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Elżbieta Zając

CONSTRUCTIONS OF LEFT NORMAL BANDOIDS

Introduction

The idea of investigation of left normal bandoids arises from the study of dissemilattices.

A dissemilattice $(B,+,\cdot)$ is a set B with two binary operations such that the reducts $(B,+)$ and $(B,\cdot)$ are semilattices and the identity

$$(x+y)\cdot z = (x\cdot z)+(y\cdot z)$$

is satisfied in $(B,+,\cdot)$.

Dissemilattices form one of the most important classes of modals - idempotent entropic algebras with additional semilattice structure (for details see [13]). Dissemilattices may be regarded as distributive lattices for which the requirement of absorption and one of distributive laws have been dropped. The name "dissemilattice" is intended to recall the presence of two semilattice structures and the fact that the one distributes over the other. The algebras are known under the name of meet-distributive bisemilattices as well ([3], [6] - [12]).

A dissemilattice which satisfies

$$x+(y\cdot z) = (x+y)\cdot(x+z)$$

is said to be distributive.

Distributive dissemilattices are Płonka sums of distributive lattices ([4], [5]). The role of the partition operation for these algebras is played by the operation $\circ$ defined as follows:

$$x\circ y := x+(x\cdot y).$$

In [12] the algebra $(B,\circ)$ obtained from a dissemilattice $(B,+,\cdot)$ was called a Hałkowska algebra. In general, the operation $\circ$ is very useful in describing the lattice and semilattice replicas of a dissemilattice $(B,+,\cdot)$. It plays an important role in the structure theory of dissemilattices ([9], [10]). Elementary properties of $(B,\circ)$ were discussed in [2]. A deeper investigation was undertaken in [12]. In particular it was shown in [12] that $(B,\circ)$ satisfies the following identities:

(B1) $x\cdot x=x$,

(B2) $x\cdot(x\cdot y) = x\cdot y$,

(B3) $x\cdot(y\cdot z) = (x\cdot y)\cdot(y\cdot z)$,

(B4) $(x\cdot y)\cdot z = (x\cdot z)\cdot y$,

(B5) $(x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z)$,

(B6) $x\cdot(y\cdot z) = x\cdot(y\cdot(x\cdot z))$.

A groupoid $(B,\cdot)$ satisfying the identities (B1)-(B6) was called in [12] a left normal bandoid. In the present paper left normal bandoids will often be called briefly bandoids. As was shown in [12], the variety of left normal bandoits is generated by the class of all Hałkowska algebras of dissemilattices. The name of left normal bandoids is motivated by the observation that they form a non-associative generalization of left normal bands. In [15] it was shown that the only proper subvarieties of the variety of left normal bandoids are the four varieties of left normal bands.

This paper is first from a series of papers studying the structure of left normal bandoids. We describe here some constructions of left normal bandoids. The paper is divided into three sections. The first one is a collection of basic notions and results about left normal bandoids, mainly from [12]. Section 2 presents a general method of constructing left normal bandoits from a family of semilattices with units and homomorphisms between them, generalizing the construction given in [12]. In Section 3 the general construction from Section 2 is specified to construct a left normal bandoid from a family of principal ideals of a complete lattice $(L,+,\cdot)$ satisfying the distributive law

$$x\cdot\sum(y_i \mid i\in I) = \sum(x\cdot y_i \mid i\in I).$$

In subsequent papers we will use these constructions to study subdirectly irreducible left normal bandoids.

For more information concerning left normal bandoids we refer the reader to [12]. [13], [15] and for basic algebraic concepts to [1].

1. Preliminaries

The aim of this section is to recall some known facts concerning left normal bandoids which will be used in the later sections. First we recall some definitions. (See [12], [13]).

Let $\underline{B} = (B,\cdot)$ be a left normal bandoid. For x,y in B denote $xy := x\cdot y$. For x in B, the mappings $R(x):B \longrightarrow B$; $y \mapsto yx$ and $L(x):B \longrightarrow B$; $y \mapsto xy$ are known as right and left multiplications respectively. As was remarked in [12], the submonoid $(T(B),\circ,id_B)$ of the monoid of all mappings from B to B (under functional composition) generated by $\{R(x) \mid x\in B\}$ is a semilattice with unit. For each x in B, the mapping $L(x):(T(B),\circ) \longrightarrow (B,\cdot)$; $\alpha \mapsto \alpha x$ is a homomorphism ([12, Proposition 2.1]). The image $xT(B)$ of $T(B)$ under the homomorphism $L(x)$ is called the orbit of x in B. Since $T(B)$ is a semilattice with unit and $L(x):(T(B),\circ)$; $\alpha \mapsto \alpha x$ is a homomorphism, the following remark is evident.

