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K. Głazek, A. Kośliński

ON THE PŁONKA DECOMPOSITION OF GRAPHS AND RELATED ALGEBRAS

1. Introduction

In this paper we will consider undirected graphs (without multiple edges and loops) and some general algebras. The terminology used here is more or less standard. In 1971 Jerzy Płonka defined (see [22]) the notion of the sum of a direct system of graphs, which was investigated by him, Raczko and by Kośliński (see [22], [20], [27], [14] and [15]). Decompositions of graphs into the Płonka sum are still not investigated enough. As is well known (see, e.g. [1]), different kinds of decompositions of graphs into some simple ones have very important applications; also the kind of decomposition considered seems to have some interesting applications. Unfortunately, decompositions into the Płonka sums of graphs were missed in the monography of Bosak [1].

In [14] the second author started to investigate decompositions of n-angles and some other graphs into the two-component Płonka sums. He obtained, among other results, all such decompositions of pentagons, but this method was not simply applicable to hexagons. The problem of finding all decompositions of a hexagon (with some diagonals) was posed by J. Płonka. It has been open for more than fifteen years. Such investigations belong to a large class of constructive enumeration problems, which are very important in combinatorics and its applications (see, e.g. [2]-[5], [7]-[9], [11], [13], [16], [23], [24], [28] and [32]).

In particular, the problem of determining the number of graphs with a given property was investigated by several authors beginning with A. Cayley ([2]-[4]), J.H. Redfield [28] and G. Pólya [23] and it is very important with respect to different applications (for example in chemistry); see also the monographs of G. Pólya and R.C. Read [24] and F. Harary and E. Palmer [13].

In this paper we present a full description of decompositions of hexagons into two-component Płonka sums which was found by the first author. This method is general and can be applicable to other n-angles. The paper is only one of initial steps in investigations of such decompositions. For the sake of completeness, we recall some known general results and some special ones for n-angles (with some diagonals) of the second author which appeared in a semi-publication [15] only.

Finally, we give an algebraic approach (proposed by the first author in 1988, and reported by him during the Jadwisin Conference on Quasigroups and Universal Algebra in 1989). This idea is based on association of some universal algebras $\mathfrak{A}=(A;\mathbb{F})$ with graphs $\Gamma=(A;\rho)$. These algebras are more rich than groupoids (on the set A of vertices or on $A\cup\{\infty\}$), considered by several authors (e.g. [6], [17], [25], [26], [30]). The algebras will be called full graph algebras and seem to be more fruitful than the above-mentioned "graph--algebras" (rather graph-groupoids), but unfortunately the set of term operations of these algebras is difficult to be determined. We consider here some variant of these algebras in which term operations are additionally compatible with some partial endomorphism (i.e. these graph-algebras can be treated as graph-algebras of mixed graphs).

The authors are grateful to Professor Jerzy Płonka for fruitful discussions about considered decompositions of graphs.

2. Preliminaries

Let A be a non-empty finite set, say $A = \{a_0, a_1, \ldots, a_m\}$. By a graph Γ with the set A of vertices we will understand (see, e.g. [12], [18]) a relational system $\Gamma=(A;\rho)$, where $\rho\subset A\times A$ is a symmetric relation (i.e. $(a_i,a_j)\in\rho \Leftrightarrow (a_j,a_i)\in\rho$).

If $(a_i,a_j)\in\rho$ then the pair (a_i,a_j) is called an (undirected) edge. We will mostly consider graphs without loops, i.e. $(a,a)\notin\rho$ for every $a\in A$. We will write $a\rho b$ instead of $(a,b)\in\rho$.

In Section 5 we will fix some special enumeration of vertices, i.e. such graphs can be considered as special semi-labelled graphs.

In [22] J. Płonka defined the notion of the sum of a direct system of graphs (using a more general definition of directed graphs in the sense of [33]) by analogy to his construction for universal algebras. He also considered a general case of relational systems [21]. One can prove (see [10]) that so-called regular formulas are preserved by the construction of the Płonka sum.

Since in [20] it was proved that if a graph is decomposable into a sum of a direct system of graphs; then this graph is also decomposable into a two-component sum; we will consider only this special case. We recall the definition of the two-component Płonka sum of undirected graphs:

Let $\Gamma_1=(A_1;\rho_1)$ and $\Gamma_2=(A_2;\rho_2)$ be two graphs such that $A_1\cap A_2=\emptyset$, and assume that there exists a homomorphism $h:A_1\longrightarrow A_2$ of the graph Γ_1 into the graph Γ_2. Then the graph $\Gamma=(A;\rho)$ is said to be the Płonka sum of Γ_1 and Γ_2 if $A=A_1\cup A_2$ and $\rho=\rho_1\cup\rho_2\cup\rho_0$, where ρ_0 is a symmetric binary relation defined for $a\in A_1$ and $b\in A_2$ by

(*) $a\rho_0 b \Leftrightarrow h(a)\rho_2 b.$

In this case we will write $\Gamma=S(\Gamma_1,\Gamma_2;h)$ and we will call the triple $(\Gamma_1,\Gamma_2;h)$ a decomposition (or a decomposing system) of Γ. The graph Γ is said to be decomposable if there exist graphs Γ_1, Γ_2 and a graph homomorphism $h:\Gamma_1\longrightarrow\Gamma_2$ such that $\Gamma=S(\Gamma_1,\Gamma_2;h)$. Remark that the graph $\Gamma=S(\Gamma_1,\Gamma_2;h)$ together with additional directed edges $(a,h(a))$, for every $a\in A_1$, can be also considered as a mixed graph.

In this paper, heavy lines in the figures mean edges of graphs Γ_1 and Γ_2 , fine lines mean additional edges determined by the construction of the Płonka sum $S(\Gamma_1,\Gamma_2;h)$, broken lines connect a vertex $a \in A_1$ with its homomorphic image $h(a)$, and heavy dots mean the vertices of the graph Γ_1.

3. Some properties of decomposable graphs

By a slight modification of Theorem 2 in [27] we obtain

Proposition 1. If φ_i is an isomorphism of Γ_i onto Γ'_i, i=1,2, such that

$(^*_*)$ $\quad h' \cdot \varphi_1 = \varphi_2 \cdot h$

(where $\cdot$ is the composition of mappings), then $S(\Gamma_1,\Gamma_2;h)$ is isomorphic to $S(\Gamma'_1,\Gamma'_2;h')$ and the required isomorphism is pieced out from φ_1 and φ_2 (or a "sum" of φ_1 and φ_2 in another terminology).

Proposition 1 allows us to reduce investigations of graph isomorphism to simpler cases.

The decompositions $(\Gamma_1,\Gamma_2;h)$ and $(\Gamma'_1,\Gamma'_2;h')$ are said to be isomorphic if the assumptions of Proposition 1 are fulfilled.

