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CONNECTED PARTITIONS AND CYCLE MATROIDS OF GRAPHS

Introduction

In [5, p.57] it was shown that:

(w) The partially ordered set of all partitions of an n-element set is a geometric lattice which is isomorphic to the lattice of closed sets of the cycle matroid of the complete graph K_n.

This is not true if we consider an arbitrary graph G, since in general we have more partitions of V(G) than closed sets of the cycle matroid of G. However, we can ask which interesting partitions of the vertex set of an arbitrary connected simple graph should be chosen to obtain from them the lattice isomorphic to the lattice of closed sets of the cycle matroid of G. In this paper we answer this question.

We say that a partition P of V(G) is connected if every class of P induces a connected subgraph of G. We denote by $\mathcal{P}_C(G)$ the set of all connected partitions of V(G).

Let M(G) denote the cycle matroid of G and let $(L_\sigma(G);\leq)$ be the lattice of closed sets in M(G). In Section 1 we prove that the poset $(\mathcal{P}_C(G);\leq)$ is isomorphic to the geometric lattice $(L_\sigma(G);\leq)$. This result is a generalization of (w) since in a complete graph K_n every partition of $V(K_n)$ is connected. Let us also note that we do not assume G to be finite.

In Section 2 we study the problem how the lattice $(\mathcal{P}_C(G);\leq)$ is situated in the lattice $(\mathcal{P}(G);\leq)$, where $\mathcal{P}(G)$ denotes the set of all partitions of V(G).

Another class of special partitions of a connected graph was described in [2] and [3].

1. A representation theorem for the poset of connected partitions of a graph

We shall consider only simple, connected graphs. Let $G=(V(G);E(G))$ be a graph. An arbitrary partition of the vertex set $V(G)$ will be called a partition of G. A partition P of G will be called connected if for every class $A\in P$ the subgraph $\langle A\rangle$ induced by A in G is connected.

We denote by $\mathcal{P}(G)$ the set of all partitions of G and by $\mathcal{P}_c(G)$ the set of all connected partitions of G. For $P_1,P_2\in\mathcal{P}(G)$ we write as usual $P_1\le P_2$ if for every $A\in P_1$ there exists $B\in P_2$ such that $A\subseteq B$. It is known that $\mathcal{P}(G)$ with relation $\le$ is a complete lattice. Obviously, the set $\mathcal{P}_c(G)$ with respect to $\le$ is a poset.

Let G be a connected graph. Consider an operator $\sigma_G: 2^{E(G)}\longrightarrow 2^{E(G)}$ defined as follows:

For $X\subseteq E(G)$, let $\sigma_G(X)=X\cup X^D$, where X^D is the set of all edges $e\in E(G)$ such that there exists a simple cycle $C=(e_1,e_2,\ldots,e_n)$ in G with $e=e_n$, $E(C)\cap X=\{e_1,\ldots,e_{n-1}\}$.

It is known that σ_G is a closure operator having two properties:

(EP) If $x,y\in E(G)$, $y\notin\sigma_G(X)$, $y\in\sigma_G(X\cup\{x\})$, then $x\in\sigma_G(X\cup\{y\})$ (the exchange property),

(FP) For $X\subseteq E(G)$ and $x\in E(G)$ it follows that $x\in\sigma_G(X)$ implies $x\in\sigma_G(F)$ for some finite subset F of X (the finite character property).

Thus the pair $M(G)=(E(G);\sigma_G)$ is a matroid, called the cycle matroid induced by G (see[5]).

We shall write σ instead of σ_G if there is no danger of confusion.

We say that a set $X\subseteq E(G)$ is closed if $\sigma(X)=X$.

It is known that the set $L_\sigma(G)$ of all closed sets in M(G) is a complete lattice with respect to inclusion (see [5], Chapter 20).

Let P be a partition of a graph G and $A\in P$. We put:

$$E(A) = \{x\in E(G):\ x\in E(\langle A\rangle)\},$$

$$E(P) = \bigcup_{A\in P} E(A).$$

Lemma 1. For every $P\in\mathcal{P}_c(G)$, the set $E(P)$ is closed in $M(G)$.

