi)\notin Id(D)$,

(d) $x=y$,

where x,y are variables and ϕ, ψ are terms of type τ different from a single variable.

Let $(x=\phi)\in Id(K)$ and $(x=\phi)\in Id(D)$.

Subsituting x for all variables in $x=\phi$ we obtain $(x=x+x)\in$ $\in Id(K)$ or $(x=x\cdot x)\in Id(K)$.

Thus from (i) it follows that $K\leq D$. So $K=D$ or $K=T$, because D is equationally complete.

Let $(\phi=\psi)\in Id(K)$ and $(\phi=\psi)\notin Id(D)$. It is easy to see that

$$(\phi^*+\phi^*=\psi^*+\psi^*)\in Id(K).$$

From Marczewski's lemma (see[3]) it follows that there exists a product p being a component of the term ϕ^* such that in every product being a component of the term ψ^* there exists a variable which does not occur in p. Let $x_1,\dots,x_n$ be all such variables. Then we have the following identities in K:

$$p+x_1+\dots+x_n = p+x_1+\dots+x_n+\psi^*+\psi^* = p+x_1+\dots+x_n+\phi^*+\phi^* =$$
$$= x_1+\dots+x_n+\phi^*+\phi^* = x_1+\dots+x_n+\psi^*+\psi^* = x_1+x_1+\dots+x_n.$$

(we repeat x_1 for case $n=1$).

Substituting x for all variables occurring in p in $p+x_1+\dots+x_n=x_1+x_1+x_2+\dots+x_n$ and y for $x_1,\dots,x_n$ we obtain

$$(x+y=y+y)\in Id(K).$$

Thus

$$(x+x=y+y)\in Id(K).$$

We have

$$x+y = (x+y)+(x+y) = (x+x)+(y+y) = (z+z)+(v+v) = z+v,$$
$$x\cdot y = (x+x)\cdot(y+y) = (z+z)\cdot(v+v) = z\cdot v.$$

So $(x+y = z+v)$, $(x\cdot y = z\cdot v)\in Id(K)$.

We have also $x\cdot y = x\cdot y+x\cdot y = u\cdot u+v\cdot v = u+v$. Thus $(x\cdot y=u+v)\in$ $\in Id(K)$.

From Lemma 1(b) it follows that $K\leq V(N(T))$, and from Lemma 2 we have

$$K=V(N(T)) \text{ or } K=T.$$

If $(x=\phi)\in Id(K)$ and $(x=\phi)\notin Id(D)$, then

$$(x=x\cdot y)\in Id(K) \quad \text{or} \quad (x=x+y)\in Id(K)$$

(from Marczewski's lemma). So, $(x=y)\in Id(K)$ and $K=T$. The case

(d) is obvious, i.e. K=T. Thus the varieties V(N(D)), D, N(T), T are all different subvarieties of V(N(D)) and it is easy to see that

$$Id(D)\cap N(T)=N(D).$$

So the lattice L(V(N(D))) has the diagram as in figure 2.

Theorem. The lattice L(V(Ex(D))) of all subvarieties of the variety V(Ex(D)) has the following diagram,

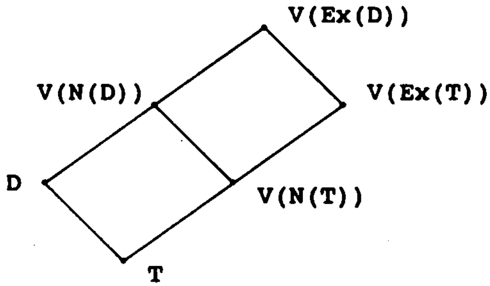

Figure 3.

i.e. L(V(Ex(D))) is isomorphic to the direct product of a three element chain and a two element chain.

Proof. An arbitrary subvariety K of V(Ex(D)) different from V(Ex(D)) satisfies at least one of the following identities:

(a) $x=\phi_1\cdot\phi_2$ and $(x=\phi_1\cdot\phi_2)\in Id(D)$,
(b) $x=\phi_1+\phi_2$ and $(x=\phi_1+\phi_2)\in Id(D)$,
(c) $x=\phi_1\cdot\phi_2$ and $(x=\phi_1\cdot\phi_2)\notin Id(D)$,
(d) $x=\phi_1+\phi_2$ and $(x=\phi_1+\phi_2)\notin Id(D)$,
(e) $x=y$,
(f) $\phi_1+\phi_2=\psi_1+\psi_2$ and $(\phi_1+\phi_2=\psi_1+\psi_2)\notin Id(D)$,
(g) $\phi_1\cdot\phi_2=\psi_1\cdot\psi_2$ and $(\phi_1\cdot\phi_2=\psi_1\cdot\psi_2)\notin Id(D)$,
(h) $\phi_1+\phi_2=\psi_1\cdot\psi_2$ and $(\phi_1+\phi_2=\psi_1\cdot\psi_2)\notin Id(D)$,
(i) $\phi_1+\phi_2=\psi_1\cdot\psi_2$ and $(\phi_1+\phi_2=\psi_1\cdot\psi_2)\in Id(D)$,