From this point of view the following two decompositions of a hexagon (with diagonals) are isomorphic:

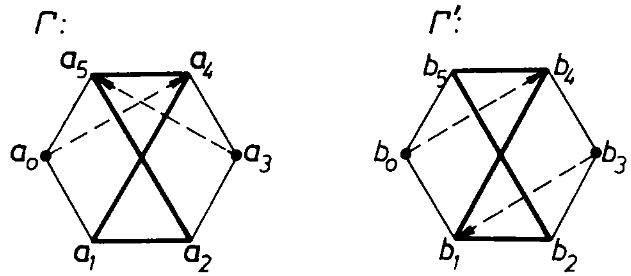


Fig. 1

Indeed, we can take $\varphi_1 : A_1 \longrightarrow A'_1$, $\varphi_2 : A_2 \longrightarrow A'_2$ defined in the following way:

$$\varphi_1(a_0)=b_0 \ , \ \varphi_1(a_3)=b_3,$$

$$\varphi_2(a_2)=b_2,\ \varphi_2(a_4)=b_4,\ \varphi_2(a_1)=b_5 \text{ and } \varphi_2(a_5)=b_1.$$

But the converse implication in Proposition 1 does not hold because, for example, the triangle $A=\{a_0,a_1,a_2\}$ with a loop is decomposable into the Płonka sum in two different ways:

$$A = \{a_0\}\cup\{a_1,a_2\} \quad\text{and}\quad A = \{a_0,a_1\}\cup\{a_2\}.$$

Fig. 2

Consider now the direct system $(\Gamma_1,\Gamma_2;h)$ of two graphs $\Gamma_1=(A_1;\rho_1)$ and $\Gamma_2=(A_2;\rho_2)$, where $h:A_1\longrightarrow A_2$ is a homomorphism of the graph Γ_1 into the graph Γ_2. We define a new system $(\bar{\Gamma}_1,\Gamma_2;\bar{h})$ by putting $\bar{\Gamma}_1=(A_1/\sim,\bar{\rho}_1)$, where $\sim$ is the equivalence relation defined on A_1 by:

$$a\sim b \quad\text{iff}\quad h(a) = h(b),$$

$A_1/\sim$ is the set of equivalence classes $[a]_\sim$ $(a\in A_1)$, and $\bar{\rho}_1$ is defined by

$$[a]_\sim\bar{\rho}_1[b]_\sim \Leftrightarrow \left(\exists a_1\in[a]_\sim\right)\left(\exists b_1\in[b]_\sim\right)(a_1\rho b_1).$$

Then we have an easy modification of Theorem 4 in [27].

Proposition 2. The mapping $\varphi:A_1\cup A_2\longrightarrow A_1/\sim\cup A_2$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } x\in A_2, \\ [x]_\sim & \text{if } x\in A_1 \end{cases}$$

is an epimorphism of the Płonka sum $S(\Gamma_1,\Gamma_2;h)$ onto $S(\bar{\Gamma}_1,\Gamma_2;\bar{h})$.

For example, the required epimorphism of the graph Γ onto the graph Γ' exists, as is shown in the following figure:

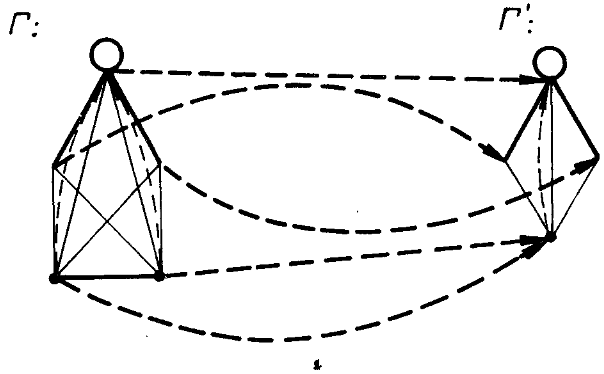


Fig. 3

Similarly, the graph Γ′ is an epimorphic image of the graph Γ in Figure 4:

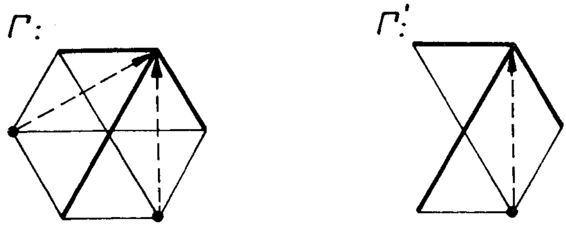


Fig. 4

It seems to be interesting to know a solution of the following

Problem 1. What kind of properties of graphs are preserved by the Płonka sum of graphs?

Up to now we know a few simple facts only. For example, we recall

Proposition 3 (cf. Theorems 9 and 10 of [27]). If Γ_1 and Γ_2 are both connected (or both cliques with all loops, i.e. $\rho_i = A_i \times A_i$ $(i=1,2)$), then also $S(\Gamma_1,\Gamma_2;h)$ has this property.

In connection with this we remark that if Γ_1 and Γ_2 are complete, but Γ_2 is without one loop, say $(a,a)\notin\rho$ for some $a\in A_2$, and $a=h(b)$, $b\in A_1$, then $S(\Gamma_1,\Gamma_2;h)$ is not complete because $(b,h(b))\notin\rho$. There are also two planar graphs Γ_1, Γ_2 and a homomorphism $h:\Gamma_1\longrightarrow\Gamma_2$ such that $S(\Gamma_1,\Gamma_2;h)$ is not planar.

A graph Γ is said to be decomposable into a two-component Płonka sum if there exist graphs Γ_1 and Γ_2 and a homomorphism $h:\Gamma_1\longrightarrow\Gamma_2$ such that $\Gamma=S(\Gamma_1,\Gamma_2;h)$.

For example, the following graphs (n-angles with diagonals in Figure 5) are not decomposable:

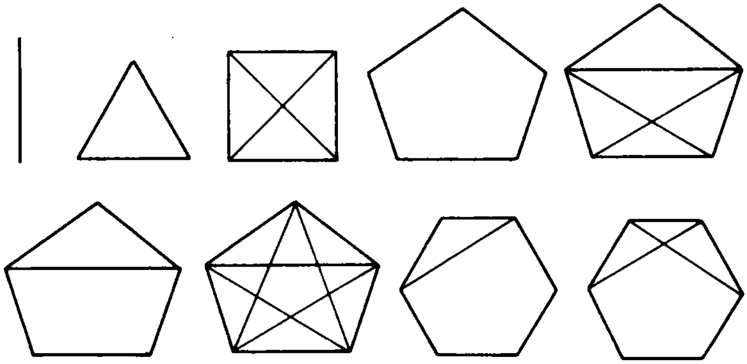


Fig. 5

On the other hand, it is easy to check (cf.[14]) that the following pentagons with diagonals are decomposable:

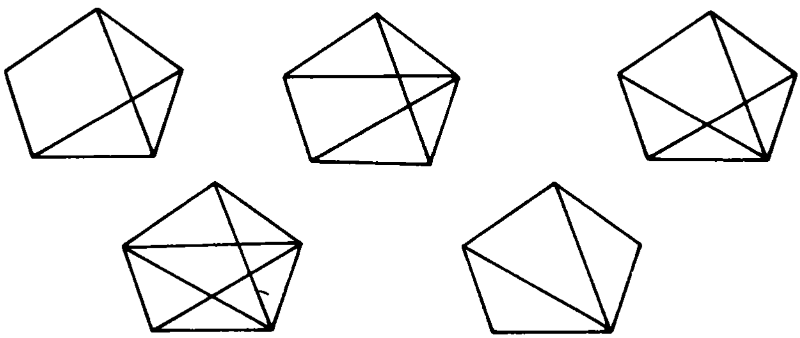


Fig. 6

It is easy to see that investigations of decomposability of graphs can be reduced to decomposability of connected graphs, because in [15] the following was observed:

Proposition 4. A graph is decomposable iff it has at least one decomposable connected component or contains two isolated vertices.

In the following we will consider simple graphs only (i.e. undirected without multiple edges and without loops).

Let a graph $\Gamma=(A;\rho)$ be the Płonka sum $S(\Gamma_1,\Gamma_2;h)$ of two graphs $\Gamma_1=(A_1;\rho_1)$ and $\Gamma_2=(A_2;\rho_2)$, where $h:A_1\longrightarrow A_2$ is the required homomorphism of the graph Γ_1 into Γ_2. Then we have some simple facts (comp. [14]), the proof of which is left to the reader:

(i) $(\forall a\in A_1)(\forall b\in A)(a\rho b \Longrightarrow h(a)\rho b)$.

(ii) $(\forall a,b\in A_1)(h(a)\rho h(b) \Longrightarrow a\rho h(b)$ and $h(a)\rho b)$.

(iii) $(\forall a,b\in A_1)(a\rho b \Longrightarrow h(a)\neq h(b))$.

(iv) $(\forall a\in A_1)((a,h(a))\notin\rho)$.

(v) If A_1 is connected and $|A_1|>1$, then $|A_2|>1$.

(vi) If A is connected, then A_2 does not contain isolated vertices.

Remark that properties (iii)-(vi) hold for the graph Γ without loops.

(vii) The degree of each $a\in A_1$ is not greater than the degree of $h(a)$.

(viii) $(\forall a\in A_1)(\forall b_1,b_2\in A_2,\ b_1\neq b_2)(b_1\rho a,\ b_2\rho a \Longrightarrow (a,b_1,h(a),b_2)$ is the four-vertex cycle).

(ix) $(\forall a_1,a_2\in A_1)(a_1\rho a_2 \Longrightarrow (a_1,h(a_2),h(a_1),a_2)$ is the four-vertex cycle).

From (viii) and (ix) we infer easily

Proposition 5. If Γ is a decomposable connected graph without hanging vertices, then Γ contains the four-vertex cycle as a subgraph.

Corollary 1 (Płonka [20]). A simple cycle of length n is decomposable iff n=4.

Now we observe

Lemma 1. If a graph $\Gamma=(A;\rho)$ contains two hanging vertices incident with the same vertex, then Γ is decomposable.

Indeed, let a and b be two hanging vertices incident with $c\in A$, i.e. $a\rho c$ and $b\rho c$. We take $A_1=\{a\}$, $A_2=A\setminus\{a\}$, $\rho_i=\rho|_{A_i}$ $(i=1,2)$, and we define $h(a)=b$. Then we conclude easily that Γ is decomposable into the Płonka sum of its subgraphs $(A_1;\rho_1)$, $(A_2;\rho_2)$ with h as the required homomorphism.

The situation considered in Lemma 1 is illustrated in Figure 7:

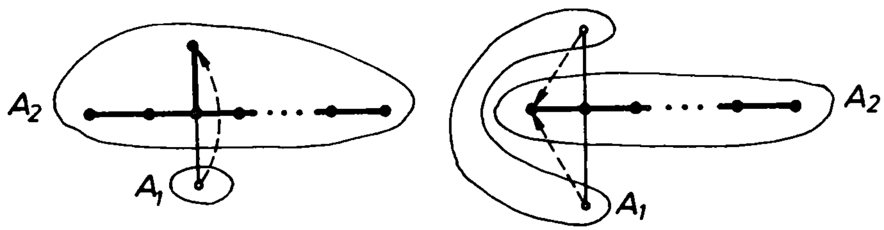


Fig. 7

4. Decomposable trees

Now we will characterize trees which are decomposable into Płonka sums. This material is partially covered by the semi-publication [15] but for completeness we will give full proofs (which are different and simpler than those in [15]).

Lemma 2. If $\Gamma=(A;\rho)$ is a tree, which is decomposable into the Płonka sum $S(\Gamma_1,\Gamma_2;h)$, where $\Gamma_i=(A_i;\rho_i)$ $(i=1,2)$, then A_1 contains hanging vertices only (consequently $\rho_1=\emptyset$).

Indeed, if the degree of a vertex $a\in A_1$ is greater than 1, then by (viii) and (ix), the considered graph Γ contains a cycle, which contradicts the assumption.

Summarizing our consideration in Section 3 and in this section we have

Theorem 1. A tree is decomposable if and only if it contains two hanging vertices incident with the same one vertex, or it is a four-vertex chain.

Proof. First, observe that if a tree Γ contains two hanging vertices incident with the same vertex, then Γ is decomposable by Lemma 1.

If now A is a four-vertex path (a,c,d,b), i.e. $a\rho c$, $c\rho d$, $d\rho b$, then we can take $A_1=\{a,b\}$, $A_2=A\setminus\{a,b\}$, $\rho_i=\rho|_{A_i}$, i=1,2, and we define the required homomorphism $h:A_1\longrightarrow A_2$ by the equalities h(a)=d, h(b)=c.

Finally, assume that there are no two hanging vertices of a tree $\Gamma=(A;\rho)$ incident with some one vertex of A. Let Γ be decomposable into two graphs $\Gamma_1=(A_1;\rho_1)$, $\Gamma_2=(A_2;\rho_2)$ and $h:A_1\longrightarrow A_2$ be a homomorphism required in the definition of the Płonka sum. Of course, $|A|>2$ and the degree of each $a\in A$ is at most 2. If $a_1\in A_1$ and $a_2\rho a_1$, then, by Lemma 2, we have $a_2\in A_2$, and, by (i) and (ii), $h(a_1)\rho a_2 \neq h(a_1)$. Put $h(a_1)=a_3$. By our assumption, a_3 is not a hanging vertex. Let $a_4\neq a_2$ and $a_4\rho a_3$. If $a_4\in A_2$, then by the definition of the Płonka sum we have $a_1\rho a_4$, which is impossible. Therefore $a_4\in A_1$ and, by Lemma 2, a_4 is a hanging vertex. Since $a_3\rho h(a_4)$, we infer that $a_1\rho h(a_4)$ and $h(a_4)=a_2$.