1.1. Remark [12]. For x in B, $(xT(B),\cdot,x)$ is a semilattice with unit. ■

1.2. Remark [12]. B is the union of the orbits of its generators. ■

1.3. Proposition ([12, Proposition 2.2]). For x, y in B, the mapping $L(x):(yT(B),\cdot) \longrightarrow (xT(B),\cdot)$; $z \mapsto xz$ is a semilattice homomorphism. ■

Let $(L(B),\circ,id_B)$ be the submonoid of the monoid of all mappings from B to B generated by $\{L(x) \mid x\in B\}$. The next corollary follows by simple induction and Proposition 1.3.

1.4. Corollary. For x, y in B and α in L(B), the mapping $L(\alpha):(yT(B),\cdot) \longrightarrow ((\alpha y)T(B),\cdot)$; $z \mapsto \alpha z$ is a semilattice homomorphism. ∎

1.5. Proposition ([2, Proposition 2.3]). The relation ≤ on B defined by

$$x \le y \Leftrightarrow x = yx$$

is a partial order on B, and $x \le y$ if and only if the orbit of x is contained in the orbit of y. ∎

Note that the relation ≤ restricted to an orbit of $(B,\cdot)$ is the usual partial order of the semilattice $(xT(B),\cdot)$. Evidently, if $(B,\cdot)$ is a left zero semigroup, then $(B,\le)$ is an antichain.

In the sequel we use the following notation. Instead of $x \le y$ we sometimes write $y \ge x$. For $x \le y$ and $x \ne y$ we use the symbol $x < y$ or $y > x$.

1.6. Remark. For x, y in B the following conditions are equivalent:

(i) $x \in yT(B)$,

(ii) $x = yx$,

(iii) $x = yz$ for some $z \in B$,

(iv) $x \le y$,

(v) $xT(B) \subseteq yT(B)$.

Proof. Observe that (i) implies (ii) since y plays the role of unit in $(yT(B),\cdot)$. The implication (ii) ⟹ (iii) is obvious. By definition of an orbit, (iii) implies (i). So (i) ⇔ (ii) ⇔ (iii). Moreover, in view of Proposition 1.5, (ii) ⇔ (iv) ⇔ (v). ∎

1.7. Remark. For x, y, z in B, $x \le y$ implies $zx \le zy$ and $xz \le yz$.

Proof. Let $x \le y$. Note that $zx \le zy$ by Proposition 1.3. Furthermore

$$(yz)(xz) = (yx)z \quad \text{by (B5)}$$
$$= xz \quad \text{since } x \le y.$$

Consequently $xz \le yz$.

1.8. Proposition. Let $x, x_1, y \in B$ with $x_1 \le x$. Then

$$x_1 y = x_1(xy) = x_1 \cdot L(x) y.$$

Proof. Note that, since $(xT(B), \cdot)$ is a semilattice and $xy, x_1 \in xT(B)$, we have

$$x_1(xy) = (xy)x_1.$$

Hence, using (B4) and the fact that $x_1 = xx_1$, we obtain

$$x_1(xy) = (xx_1)y = x_1 y.$$

The equality $x_1(xy) = x_1 \cdot L(x)y$ holds by definition of $L(x)$. ∎

In the sequel, right normalized products $(x_1(x_2 \ldots \ldots (x_{n-1}x_n) \ldots))$ of elements of a left normal bandoid will be written simply as $x_1 x_2 \ldots x_{n-1} x_n$.

1.9. Proposition [12, Lemma 4.6]. For elements $x_1 x_2 \ldots x_n$ of left normal bandoid, and for $1 \le i \le n$, one has

$$x_i x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_n = x_i x_1 \ldots x_{i-1} x_{i+1} \ldots x_n. \qquad ∎$$

1.10. Proposition ([12, Corollary 4.4]). Left normal bandoids generated by pairs of elements are associative. ∎

1.11. Corollary. For all x, y in B

$$(xy)x = (xy)y = x(yx) = xy.$$

Proof. Propositions 1.9 and 1.10. ∎

2. <u>A construction of left normal bandoids from semilattices and homomorphisms between them</u>

In this section we describe a method of constructing left normal bandoids from a family of semilattices and homomorphisms between them, and show that each left normal bandoid may be represented using this method. The last result was essentially proved in [12], but the general construction was not described there.