Proof. Let $e\in\sigma(E(P))$. We have to show that if $e\in(E(P))^D$, then $e\in E(P)$. Let $e\in(E(P))^D$. Then there exists a simple cycle $C = (e_1,e_2,\ldots,e_n)$, such that $e=e_n$ and $e_1,\ldots,e_{n-1}\in E(P)$. Since $e_1,\ldots,e_{n-1}$ form a path in G, so there exists a class $A\in P$ such that $e_1,\ldots,e_{n-1}\in E(A)$. But $\langle A\rangle$ is an induced subgraph of G, consequently $e\in E(A)$. Thus $\sigma(E(P))=E(P)$.

Lemma 2. If $X\in L_\sigma(G)$, then there exists $P\in\mathcal{P}_c(G)$ such that $E(P)=X$.

Proof. Let X be closed in $M(G)$. We define in $V(G)$ a relation $R(X)$ putting for $u,v\in V(G)$:

$$(1)\quad \begin{cases} uR(X)v \text{ if } u=v \text{ or there exists a simple chain } u_0,u_1,\ldots,u_n \\ \text{such that } u_0=u,\ u_n=v \text{ and all edges } \{u_i,u_{i+1}\} \text{ for} \\ i=0,1,\ldots,n-1 \text{ belong to } X. \end{cases}$$

Obviously, $R(X)$ is an equivalence relation, so it induces a connected partition $P(X)$ in G. We show that $E(P(X))=X$. Evidently, $X\subseteq E(P(X))$. Let $\{u,v\}\in E(P(X))$. So there exists $A\in P(X)$ such that $u,v\in A$. Hence there exists a simple chain of the form (1). If $n=1$, then $\{u,v\}\in X$ since there are no parallel edges in G. Otherwise, we get a simple cycle $u_0,u_1,\ldots$ $\ldots,u_n$ such that $\{u_i,u_{i+1}\}\in X$ for $i=0,1,\ldots,n-1$. Since X is closed, so $\{u_0,u_n\}\in X$.

Lemma 3. For every partition $P\in\mathcal{P}(G)$, there exists the unique partition P^c such that $P^c\in\mathcal{P}_c(G)$ and $E(P^c)=E(P)$.

Proof. Let $P=\{A_i\}_{i\in I}$ be a partition of the graph G. Let P^c consists of all $A\subseteq V(G)$ for which there exists $i\in I$ such that $\langle A\rangle$ is a component of $\langle A_i\rangle$. Obviously, P^c is connected, $P^c\le P$ and $E(P^c)=E(P)$.

Let $P'\in\mathcal{P}_c(G)$, $E(P')=E(P^c)$ and $P'\ne P^c$. Then there exists $v\in V(G)$ such that $[v]_{P'} = [v]_{P^c}$. Put $A=[v]_{P^c}$, $A'=[v]_{P'}$ and let $w\in A\setminus A'$. If there is $w\in A'\setminus A$, then the proof is analogous.

Since the graph <A> is connected, so there exists a simple chain $w,\dots,w_0,v_0,\dots,v$ such that $w_0\in A\setminus A'$, $v_0\in A\cap A'$ and $\{w_0,v_0\}\in E(G)$. Consequently, $\{w_0,v_0\}\in E(P^c)\setminus E(P')$, a contradiction. Thus we have $P'=P^c$.

Lemma 4. If $P_1,P_2\in\mathcal{P}_c(G)$, then $P_1\le P_2$ iff $E(P_1)\subseteq E(P_2)$.

Proof. If $P_1\le P_2$ and $e\in E(P_1)$, then obviously $e\in E(P_2)$. Let $E(P_1)\subseteq E(P_2)$, $u\in[v]_{P_1}$ and $u\ne v$. It is enough to show that $u\in[v]_{P_2}$. Since P_1 is connected, so there exists a chain $u=u_1$, $e_1,u_2,\dots,u_{n-1},e_{n-1},u_n=v$ in G such that $u_1,u_2,\dots,u_n\in[v]_{P_1}$. Hence $e_1,\dots,e_{n-1}\in E(P_1)$ and $e_1,\dots,e_{n-1}\in E(P_2)$. From the definition of E(P) it follows that $\{u_i,u_j\}\in E(P_2)$ implies $u_i\in[u_j]_{P_2}$. Thus $u_1,u_2,\dots,u_n\in[v]_{P_2}$.

Next theorem follows by Lemmas 1-4.