Suppose there exists $a_5\in A_2$ such that $a_5\neq a_2,a_3$ and $a_3\rho a_5$, or $a_2\rho a_5$. Then $a_1\rho a_5$ or $a_4\rho a_5$, which is impossible. Let now $a_5\in A_1$ and $a_5\neq a_1,a_4$. Then $h(a_5)\neq a_2,a_3$ and for $h(a_5)$ we get the above-considered case. Therefore $A=\{a_1,a_2,a_3,a_4\}$ and Γ is a simple path. Thus Theorem 1 is proved.

Corollary 2 (Płonka [20]). A simple path of length n is decomposable iff n=3 or n=4.

5. Decomposable n-angles with some diagonals

Let $\Gamma=(A;\rho)$ be a (simple) graph. If $A=\{a_0,a_1,\dots,a_{n-1}\}$ (where $a_i\neq a_j$ if $i\neq j$) and $a_i\rho a_{i+1}$ for each $i=0,1,\dots,n-2$, and $a_{n-1}\rho a_0$, then $\Gamma=(A;\rho)$ is called an n-angle (with or without diagonals). Therefore $\Gamma=(A;\rho)$ with $A=\{a_0,a_1,\dots$ $\dots,a_{n-1}\}$ is an n-angle if Γ is a Hamiltonian graph with a distinguished Hamiltonian cycle $(a_0,a_1,\dots,a_{n-1})$. A pair (a_i,a_j), where $a_i,a_j\in A$, $j\neq i-1,i,i+1$, is called a diagonal if $a_i\rho a_j$. An n-angle has at most $\frac{1}{2}n(n-3)$ diagonals. A diagonal

(a_i, a_j) of a 2n-angle is called main if $j=i+n \pmod{2n}$. Other diagonals are called short. An n-angle is called 1-saturated if there exists $a_i \in A$ such that $a_i \rho a_j$ for all $j \neq i$. In this case we will also say that Γ is complete at a vertex a_i. An n-angle is called k-saturated if it is complete at some k vertices. An n-angle is called strictly k-saturated if it is k-saturated and has no more diagonals. Of course, an n-saturated n-angle is a (simple) complete graph K_n with n vertices. An n-angle is said to be k-nonsaturated if there are k vertices with no diagonals.

Now we recall some general statements about n-angles $\Gamma=(A;\rho)$ which are decomposable into a Płonka sum $S(\Gamma_1,\Gamma_2;h)$ of two graphs $\Gamma_1=(A_1;\rho_1)$ and $\Gamma_2=(A_2;\rho_2)$ with the required graph homomorphism $h:A_1 \longrightarrow A_2$. By using the general properties (i)-(ix), Proposition 5 and the following Lemmas 3 and 4 we can easily prove (follow by [14]) more results on decompositions of n-angles:

Lemma 3. Let $\Gamma=(A;\rho)=S(\Gamma_1,\Gamma_2;h)$. If $a,b,c \in A_1$ are pairwise different, $a\rho_1 b$ and $a\rho_1 c$, then there exist two vertices of degree at least three which are not connected by some edge (namely, a and h(a)).

Indeed, by (iii) we get $h(a) \neq h(b)$ and $h(a) \neq h(c)$. Moreover, by the definition of homomorphism, $h(a)\rho_2 h(b)$, $h(a)\rho_2 h(c)$ and, finally, $h(a)\rho b$ and $h(a)\rho c$. Therefore the vertices a and h(a) are of degree at least 3; moreover - because of looplessness by (iv) - for the pair $(a,h(a))$ the relation ρ does not hold.

Corollary 3. If an n-angle $\Gamma=(A;\rho)=S(\Gamma_1,\Gamma_2;h)$ has at most one diagonal, then no vertex of degree 3 is in A_1, and no three successive vertices belong to A_1.

Lemma 4. If $\Gamma=(A;\rho)=S(\Gamma_1,\Gamma_2;h)$ is an n-angle, then $|A_2|>1$, A_2 is connected and every two successive vertices belonging to A_i are connected by an edge in Γ_i, $i=1,2$.

Indeed, the first statement is implied by looplessness, the second and the third ones follow immediately because in Γ

we can only obtain additional edges (which are not in Γ_1 and Γ_2) which connect some vertices of A_2 with some vertices of A_1.

Recall that J. Płonka [20] proved that a simple cycle is decomposable iff it is the four-vertex cycle (see Corollary 1). But any full n-angle (i.e. with all possible diagonals, or - in other words - the complete graph K_n) is indecomposable. We observed in 1973 that an n-angle with one diagonal is decomposable iff this diagonal is main and n=4 or n=6 (see [14]). It is easy to see that every n-angle without exactly one diagonal is decomposable. Indeed, if this diagonal is (a_i,a_j), $j\neq i-1,i,i+1$, then we can take $A_1=\{a_i\}$, $A_2=A\setminus\{a_i\}$ and define $h(a_i)=a_j$. More generally one can prove ([14]) that for every pair (n,k), where $n>4$ and $1<k<\frac{1}{2}n(n-3)$, there exists an n-angle with k diagonals which is decomposable into a Płonka sum of its subgraphs.

In [14] it was proved that a one-nonsaturated n-angle (with n>2) is decomposable iff n=4. However, it was proved in [14] that a strictly one-saturated n-angle is decomposable iff n=4 or n=5.

A 2n-angle has an entered n-angle if $a_k\rho a_{k+_{2n}2}$ for all even k or for all odd k, where $+_{2n}$ denotes addition modulo 2n. It is known (see [14]) that a 2n-angle Γ with an entered n-angle is decomposable iff n=3. Indeed, the hexagon with three "short" diagonals is decomposable in the following way (see the left part of Figure 8):

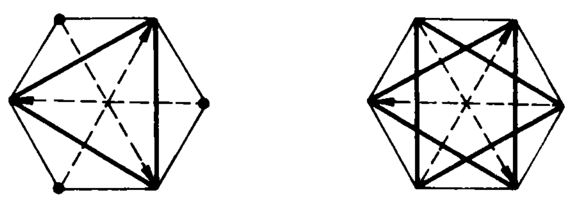


Fig. 8

Similarly we can obtain a decomposition (unique with respect to isomorphism) of the hexagon with six (all) short diagonals.

In the next section we will consider all possible decompositions of hexagons with some diagonals. For this purpose we will give some additional conventions and definitions.