2.1. Definition. A semilattice system is a triple $(I, \{\underline{S}_i\}_{i \in I}, \{\phi_{ij}\}_{i,j \in I})$, where for each i, j, k in I

(i) $\underline{S}_i = (S_i, \cdot, 1)$ is a semilattice with unit;

(ii) $\phi_{ij} : (S_i, \cdot) \longrightarrow (S_j, \cdot)$ is a semilattice homomorphism;

(iii) $\phi_{ik|S_i\cap S_j} = \phi_{jk|S_i\cap S_j}$;

(iv) $x\cdot\phi_{ki}y = x\cdot\phi_{kj}y$ for every x in $S_i\cap S_j$;

(v) $\phi_{ii} = id_{S_i}$. ∎

Note that if $x\in S_i\cap S_j$, $y\in S_k\cap S_l$ for some i, j, k, l in I, then $x\cdot\phi_{ki} = x\cdot\phi_{lj}y$. Indeed, by 1.1 (iv) $x\cdot\phi_{ki}y = x\cdot\phi_{kj}y$, and by 1.1(iii) $\phi_{kj}y = \phi_{lj}y$. Thus $x\cdot\phi_{ki}y = x\cdot\phi_{lj}y$. This justifies the following

2.2. Definition. Let $\mathcal{S} = (I,\{\underline{S}_i\}_{i\in I},\{\phi_{ij}\}_{i,j\in I})$ be a semilattice system. Define a binary operation on the set $S = \bigcup_{i\in I} S_i$ as follows. For $x\in S_i$, $y\in S_j$, $i,j\in I$,

$$x\cdot y := x\cdot\phi_{ji}y,$$

where the symbol $\cdot$ on the right hand side denotes multiplication in $\underline{S}_i$. Then the groupoid $\underline{S}=(S,\cdot)$ is called the sum of the system $\mathcal{S}$. ∎

Note that for x, y in S_i, $x\cdot y$ in $\underline{S}$ coincides with $x\cdot y$ in $\underline{S}_i$.

2.3. Theorem. The sum $\underline{S}$ of a semilattice system $(I,\{\underline{S}_i\}_{i\in I},\{\phi_{ij}\}_{i,j\in I})$ is a (left normal) bandoid if and only if for all i, j, k in I, x in S_i

(i) $x\cdot\phi_{ji}\phi_{ij}x = x\cdot\phi_{ji}1_j$

and

(ii) $\phi_{ji}\phi_{ij}\phi_{ki} = \phi_{ji}\phi_{kj}$.

2.4. Remark. Note that condition (i) is equivalent to $x\cdot(1_j\cdot x)=x\cdot 1_j$ and the condition (ii) to $1_i\cdot(1_j\cdot(1_i\cdot y))= =1_i\cdot(1_j\cdot y)$ for all $y\in S_k$.

Proof. (⇒) First we prove that the condition (i) is satisfied. Let $i,j\in I$ and $x\in S_i$. As a consequence of the assumption that $\underline{S}$ is a bandoid, and by Corollary 1.11, we get $x\cdot(1_j\cdot x)=x\cdot 1_j$. By Remark 2.4 this completes the proof of (i).

To prove (ii), assume that $i,j,k\in I$ and $y\in S_k$. Since $\underline{S}$ is a bandoid, by Proposition 1.9 we have $1_i\cdot(1_j\cdot(1_i\cdot y))=1_i\cdot(1_j\cdot y)$. Thus, by Remark 2.4, (ii) holds.

(⇐) Let $i,j,k \in I$, $x \in S_i$, $y \in S_j$, $z \in S_k$. Using Definition 2.2 one obtains

$x\cdot x = x\cdot\phi_{ii}x = x$, since $\phi_{ii} = id_S$ and $\underline{S}_i$ is a semilattice, and

$x\cdot(x\cdot y) = x\cdot(x\cdot\phi_{ji}y) = x\cdot\phi_{ji}y$ since $\underline{S}_i$ is a semilattice

$= x\cdot y.$

Further

$(x\cdot y)\cdot(y\cdot z) = (x\cdot\phi_{ji}y)\cdot\phi_{ji}(y\cdot\phi_{kj}z)$

$= (x\cdot\phi_{ji}y)\cdot(\phi_{ji}y\cdot\phi_{ji}\phi_{kj}z)$ since ϕ_{ji} is a homeomorphism