Theorem 1. If G is a connected simple graph, then the mapping $\varphi:\mathcal{P}_c(G)\longrightarrow L_\sigma(G)$ defined by $\varphi(P)=E(P)$ is an isomorphism of the poset $(\mathcal{P}_c(G);\le)$ and the lattice $(L_\sigma(G);\subseteq)$.

Corollary 1. The poset $(\mathcal{P}_c(G);\le)$ is a geometric lattice.

Corollary 2. Let G_1 and G_2 be two connected simple graphs. Then $(\mathcal{P}_c(G);\le)$ is isomorphic to $(\mathcal{P}_c(G_2);\le)$ iff $M(G_1)$ is isomorphic to $M(G_2)$.

Let us observe that the notion of a connected partition can be extended to an arbitrary (not necessarily connected and simple) graph. Then we have

Corollary 3. For every graph G the lattice $(\mathcal{P}_c(G);\le)$ is isomorphic to the lattice $(L_\sigma(G^*);\subseteq)$, where G^* is a simple graph obtained from G by canceling loops and substituting every set of parallel edges by one of them.

2. Properties of the lattice of connected partitions of a graph

For $P_1,P_2\in\mathcal{P}(G)$ we denote by $P_1\vee P_2$ the least upper bound (join) and by $P_1\wedge P_2$ the greatest lower bound (meet) of P_1 and P_2. If $B\subseteq V(G)$, then $\bigvee B$ and $\bigwedge B$ denote the least upper bound and the greatest lower bound of elements of B. Finally, we denote by P(G) the lattice $(\mathcal{P}(G);\vee,\wedge)$.

Lemma 5. Let $\{P_i\}_{i\in I}$ be a family of connected partitions of G and let $u\in[v]_{\bigvee_{i\in I} P_i}$ for some $u,v\in V(G)$. Then there exists a sequence $P_{k_1},\dots,P_{k_m}$ of partitions in $\{P_i\}_{i\in I}$ and there exists a chain $v_0,e_1,v_1,\dots,e_n,v_n$ in G such that $n\geq 0$, $v_0=v$, $v_n=u$ and $v_0,v_1,\dots,v_n$ belong to the same class of the partition $P_{k_1}\vee\dots\vee P_{k_m}$.

Proof. If $u=v$, then the proof is trivial. Let $u\neq v$. It is known (see [1]) that $u\in[v]_{\bigvee_{i\in I} P_i}$ implies that there exist $k_1,\dots,k_m\in I$, $w_0,w_1,\dots,w_m\in V(G)$, $(m>0)$, such that $w_0=v$, $w_m=u$, $w_{s+1}\in[w_s]_{P_{k_{s+1}}}$ for $s=0,1,\dots,m-1$. However, $P_{k_{s+1}}$ is connected, so there exists a chain $x_s^0,e_s^1,x_s^1,\dots,e_s^{n(s)},x_s^{n(s)}$ such that $x_s^0=w_s$, $x_s^{n(s)}=w_{s+1}$ and $x_s^0,x_s^1,\dots,x_s^{n(s)}\in[w_s]_{P_{k_{s+1}}}$. So the chain

$$v=x_0^0,e_0^1,x_0^1,\dots,e_0^{n(0)},x_0^{n(0)} = x_1^0,e_1^1,\dots,x_1^{n(1)} =$$
$$= x_2^0,\dots,e_{m-1}^{n(m-1)},x_{m-1}^{n(m-1)} = u$$

satisfies the requirements of the lemma for $n=n(0)+\dots+n(m-1)$. In fact, $u\in[v]_P$, where $P=P_{k_1}\vee\dots\vee P_{k_m}$. Further, $x_s^j\in[v]_{P^s}$, where $P^s = P_{k_1}\vee\dots\vee P_{k_{s+1}}$, $[v]_{P^s}\subseteq[v]_P$, since $P^s\leq P$.

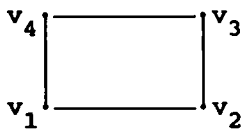
From Lemma 5 next Corollary and Theorem follow.

Corollary 4. If $\{P_i\}_{i\in I}$ is a family of connected partitions of G, then $\bigvee_{i \in I} P_i \in \mathcal{P}_c(G)$.

Theorem 2. The set $\mathcal{P}_c(G)$ is a join subsemilattice of the semilattice $(\mathcal{P}(G);\vee)$.