Consider an n-angle $\Gamma=(A;\rho)=S(\Gamma_1,\Gamma_2;h)$ with $A=\{a_0,a_1,\ldots,a_{n-1}\}$. We will assume (without loss of generality) the following:

1^o. $(a_0,a_1,\ldots,a_{n-1})$ is a distinguished Hamiltonian cycle (or circuit in another terminology,

2^o. $a_0\in A_1$ and $a_{n-1}\in A_2$.

Two decompositions of an n-angle (treated as a regular n-angle) are said to be essentially different if they are not isomorphic by any axis-symmetry which preserves property 2^o. Therefore an n-angle will be considered here as "semi-labelled" (for labelled graphs see, e.g., [8] and [9]). From this point of view, the two decompositions of Γ and Γ' in Figure 3 are essentially different (but they are isomorphic), and the decomposition of Γ is not essentially different from the decomposition of the graph Γ'' in Figure 9:

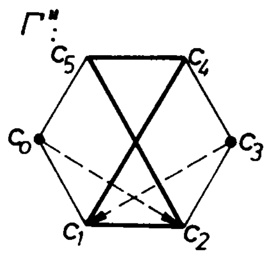


Fig. 9

Let $A=A_1\cup A_2=\{a_0,\ldots,a_{n-1}\}$, $A_1\cap A_2=\emptyset$, $a_0\in A_1$, $a_{n-1}\in A_2$. Further, let $\Gamma_1=(A_1;\rho_1)$ and $\Gamma_2=(A_2;\rho_2)$ be two (undirected) graphs (without loops), and let $h:A_1\longrightarrow A_2$ be a fixed graph homomorphism of Γ_1 into Γ_2. Then the graph $\Gamma=S(\Gamma_1,\Gamma_2;h)$ is called primitive if the relations ρ_1 and ρ_2 are minimal in order to obtain the Hamiltonian cycle $(a_0,a_1,\ldots,a_{n-1})$ in Γ. That is, if we omit one edge $(a_i,a_j)\in\rho_1\cup\rho_2$, then the

Hamiltonian cycle $(a_0,a_1,\ldots,a_{n-1})$ will be destroyed. For example, the first graph in Figure 8 is primitive but the second one is not, because we can omit an arbitrary edge which connects vertices of A_1.

6. Enumeration of decompositions of hexagons

The general procedure to find all possible essentially different decompositions of an n-angle is the following:

I. Fix a Hamiltonian cycle $(a_0,a_1,\ldots,a_{n-1})$;

II. Look for all possible decompositions $A=A_1\cup A_2$ of the set $A=\{a_0,a_1,\ldots,a_{n-1}\}$ of vertices with property 2^o;

III. Find all possible "primitive" graphs with fixed mapping $h:A_1\longrightarrow A_2$ as the homomorphism required in the definition of $\Gamma=S(\Gamma_1,\Gamma_2;h)$;

IV. Enumerate all essentially different decompositions of graphs $\Gamma=S(\Gamma_1,\Gamma_2;h)$, which can be obtained by adding some additional admissible edges between vertices of A_1 or between vertices of A_2;

V. Take an account of all such possibilities.

If we use this procedure for hexagons we obtain

Theorem 2. Let $\Gamma=(A;\rho)=S(\Gamma_1,\Gamma_2;h)$ be a hexagon with some diagonals and a fixed Hamiltonian cycle $(a_0,a_1,\ldots,a_5)$. Suppose $a_0\in A_1$ and $a_5\in A_2$. Then the numbers of possible essentially different decompositions are the following:

a) 44 if $|A_1|=1$,

b) 40 if $|A_1|=2$,

c) 22 if $|A_1|=3$,

d) 5 If $|A_1|=4$.

Proof. First, observe that under our assumptions - taking into account general properties of Płonka sums (see Sections 3 and 5) - we have only the following possibilities for A_1 and h:

1) $A_1=\{a_0\}$, $h(a_0)=a_3$,

2) $A_1=\{a_0\}$, $h(a_0)=a_2$,

2') $A_1=\{a_0\}$, $h(a_0)=a_4$,

3) $A_1=\{a_0,a_1\}$, $h(a_0)=a_4$ and $h(a_1)=a_3$,

4) $A_1=\{a_0,a_1\}$, $h(a_0)=a_3$ and $h(a_1)=a_4$,

5) $A_1=\{a_0,a_2\}$, $h(a_0)=h(a_2)=a_4$,

5′) $A_1=\{a_0,a_4\}$, $h(a_0)=h(a_4)=a_2$,

6) $A_1=\{a_0,a_2\}$, $h(a_0)=a_3$, $h(a_2)=a_5$,

6′) $A_1=\{a_0,a_4\}$, $h(a_0)=a_3$, $h(a_4)=a_2$,

7) $A_1=\{a_0,a_2\}$, $h(a_0)=a_4$, $h(a_2)=a_5$,

7′) $A_1=\{a_0,a_4\}$, $h(a_0)=a_2$, $h(a_4)=a_1$,

8) $A_1=\{a_0,a_2\}$, $h(a_0)=a_3$, $h(a_2)=a_4$,

8′) $A_1=\{a_0,a_4\}$, $h(a_0)=a_3$, $h(a_4)=a_2$,

9) $A_1=\{a_0,a_3\}$, $h(a_0)=a_4$, $h(a_3)=a_5$,

9′) $A_1=\{a_0,a_3\}$, $h(a_0)=a_2$, $h(a_3)=a_1$,

10) $A_1=\{a_0,a_3\}$, $h(a_0)=a_4$, $h(a_3)=a_1$,

10′) $A_1=\{a_0,a_3\}$, $h(a_0)=a_2$, $h(a_3)=a_5$,

11) $A_1=\{a_0,a_1,a_2\}$, $h(a_0)=a_4$, $h(a_1)=a_3$, $h(a_2)=a_4$,

11′) $A_1=\{a_0,a_1,a_2\}$, $h(a_0)=a_4$, $h(a_1)=a_5$, $h(a_2)=a_4$,

12) $A_1=\{a_0,a_1,a_2\}$, $h(a_0)=a_4$, $h(a_1)=a_3$, $h(a_2)=a_5$,

12′) $A_1=\{a_0,a_1,a_2\}$, $h(a_0)=a_3$, $h(a_1)=a_5$, $h(a_2)=a_4$,

13) $A_1=\{a_0,a_1,a_2\}$, $h(a_0)=a_3$, $h(a_1)=a_4$, $h(a_2)=a_5$,

14) $A_1=\{a_0,a_1,a_3\}$, $h(a_0)=a_4$, $h(a_1)=h(a_3)=a_5$,

14′) $A_1=\{a_0,a_1,a_4\}$, $h(a_0)=h(a_4)=a_2$, $h(a_1)=a_3$,

15) $A_1=\{a_0,a_1,a_3\}$, $h(a_0)=a_2$, $h(a_1)=a_4$, $h(a_3)=a_5$,

15′) $A_1=\{a_0,a_1,a_4\}$, $h(a_0)=a_3$, $h(a_1)=a_5$, $h(a_4)=a_2$,

16) $A_1=\{a_0,a_1,a_3\}$, $h(a_0)=a_2$, $h(a_1)=a_5=h(a_3)$,

16′) $A_1=\{a_0,a_1,a_4\}$, $h(a_0)=a_2$, $h(a_1)=a_5$, $h(a_4)=a_2$,

17) $A_1=\{a_0,a_2,a_3\}$, $h(a_0)=a_4$, $h(a_2)=a_4$, $h(a_3)=a_5$,

17′) $A_1=\{a_0,a_3,a_4\}$, $h(a_0)=a_2$, $h(a_3)=a_1$, $h(a_4)=a_2$,

18) $A_1=\{a_0,a_2,a_3\}$, $h(a_0)=a_4$, $h(a_2)=a_5$, $h(a_3)=a_1$,

18′) $A_1=\{a_0,a_3,a_4\}$, $h(a_0)=a_2$, $h(a_3)=a_5$, $h(a_4)=a_1$,

19) $A_1=\{a_0,a_2,a_3\}$, $h(a_0)=a_4$, $h(a_2)=a_4$, $h(a_3)=a_1$,

19′) $A_1=\{a_0,a_3,a_4\}$, $h(a_0)=a_2$, $h(a_3)=a_5$, $h(a_4)=a_2$,

20) $A_1=\{a_0,a_2,a_4\}$, $h(a_0)=a_3$, $h(a_2)=a_5$, $h(a_4)=a_1$,

21) $A_1=\{a_0,a_1,a_2,a_3\}$, $h(a_0)=a_4=h(a_2)$, $h(a_1)=a_5=h(a_3)$,

22) $A_1=\{a_0,a_1,a_3,a_4\}$, $h(a_0)=a_2=h(a_4)$, $h(a_1)=a_5=h(a_3)$.

But by using suitable axis-symmetries we can observe that the possibilities denoted by (k) and (k′) are dual and lead to non-essentially different decompositions. For example, cases (2) and (2′) give isomorphic decompositions by symmetry with respect to the axis (a_0,a_3). Similarly, taking into account symmetry with respect to the axis (a_1,a_4) we observe that possibilities (7) and (8) are not essentially different.

Moreover possibilities (17), (18) and (19) can be obtain by suitable axis-symmetries from (14), (15) and (16), respectively.

Therefore we have the following 18 essentially different primitive graphs:

For $|A_1|=1$:

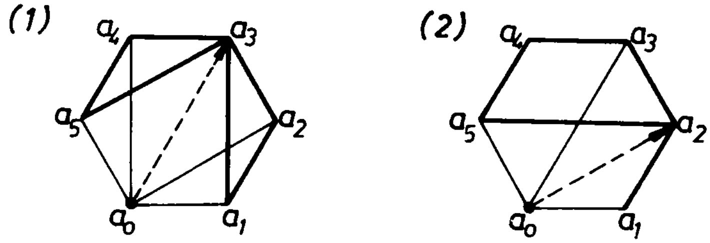


Fig. 10

For $|A_1|=2$:

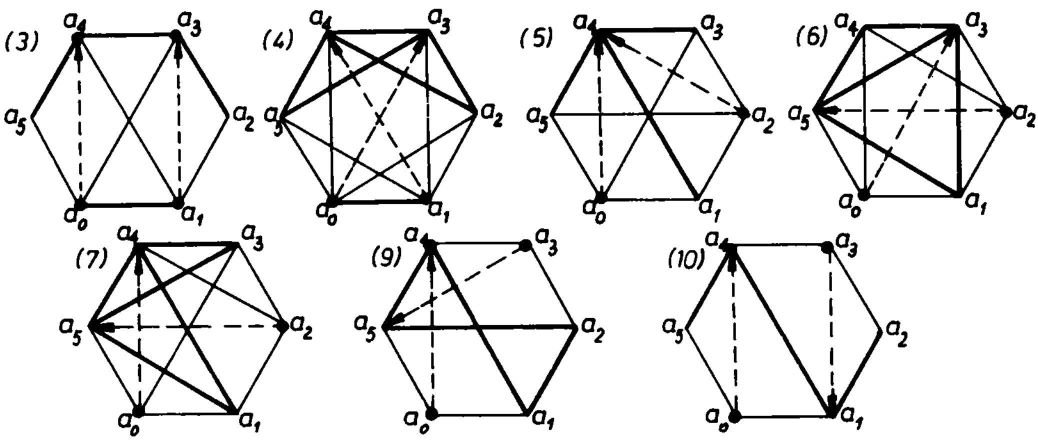


Fig. 11

For $|A_1|=3$:

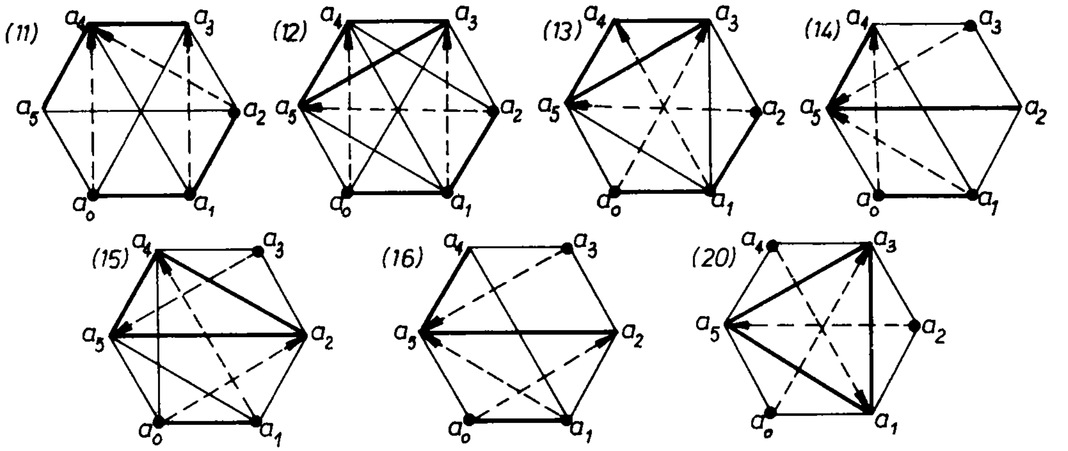


Fig. 12

For $|A_1|=4$:

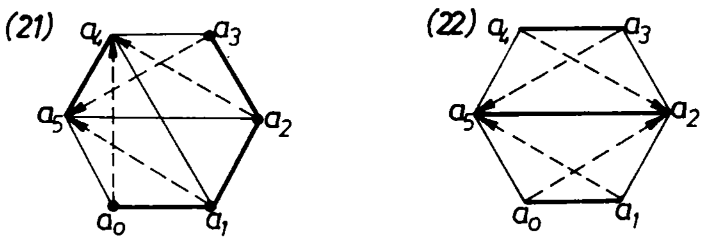


Fig. 13

From cases (1)-(7), (9)-(16) and (20)-(22), adding admissible edges between vertices of A_1 and between vertices of A_2 and by using suitable axis-symmetries, we obtain the following numbers of all possibilities and essentially different ones, respectively:

Cases	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(9)	(10)
All poss.	16	32	8	2	8	4	4	8	16
Ess. diff. poss.	12	32	6	2	6	4	4	6	12

(11)	(12)	(13)	(14)	(15)	(16)	(20)	(21)	(22)
2	2	2	4	4	4	8	2	4
2	2	2	4	4	4	4	2	3

By counting together the numbers of essentially different possibilities of cases (1) and (2), of cases (3)-(10), of cases (11)-(16) and (20), and, finally, of cases (21) and (22) we obtain numbers 44, 40, 22 and 5, which completes the proof of Theorem 2.

It is easy to observe, by the remark after Proposition 1, that six (essentially different possibilities of case (10) are (pairwise) isomorphic with those of case (9). The possibilities in (16) are isomorphic to suitable possibilities in (14), too. The number of all possibilities, under our assumptions 1^{o} and 2^{o} (i.e. treated as "semi-labelled" graphs), is 248. This number for any n-angle should be even. Moreover the number of all decompositions of labelled hexagons is 2976.

By analyzing the proof of Theorem 2 one can verify (we leave this to the reader)

Theorem 3. There are only

α) 10 essentially different decompositions of the hexagon with 3 main diagonals;

β) 9 essentially different decompositions of the hexagon with 2 main diagonals;

γ) 2 essentially different decompositions of the hexagon with one main diagonal;

δ) 4 essentially different decompositions of the hexagon with two entered triangles.

We illustrate these possibilities in pictures (see Figures 16-19):

α) 3 main diagonals:

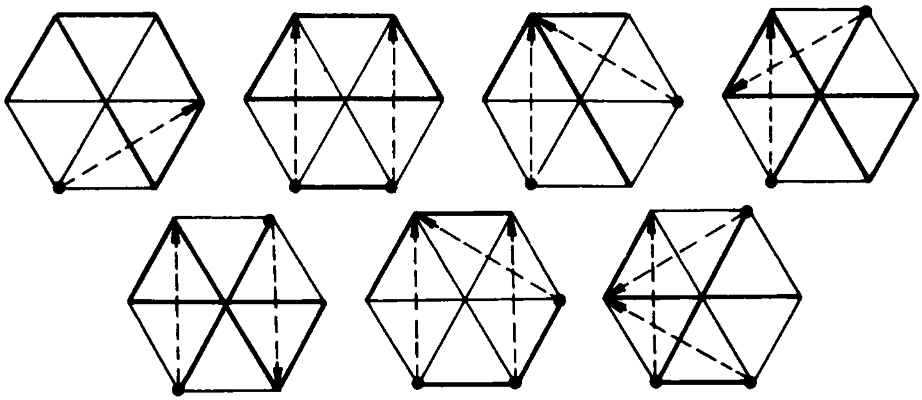
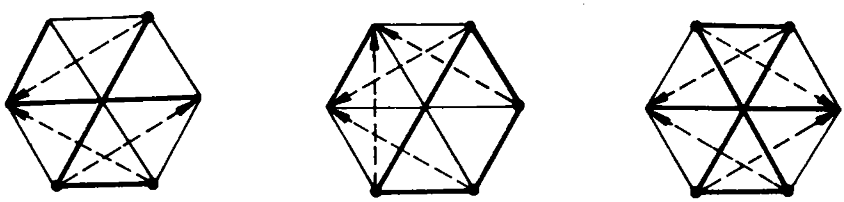


Fig. 14



β) 2 main diagonals:

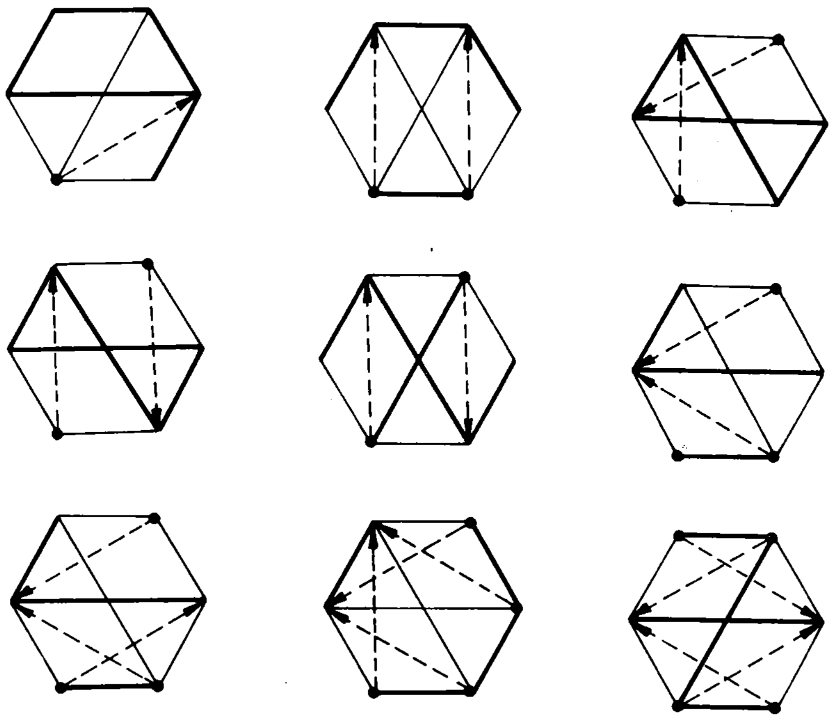


Fig. 15

γ) 1 main diagonal:

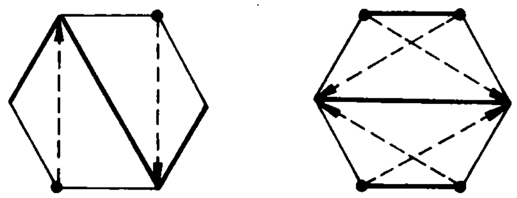


Fig. 16

δ) two entered triangles:

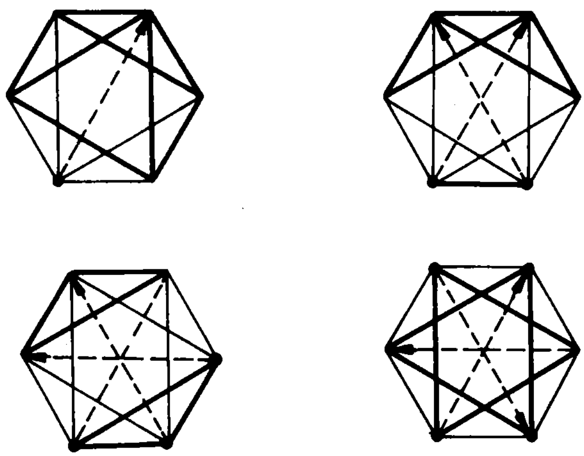


Fig. 17

In the context of our results the following problem seems to be interesting but difficult:

Problem 2 (J. Płonka). Characterize n-angles which are decomposable into Płonka sums of graphs.