$= x\cdot(\phi_{ji}y\cdot\phi_{ji}\phi_{kj}z)$ since $\underline{S}_i$ is a semilattice

$= x\cdot\phi_{ji}(y\cdot\phi_{kj}z)$ since ϕ_{ji} is a homomorphism

$= x\cdot(y\cdot z).$

$(x\cdot y)\cdot z = (x\cdot\phi_{ji}y)\cdot\phi_{ki}z$

$= (x\cdot\phi_{ki}z)\cdot\phi_{ji}y$ since $\underline{S}_i$ is a semilattice

$= (x\cdot z)\cdot y.$

Thus the identities (B1) - (B4) are satisfied in $\underline{S}$. To prove (B6) and (B5) we will use conditions (i) and (ii) as well.

Now $x\cdot(y\cdot(x\cdot z)) = x\cdot\phi_{ji}(y\cdot\phi_{ij}(x\cdot\phi_{ki}z))$

$= x\cdot(\phi_{ji}y\cdot(\phi_{ji}\phi_{ij}x\cdot\phi_{ji}\phi_{ij}\phi_{ki}z))$ since ϕ_{ji}, ϕ_{ij} are homomorphisms

$= \phi_{ji}y\cdot(x\cdot\phi_{ji}\phi_{ij}x)\cdot\phi_{ji}\phi_{ij}\phi_{ki}z$ since $\underline{S}_i$ is a semilattice

$= \phi_{ji}y\cdot(x\cdot\phi_{ji}1_j)\cdot\phi_{ji}\phi_{kj}z$ by (i) and (ii)

$= x\cdot(\phi_{ji}y\cdot\phi_{ji}1_j)\cdot\phi_{ji}\phi_{kj}z$ since $\underline{S}_i$ is a semilattice

$= x\cdot\phi_{ji}(y\cdot 1_j)\cdot\phi_{ji}\phi_{kj}z$ since ϕ_{ji} is a homomorphism

$= x\cdot\phi_{ji}y\cdot\phi_{ji}\phi_{kj}z$ since 1_j is the unit in $\underline{S}_j$

$= x\cdot(y\cdot z).$

It follows that (B6) holds in $\underline{S}$. It remains to prove the validity of (B5). First we prove

$x\cdot(y\cdot x) = x\cdot\phi_{ji}(y\cdot\phi_{ij}x) = x\cdot(\phi_{ji}y\cdot\phi_{ji}\phi_{ij}x)$

$= \phi_{ji}y\cdot(x\cdot\phi_{ji}\phi_{ij}x)$

$= \phi_{ji}y\cdot(x\cdot\phi_{ji}1_j)$ by (i)

$$= x\cdot\phi_{ji}y\cdot\phi_{ji}1_j$$
$$= x\cdot\phi_{ji}(y\cdot 1_j) = x\cdot\phi_{ji}y = x\cdot y.$$

So the identity

(2.3.1) $\quad x\cdot(y\cdot x)=x\cdot y$

holds in $\underline{S}$.

Now note that (B1) and (B3) imply

(2.3.2) $\quad (x\cdot z)\cdot z = (x\cdot z)\cdot(z\cdot z) = x\cdot(z\cdot z) = x\cdot z$

As a consequence of (B5), (2.3.2) and (2.3.1) we get that
$(x\cdot z)\cdot(y\cdot z) = (x\cdot z)\cdot(y\cdot((x\cdot z)\cdot z)) = (x\cdot z)\cdot(y\cdot(x\cdot z)) = (x\cdot z)\cdot y.$
Therefore (B6) holds in $\underline{S}$ as well. ∎

2.5. Theorem. Every left normal bandoid is the sum of some semilattice system $\mathcal{S} = \big(I,\{\underline{S}_i\}_{i\in I},\{\phi_{ij}\}_{i,j\in I}\big)$.