Let $\vee_c$ and $\wedge_c$ be the join and the meet operations in $\mathcal{P}_c(G)$. Denote by $\mathbb{P}_c(G)$ the lattice $(\mathcal{P}_c(G);\vee_c,\wedge_c)$. Observe that the least element of $\mathbb{P}_c(G)$ and the greatest element of $\mathbb{P}_c(G)$ coincide with the least and the greatest elements of $\mathbb{P}(G)$ respectively. Moreover, an atom of $\mathbb{P}_c(G)$ is an atom of $\mathbb{P}(G)$, since it must be of the form $\{\{v_1,v_2\}\}\cup\{\{v : v\in V(G)\setminus \setminus\{v_1,v_2\}\}$, where $v_1,v_2\in V(G)$, $v_1\neq v_2$ and $\{v_1,v_2\}\in E(G)$.

Remark 1. The meet $P_1\wedge P_2$ of two connected partitions P_1 and P_2 of G need not be a connected partition of G. For example see P_1 and P_2 in Figure 1.



$P_1 = \{\{v_1\},\{v_2,v_3,v_4\}\}$,

$P_2 = \{\{v_1,v_2,v_4\},\{v_3\}\}$.

Figure 1.

Remark 2. If $P_1,P_2\in\mathcal{P}_c(G)$, then $P_1\wedge_c P_2$ is a partition whose classes are connected components induced in V(G) by the edges from $E(P_1)\cap E(P_2)$.

Lemma 6. For every $P_1,P_2\in\mathcal{P}(G)$ we have:

1^o $E(P_1)\cap E(P_2) = E(P_1\cap P_2)$,

2^o $E(P_1)\cup E(P_2) \subseteq E(P_1\vee P_2)$.

We omit the simple proof.

Remark 3. Note that the inclusion $E(P_1\vee P_2)\subseteq E(P_1)\cup E(P_2)$ does not need to hold. For example consider partitions P_1 and $P_3=\{\{v_1,v_3,v_4\},\{v_2\}\}$ of a graph in Figure 1.

Recall that for $P\in\mathcal{P}(G)$, we denoted by P^c the unique connected partition such that $E(P^c)=E(P)$ (see Lemma 3).

From Lemmas 3 and 6 we have

Theorem 3. The mapping $\psi : \mathcal{P}(G) \longrightarrow \mathcal{P}_c(G)$, where $\psi(P)=P^c$, is an idempotent endomorphism of $(\mathcal{P}(G);\wedge)$ onto $(\mathcal{P}_c(G);\wedge_c)$.

Theorem 4. If $P_1,P_2\in\mathcal{P}_c(G)$, then P_2 covers P_1 in $\mathbb{P}_c(G)$ iff P_2 covers P_1 in $\mathbb{P}(G)$.

Proof. The sufficiency is obvious. We prove the necessity. Let P_2 covers P_1 in $\mathbb{P}_c(G)$. So there exist classes of P_2 which are unions of more then one class of P_1. If there exist two such classes A_1 and A_2, then consider a partition $P\in\mathcal{P}_c(G)$ consisting of A_1 and all classes from P_1 which are not included in A_1. Then all P_1, P, P_2 are different and $P_1<P<P_2$, contrary to the assumption. So only one class of P_2, say A_1, is a join of more one class of P_1. Let $A_1 = \bigcup_{i\in I} B_i$, where $B_i\in P_1$ and all B_i are different. Assume that $|I|>2$. Since P_2 is connected, so $\langle A_1\rangle$ is a connected subgraph of G. Hence there exist $i_0,j_0\in I$, $i_0\neq j_0$ such that $\langle B_{i_0}\cup B_{j_0}\rangle$ is connected. Consider a partition P' consisting of $B_{i_0}\cup B_{j_0}$ and all classes of P_1 different from those. Then $P'\in\mathcal{P}_c(G)$ and $P_1<P'<P_2$, a contradiction. Thus $|I|=2$ and P_2 covers P_1 in $\mathbb{P}(G)$.

Remark 4. It is known that the lattice $\mathbb{P}(G)$ is simple (see [4]), i.e. it has only trivial congruence relations. Note that $\mathbb{P}_c(G)$ does not need to be simple, for example if T_n is a tree with n vertices (n>2), then the lattice $\mathbb{P}_c(T_n)$ is a Boolean Algebra having 2^{n-1} elements.

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