where x is a variable and ϕ_1, ϕ_2, ψ_1, ψ_2 are terms of type τ. Let

$$(x=\phi_1\cdot\phi_2)\in Id(K)$$
$$(x=\phi_1\cdot\phi_2)\in Id(D).$$

Substituting x for each variable in this identity we get

$$(x=x\cdot x)\in Id(K).$$

Thus we have

$$x+x=(x+x)\cdot(x+x)=x\cdot x=x,$$

so

$$(x=x+x)\in Id(K).$$

Further

$$x\cdot(y+z)=[x\cdot(y+z)]\cdot[x\cdot(y+z)]=(x\cdot y+x\cdot z)\cdot(x\cdot y+x\cdot z)=x\cdot y+x\cdot z.$$

Therefore

$$(x\cdot(y+z)=x\cdot y+x\cdot z)\in Id(D)$$

and it is easy to see that $K\le D$ and $K=T$ or $K=D$.

Analogously, if $(x=\phi_1+\phi_2)\in Id(K)$ and $(x=\phi_1+\phi_2)\in Id(D)$, then $K=D$ or $K=T$. Let $(x=\phi_1\cdot\phi_2)\in Id(K)$ and $(\phi_1\cdot\phi_2=x)\notin Id(D)$. Then $(x=(\phi_1\cdot\phi_2)^*\cdot(\phi_1\cdot\phi_2)^*)\in Id(K)$, where $(\phi_1\cdot\phi_2)^*$ is the canonical form of the term $\phi_1\cdot\phi_2$ in D. From Marczewski's lemma it follows that in every product being a component of $(\phi_1\cdot\phi_2)^*$ there exists a variable different from x. Substituting y for all these variables we obtain

$$(x=y\cdot y)\in Id(K),\ \text{or}\ (x=x\cdot y)\in Id(K).$$

Thus $(x=y)\in Id(K)$ and $K=T$. Similarly, if $(x=\phi_1+\phi_2)\in Id(K)$ and $(x=\phi_1+\phi_2)\notin Id(D)$, then $K=T$. In case (e) it is obvious that $K=T$.

Let $(\phi_1+\phi_2=\psi_1+\psi_2)\in Id(K)$ and $(\phi_1+\phi_2=\psi_1+\psi_2)\notin Id(D)$. Then $((\phi_1+\phi_2)^*+(\phi_1+\phi_2)^*=(\psi_1+\psi_2)^*+(\psi_1+\psi_2)^*)\in Id(K)$.

In a similar way as in the proof of Lemma 3 (case (b)) we obtain

$$(x+y=z+v),\ (x\cdot y=z\cdot v)\in Id(K).$$

So from Lemma 1 (a) we have $K\le V(Ex(T))$ and using Lemma 2 we see that K is one of the varieties $V(Ex(T))$, $V(N(T))$, T.

Analogously in cases (g), (h) we have $K\le V(Ex(T))$. Let

$$(\phi_1+\phi_2=\psi_1\cdot\psi_2)\in Id(K)\quad\text{and}\quad(\phi_1+\phi_2=\psi_1\cdot\psi_2)\in Id(D).$$

Then

$$((\phi_1+\phi_2)^*+(\phi_1+\phi_2)^*=(\psi_1\cdot\psi_2)^*\cdot(\psi_1\cdot\psi_2)^*)\in Id(K).$$

So, $(x+x=x\cdot x\in Id(K))$, and

$$x\cdot(y+z)=[x\cdot(y+z)]\cdot[x\cdot(y+z)]=(x\cdot y+x\cdot z)\cdot(x\cdot y+x\cdot z)=$$
$$=(x\cdot y+x\cdot z)+(x\cdot y+x\cdot y)=x\cdot y+x\cdot z.$$

Thus $(x\cdot(y+z)=x\cdot y+x\cdot z)\in Id(K)$.

Using (i) we have $K \leq V(N(D))$.

From Lemma 3 it follows that K is one of the classes V(N(D)), V(N(T)), D, T.

We have already proved that T, V(N(T)), V(N(D)), V(Ex(T)), V(Ex(D)), D are all subvarieties of the variety V(Ex(D)). Observe that

$$Ex(T) \cap Id(D) = Ex(D),$$
$$N(D) \cap Ex(T) = Ex(D).$$

Consequently,

$$V(Ex(T)) \vee D = V(N(D)) \vee V(Ex(T)) = V(Ex(D)).$$

Thus from Lemmas 2 and 3 it follows that lattice L(V(Ex(D)) has the diagram as in figure 3.

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