Proof. Let I be a set of generators of a left normal bandoid $\underline{B}$. For $x,y\in I$, let $\phi_{xy}:=L(y):xT(B) \longrightarrow yT(B);\ z \mapsto yz$ and $\underline{S}_x:=(xT(B),\cdot,x)$. Then $\underline{B}$ is the sum of the system $\mathcal{S}=\big(I,\{\underline{S}_i\}_{i\in I},\{\phi_{ij}\}_{i,j\in I}\big)$, by Remarks 1.1, 1.2 and Propositions 1.3, 1.8. ∎

2.6. Corollary. A groupoid $(G,\cdot)$ is a left normal bandoid if and only if it is the sum of a semilattice system satisfying conditions 2.3(i) and 2.3(ii). ∎

Now we will consider left normal bandoids for which each orbit is contained in one that is maximal under set theoretical inclusion. Note that such a bandoid $\underline{B}$ is the sum of the system $\big(I,\{\underline{S}_i\}_{i\in I},\{\phi_{ij}\}_{i,j\in I}\big)$, where I is the set of all elements which determine maximal orbits, and for $x,y\in I$, ϕ_{xy} and $\underline{S}_x$ are defined as in proof of Theorem 2.5. For these bandoids a graphical representation is given by A. Romanowska and J. Smith in [12]. Orbits in a bandoid are semilattices, so products of elements lying within a single maximal orbit are given by the Hasse diagram of this orbit. To specify the remaining products, it suffices to indicate the semilattice homomorphisms between maximal orbits. The actions of these homomorphisms are indicated on the labelled Hasse diagram of the disjoint union of maximal orbits by means of arrows. If an element appears in two distinct maximal orbits, then the arrow between the element in the first orbit and the same element in the second orbit is omitted. As examples, the free left normal bandoid on {x,y} is given by

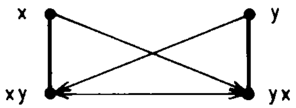

Figure 2.7

and the Hałkowska algebra of the dissemilattice ({a,b,c,d},+,·) with $d \le_{\cdot} c \le_{\cdot} b \le_{\cdot} a$ and $c \le_{+} a \le_{+} b \le_{+} d$ is given by

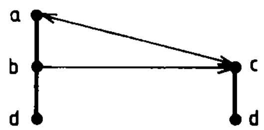

Figure 2.8

In papers of this series we use a modification of this representation. The difference is that, if maximal orbits are not disjoint, then in the modified representation their diagrams are not disjoint either. As example the left normal bandoid pictured in Figure 2.8 is given by

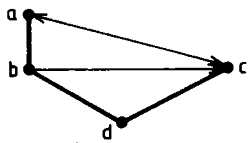

Figure 2.9

3. Constructing left normal bandoids from a complete lattice satisfying a generalized distributive law

In this section we use the general construction described in Section 2 to construct left normal bandoids from principal ideals of a complete lattice $\underline{L}=(L,+,\cdot)$ satisfying the following distributive law:

(3.1) $x\cdot\sum(y_i \mid i\in I) = \sum(x\cdot y_i \mid i\in I)$.

This construction will then play an essential role in building up subdirectly irreducible left normal bandoids.

Let $\underline{L}=(L,+,\cdot)$ be a complete lattice satisfying the distributive law (3.1). The usual partial order of the lattice

$\underline{L}$ will be denoted by $\leq_L$. The greatest and least elements in $(L,\leq_L)$ are denoted by 1 and 0 respectively.

Define a family of disjoint semilattices with units as follows. For (x,y) in $\leq_L$ let

(3.2) $\underline{S}_{xy} \cong ((x],\cdot,x)$,

where $(x]$ denotes the principal ideal of $\underline{L}$ generated by x and $\cdot$ is the meet operation of the lattice $\underline{L}$ restricted to $(x]$. If $v\in(x]$ then the corresponding element of S_{xy} is denoted by v_{xy}. Obviously $\underline{S}_{11}$ is isomorphic to $(L,\cdot,1)$. The elements of S_{11} will be written without indices, i.g., $x:=x_{11}$.

For (x,y), (z,t) in $\leq_L$, the mapping $\phi_{xyzt}:S_{xy}\longrightarrow S_{zt}$ is defined as follows:

(3.3) $\phi_{xyzt}u_{xy}:=(z\cdot\sum(v\in(y] \mid v\cdot x=u))_{zt}$.