Another interesting problem is the following.

Problem 3. What class of graphs can be built by the construction of a Płonka sum of graphs from some special graphs (e.g. triangles, n-angles, trees, etc.)?

7. Algebras related to graphs

In this section we signalize some algebraic approach to investigation of graphs, which seems to be interesting and fruitful.

Consider a binary relation θ on a set A. An operation $f:\underbrace{A\times\ldots\times A}_{n}\longrightarrow A$ is said to be compatible with θ if

$$(\forall x_1,\ldots,x_n,x'_1,\ldots,x'_n\in A)\big(x_i\theta x'_i \text{ for } i=1,\ldots,n \implies$$
$$\implies f(x_1,\ldots,x_n)\theta f(x'_1,\ldots,x'_n)\big);$$

in other words: f preserves the subset θ of $A\times A$ or θ is a subalgebra of the square of the algebra (A;f). The set of all operations (of arbitrary arities) which preserve θ is denoted by Pol $\{\theta\}$. This set Pol $\{\theta\}$ is closed under composition of operations, i.e., Pol $\{\theta\}$ forms a clone (see [31]).

The set of those operations on A which preserve a subset B of a power of A will be denoted by $\mathrm{Pol}_A\{B\}$. More generally, for an arbitrary set S of subsets of powers of A we define the set $\mathrm{Pol}_A S$ of operations by

$$\mathrm{Pol}_A S = \bigcap_{B\in S}\mathrm{Pol}_A B.$$

Proposition 6 (see [31]). $\mathrm{Pol}_A S$ is a clone for an arbitrary set S of subsets of powers of A.

Example. Let $\Gamma=(A;\rho)$ be a graph with $|A|=n$ $(\rho\subset A\times A)$. One can verify that for a determination of the algebra $\mathfrak{A}[\Gamma]=(A;\mathrm{Pol}_A\{\rho\})$ it is enough to know the set of all m-ary term operations $\mathbf{T}^{(m)}(\mathfrak{A}[\Gamma])$ of this algebra $\mathfrak{A}[\Gamma]$, with some $m\le 2^n$. Unfortunately, the definition of $\mathfrak{A}[\Gamma]$ is not effective. It will be important to answer the following

Problem 4. What is the most economical way to determine the set of fundamental operations (possibly small and consisting of operations of small arities)?

By using the notion of the weak automorphism of a general algebra (see [29]), we have some simple observation.

Proposition 7. Every automorphism of the graph $\Gamma=(A;\rho)$ is a weak automorphism of $\mathfrak{A}[\Gamma]$.

The algebra $\mathfrak{A}[\Gamma]$ will be called a full graph-algebra. This algebra $\mathfrak{A}[\Gamma]$ will be investigated in a forthcoming publication (of the first author). Such algebras $\mathfrak{A}[\Gamma]$ carry more information about graphs Γ, than graph-groupoids (or so-called Shallon algebras, or graph algebras) investigated by several authors (see, e.g., [30], [17], [25], [26]).

Problem 5. What is the connection between some properties of algebras and properties of graphs determined by the correspondence $\Gamma \longrightarrow \mathfrak{A}[\Gamma]$?

Now we propose some modification of this approach to investigations of decompositions of graphs into the Płonka sum of subgraphs.

Let $\Gamma=(A;\rho)$ be a graph, which is decomposable into a Płonka sum of its subgraphs $\Gamma_1=(A_1;\rho_1)$ and $\Gamma_2=(A_2;\rho_2)$, i.e. $\Gamma=S(\Gamma_1,\Gamma_2;h)$, where $h:A_1 \longrightarrow A_2$ is a suitable graph homomorphism. Consider a mixed graph $\overline{\Gamma}=(A;\theta)$, where $\theta=\rho\cup h^{\square}$ and $h^{\square}=\{(a,h(a)) : a\in A_1\}$. Then we can define a new "graph algebra" $\mathfrak{A}[\overline{\Gamma},A_1,A_2]$ by taking as term operations all operations on A which preserve the relation θ and the subsets A_1 and A_2. Taking into account the definition of the Płonka sum of algebras (see [19]) we can verify

Proposition 8. The algebra $\mathfrak{A}[\overline{\Gamma},A_1,A_2]$ is decomposable into a Płonka sum of its subalgebras A_1 and A_2.

Example. Consider the algebra $\mathfrak{A}[\overline{\Gamma},A_1,A_2]$ for the following decomposition of the hexagon:

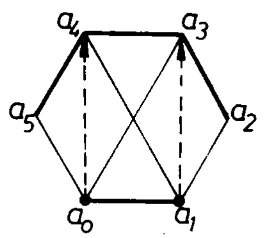


Fig. 18

Then all unary term operations of the algebra $\mathfrak{A}[\overline{\Gamma},A_1,A_2]$ are the following:

	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	$g_5(x)$	$g_6(x)$	$g_7(x)$	$g_8(x)$
a_0	a_0	a_0	a_0	a_0	a_1	a_1	a_1	a_1
a_1	a_1	a_1	a_1	a_1	a_0	a_0	a_0	a_0
a_2	a_2	a_4	a_2	a_4	a_4	a_2	a_4	a_2
a_3	a_3	a_3	a_3	a_3	a_4	a_4	a_4	a_4
a_4	a_4	a_4	a_4	a_4	a_3	a_3	a_3	a_3
a_5	a_5	a_5	a_3	a_3	a_4	a_4	a_2	a_2

We give also an example of a binary term operation of the considered algebra:

$\circ$	a_0	a_1	a_2	a_3	a_4	a_5
a_0	a_0	a_0	a_4	a_4	a_4	a_4
a_1	a_1	a_1	a_3	a_3	a_3	a_3
a_2	a_4	a_2	a_2	a_2	a_4	a_4
a_3	a_3	a_3	a_3	a_3	a_3	a_3
a_4	a_4	a_4	a_4	a_4	a_4	a_4
a_5	a_5	a_3	a_3	a_3	a_5	a_5

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Added in proof.

We have learnt of two recent papers connected with Płonka sums of graphs:

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW,
50-384 WROCŁAW, POLAND

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