3.4. Remark. For every $(x,y)\in\leq_L$

(i) $\phi_{xy11}u_{xy} = \sum(v\in(y] \mid v\cdot x=u)$

and

(ii) $\phi_{11xy}u_{11} = (x\cdot u)_{xy}$. ∎

3.5. Remark. For all $(x,y),(z,t)\in\leq_L$,

$$\phi_{xyzt}u_{xy} = \phi_{11zt}\phi_{xy11}u_{xy}. \qquad ∎$$

3.6. Remark. For every $(x,y)\in\leq_L$

$$\phi_{xy11}x_{xy} = y \qquad ∎$$

We will prove that for every $R\subseteq\leq_L$ the triple

(3.7) $\mathcal{S}(L,R) := \left(R,\{\underline{S}_{xy}\}_{(x,y)\in R},\{\phi_{xyzt}\}_{(x,y),(z,t)\in R}\right)$

is a semilattice system. Note that since all S_{xy} are pairwise disjoint we do not need to check 2.1(iii) and 2.1(iv).

3.8. Lemma. For every $(x,y)\in\leq_L$, the mapping ϕ_{xyxy} is the identity mapping on S_{xy}.

Proof. Let $(x,y)\in\leq_L$ and $z\in(x]$. By (3.3) we have

$$\phi_{xyxy}u_{xy}:=\left(x\cdot\sum(v\in(y] \mid v\cdot x=u)\right)_{xy}.$$

Since $\underline{L}$ satisfies the distributive law (3.1), this implies that

$$\phi_{xyxy}(u_{xy}) = u_{xy}. \qquad ∎$$

It remains to show that the mappings ϕ_{xyzt} are semilattice homomorphisms. First we prove the following lemma.

3.9. Lemma. For every $(x,y)\in\leq_L$ and $u,w\in(x]$,

$$\sum(v\in(y] \mid v\cdot x=u\cdot w) = (\sum(v\in(y] \mid v\cdot x=u))\cdot(\sum(v\in(y] \mid v\cdot x=w)).$$

Proof. Let us define

$$a := \sum(v\in(y] \mid v\cdot x=u\cdot w),$$
$$b := \sum(v\in(y] \mid v\cdot x=u),$$
$$c := \sum(v\in(y] \mid v\cdot x=w).$$

By distributivity (3.1) of $\underline{L}$ it follows that $a\cdot x=u\cdot w$, $b\cdot x=u$ and $c\cdot x=w$. Therefore $(b\cdot c)\cdot x=u\cdot w$, and since $b\cdot c\in(y]$, it follows that

(3.9.1) $\quad b\cdot c\leq_L a.$

Since $a+b\in(y]$ and $(a+b)\cdot x = a\cdot x+b\cdot x = u\cdot w+u = u$, we have that $a+b\leq_L b$, whence $a+b=b$. Consequently

(3.9.2) $\quad a\leq_L b.$

Analogously we prove that

(3.9.3) $\quad a\leq_L c.$

Now, (3.9.1), (3.9.2) and (3.9.3) imply $a=b\cdot c$, which completes the proof. ∎

3.10. Lemma. For all $(x,y),(z,t)\in\leq_L$, the mappings ϕ_{xyzt} are semilattice homomorphisms.

Proof. Let $(x,y),(z,t)\in\leq_L$ and $u,w\in(x]$. By definition (3.3)

$$\phi_{xyzt}(u\cdot w)_{xy} = (z\cdot\sum(v\in(y] \mid v\cdot x=u\cdot w))_{zt}.$$

Using Lemma 3.9 we obtain

$$\phi_{xyzt}(u\cdot w)_{xy} = (z\cdot(\sum(v\in(y] \mid v\cdot x=u))\cdot(\sum(v\in(y] \mid v\cdot x=w))_{zt}.$$

Since $\underline{L}$ is a lattice it follows that

$$\phi_{xyzt}(u\cdot w)_{xy}= ((z\cdot(\sum(v\in(y]\mid v\cdot x=u)))\cdot(z\cdot(\sum(v\in(y]\mid v\cdot x=w))))_{zt}.$$

Therefore, by (3.3), $\phi_{xyzt}(u\cdot w)_{xy} = \phi_{xyzt}u_{xy}\cdot\phi_{xyzt}w_{xy}$. ∎

3.11. Corollary. Let $R\subseteq\leq_L$. The triple $\mathcal{S}(L,R)$ defined by (3.7) is a semilattice system.

Proof. Lemma 3.8, Lemma 3.10. ∎

The sum of the system $\mathcal{S}(L,R)$ turns out to be a bandoid. To prove this we need the following lemma.

3.12. Lemma. Let $x,y,z\in L$ with $x\leq_L y$. Then

$$z\cdot\sum(v\in(y] \mid v\cdot x=z\cdot x) = z\cdot y.$$

Proof. Obviously $\sum(v\in(y] \mid v\cdot x=z\cdot x)\leq_L y$. Hence

$$z\cdot\sum(v\in(y] \mid v\cdot x=z\cdot x)\leq_L z\cdot y.$$

On the other hand, since $z\cdot y\in(y]$ and $(z\cdot y)\cdot x=z\cdot x$, we have

$$z\cdot\sum(v\in(y] \mid v\cdot x=z\cdot x)\geq_L z\cdot y.$$ ∎

3.13. Theorem. Let $R\subseteq\leq_L$. The sum of the system $\mathscr{S}(L,R)$ defined by (3.7) is a left normal bandoid.

Proof. In view of Theorem 2.3, it suffices to prove that the conditions (i) and (ii) of this theorem are satisfied in $\mathscr{S}(L,R)$. First we show that (i) holds.

Let $(u,w),(t,z)\in R$ and $x\in(u]$. Since $\phi_{uwtz}x_{uw}\in S_{tz}$ and t_{tz} is the unit in $\underline{S}_{tz}$, it follows that $\phi_{uwtz}x_{uw}\leq t_{tz}$ in the semilattice $(S_{tz},\leq)$. Hence, by Lemma 3.10, $\phi_{tzuw}\phi_{uwtz}x_{uw}\leq$ $\leq\phi_{tzuw}t_{tz}$ in $(S_{uw},\leq)$. Thus

(3.13.1) $\qquad x_{uw}\cdot\phi_{tzuw}\phi_{uwtz}x_{uw} \leq x_{uw}\cdot\phi_{tzuw}t_{tz}$ in $(S_{uw},\leq)$.

On the other hand

$x_{uw}\cdot\phi_{tzuw}\phi_{uwtz}x_{uw} = x_{uw}\cdot\big(u\cdot\sum\big(v\in(z]\mid v\cdot t=t\cdot\sum(s\in(w]\mid s\cdot u=x)\big)\big)_{uw}$

$\geq x_{uw}\cdot\big(u\cdot z\cdot\sum(s\in(w] \mid s\cdot u=x)\big)_{uw}$

since $z\cdot\sum(s\in(w] \mid s\cdot u=x)\in(z]$ and

$$\begin{aligned}(z\cdot\sum(s\in(w] \mid s\cdot u=x))\cdot t &= (t\cdot z)\cdot\sum(s\in(w] \mid s\cdot u=x)\\ &= t\cdot\sum(s\in(w] \mid s\cdot u=x).\end{aligned}$$

Now $x_{uw}\cdot\big(u\cdot z\cdot\sum(s\in(w] \mid s\cdot u=x)\big)_{uw}$

$= x_{uw}\cdot(z\cdot x)_{uw}$ since by (3.1) $u\cdot\sum(s\in(w] \mid s\cdot u=x)=x$

$= (x\cdot(z\cdot x))_{uw}$ by (3.2)

$= (x\cdot z)_{uw}$

$= (x\cdot u\cdot z)_{uw}$ since $x\in(u]$

$= x_{uw}\cdot(u\cdot z)_{uw}$ by (3.2)

$= x_{uw}\cdot\phi_{tzuw}t_{tz}$ since $\phi_{tzuw}t_{tz}= (u\cdot\sum(v\in(z]\mid v\cdot t=t))_{uw}=(u\cdot z)_{uw}$.

So we have

(3.13.2) $\quad x_{uw}\cdot\phi_{tzuw}\phi_{uwtz}x_{uw} \geq x_{uw}\cdot\phi_{tzuw}t_{tz}$ in $(S_{uw},\leq)$.

As a consequence of (3.13.1) and (3.13.2) we get

$$x_{uw}\cdot\phi_{tzuw}\phi_{uwtz}x_{uw} = x_{uw}\cdot\phi_{tzuw}t_{tz} \text{ in } (S_{uw},\leq).$$

This means that the condition (i) of Theorem 2.3 is satisfied in $\mathcal{S}(L,R)$.

Now we prove the condition (ii). Let $(x,z),(y,t),(u,w)\in R$ and $s\in(u]$. Then we get the following:

$$\phi_{ytxz}\phi_{xzyt}\phi_{uwxz}s_{uw} = \phi_{ytxz}\phi_{xzyt}\big(x\cdot\textstyle\sum(v\in(w] \mid v\cdot u=s)\big)_{xz}$$

$= \phi_{ytxz}\phi_{xzyt}(x\cdot r)_{xz}$ where $r:=\sum(v\in(w] \mid v\cdot u=s)$

$= \phi_{ytxz}(y_{yt}\cdot\phi_{xzyt}(x\cdot r)_{xz})$ since y_{yt} is the unit in $\underline{S}_{yt}$

$= \phi_{ytxz}(y_{yt})\cdot\phi_{ytxz}\phi_{xzyt}(x\cdot r)_{xz})$ by Lemma 3.10

$=\big(x\cdot\sum(v\in(t] \mid v\cdot y=y)\big)_{xz}\cdot\big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r))\big)_{xz}$

$=\big(x\cdot\sum(v\in(t] \mid v\cdot y=y)\big)_{xz}\cdot\sum(v\in(t] \mid v\cdot y=y\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r))\big)_{xz}$

$= \big(x\cdot t\cdot\sum(v\in(t] \mid v\cdot y=y)\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r))\big)_{xz}$ since $y\cdot t=y$

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=x\cdot y)\cdot\sum(v\in(t] \mid v\cdot y=y\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r))\big)_{xz}$

since by Lemma 3.12, $x\cdot\sum(v\in(t] \mid v\cdot y=x\cdot y) = x\cdot t$

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=x\cdot y\cdot y\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r))\big)_{xz}$ by Lemma 3.9

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot x\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r))\big)_{xz}$

$=\big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot x\cdot r)\big)_{xz}$

since by distributivity (3.1), $x\cdot\sum(v\in(z] \mid v\cdot x=x\cdot r) = x\cdot r$

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot x\cdot y\cdot r)\big)_{xz}$

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot x)\cdot\sum(v\in(t] \mid v\cdot y=y\cdot r)\big)_{xz}$ by Lemma 3.9

$= \big(x\cdot t\cdot\sum(v\in(t] \mid v\cdot y=y\cdot r)\big)_{xz}$ since,

by Lemma 3.12, $x\cdot\sum(v\in(t] \mid v\cdot y=x\cdot y) = x\cdot t$

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot r)\big)_{xz}$ since $\sum(v\in(t] \mid v\cdot y=y\cdot r) \leq_L t$

$= \big(x\cdot\sum(v\in(t] \mid v\cdot y=y\cdot\sum(v\in(w] \mid v\cdot u=s))\big)_{xz}$ by definition of r

$= \phi_{ytxz}\phi_{uwyt}s_{uw}$ by definition (3.3).

By Theorem 2.3 this completes the proof. ∎

A left normal bandoid that is the sum of a semilattice system $\mathcal{S}(L,R)$ will be denoted by $\underline{B}(L,R)$. The set of elements of this bandoid will be denoted by B(L,R).

3.14 Example. Let $\underline{L}$ be the lattice pictured below

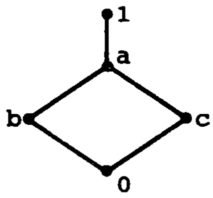

Figure 3.15.

and let $R = \{(1,1),(b,1),(c,1)\}$. The bandoid $\underline{B}(L,R)$ is presented in the picture below:

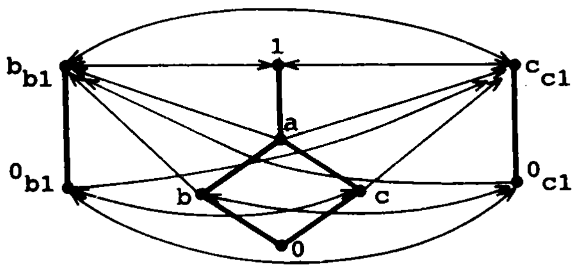

Figure 3.16. ∎

3.17. Example. Let $\underline{L}$ be the four element chain $\{1,a,b,0\}$ with $0\le_L b\le_L a\le_L 1$ and $R=\{(1,1),(a,1),(b,a)\}$. The bandoid $\underline{B}(L,R)$ is pictured below:

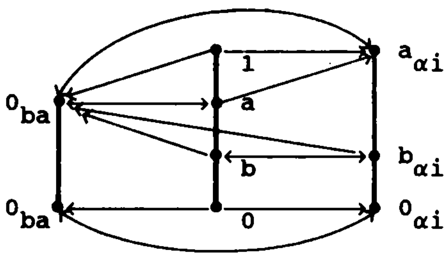

Figure 3.18.

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INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY OF KIELCE, 25-406 KIELCE, POLAND.

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