DEMONSTRATIO MATHEMATICA
Vol. XXIV No 1-2 1991

Jerzy Płonka

THE ALGEBRAIC SUM OF A SEMILATTICE ORDERED SYSTEM OF RELATIONAL SYSTEM

Dedicated to J. Mycielski

0. Preliminaries

By a type of relational systems we mean a mapping $\tau : R \longrightarrow N$, where R is a set of relational symbols and N is the set of positive integers. By a relational system of type τ we mean a pair $\mathfrak{A} = (A;R^{\mathfrak{A}})$, where A is a non-empty set and $R^{\mathfrak{A}}$ is the set of relations on A being realizations of symbols from R i.e. for each $r^{\mathfrak{A}} \in R^{\mathfrak{A}}$ the arity of $r^{\mathfrak{A}}$ is equal to $\tau(r)$. If $\langle a_1,\ldots,a_{\tau(r)}\rangle \in r^{\mathfrak{A}}$, we shall write simply $r^{\mathfrak{A}}(a_1,\ldots$ $\ldots,a_{\tau(r)})$. A relation $r^{\mathfrak{A}}$ is a function on variables $x_1,\ldots$ $\ldots,x_{\tau(r)-1}$ if for every $a_1,\ldots,a_{\tau(r)-1} \in A$ there exists the unique $a \in A$ such that $r^{\mathfrak{A}}(a_1,\ldots,a_{\tau(r)-1},a)$. In this case we shall say shortly that $r^{\mathfrak{A}}$ is a function.

Let $\mathfrak{A} = (A;R^{\mathfrak{A}})$ and $\mathfrak{B} = (B;R^{\mathfrak{B}})$ be two relational systems of type τ. A mapping $h : A \longrightarrow B$ is a homomorphism if for every $r \in R$, $a_1,\ldots,a_{\tau(r)} \in A$, $r^{\mathfrak{A}}(a_1,\ldots,a_{\tau(r)})$ implies $r^{\mathfrak{B}}(h(a_1),\ldots,h(a_{\tau(r)}))$.

In [2] and [3] a construction of algebras was defined, called the sum of a direct system of algebras. This construction was based on the notion of a semilattice. In [4] a similar construction was defined for relational systems.

In this paper we define a uniform construction, called the algebraic sum of a semilattice ordered system of relational systems, where we do not distinguish between functions and relations. The algebraic sum is a generalization of that from [2] and [3], and is connected with that from [4] (see Remark 1 and (v)).

The problem of finding such a generalization was suggested to me by J. Mycielski.

1. Algebraic sum

By a semilattice ordered system of relational systems of type τ we mean a triple

$$\mathcal{A} = \left((I;\le),\ \{\mathfrak{A}_i\}_{i\in I}, \{h_i^j\}_{i,j\in I;\ i\le j}\right)$$

satisfying the following three conditions:

(i) $(I;\le)$ is a join-semilattice, i.e. for every $i,j\in I$ there exists their supremum $i\vee j$. If there are unary symbols in R then $(I;\le)$ has the least element ω;

(ii) every $\mathfrak{A}_i$ is a relational system of type τ, where $\mathfrak{A}_i = (A_i;R^i)$ and for every $i,j\in I$, if $i\ne j$ then $A_i\cap A_j = \emptyset$;

(iii) for each $i\le j$, h_i^j is a homomorphism of $\mathfrak{A}_i$ into $\mathfrak{A}_j$, where h_i^i is the identity map and if $i\le j\le k$ then $h_j^k\circ h_i^j = h_i^k$.

We define a new relational system $S(\mathfrak{A}) = \left(\bigcup_{i\in I} A_i;R^S\right)$ of type τ, where for every $r\in R$, $a_1\in A_{i_1},\dots,a_{\tau(r)}\in A_{i_{\tau(r)}}$ and $k = \sup\{i_1,\dots,i_{\tau(r)-1}\}$ we have

$$r^S(a_1,\dots,a_{\tau(r)}) \ \text{ iff } \ i_{\tau(r)} = k \ \text{ and}$$
$$r^k\left(h_{i_1}^k(a_1),\dots,h_{i_{\tau(r)-1}}^k(a_{\tau(r)-1}),a_{\tau(r)}\right).$$

The relational system $S(\mathcal{A})$ will be called the algebraic sum of $\mathcal{A}$. We have the following propositions (iv) and (v).

(iv) If $\tau(r)=1$ for some $r\in R$ then $r^S=r^\omega$. If $\tau(r)=2$

for some $r\in R$ then $r^S=\bigcup_{i\in I} r^i$.

Remark 1. In [4] a sum of a direct system of relational systems was defined, where for a given semilattice ordered system $\mathcal{A}$ of relational systems there was the following definition concerning $S(\mathcal{A})$. If $r\in R$, $a_j\in A_{i_j}$, $j=1,\dots,\tau(r)$, $q = \sup\{i_1,\dots,i_{\tau(r)}\}$ then

(1.1) $\quad r^S(a_1,\dots,a_{\tau(r)})$ iff $r^q(h^q_{i_1}(a_1),\dots,h^q_{i_{\tau(r)}}(a_{\tau(r)}))$.

So the construction in [4] does not distinguish the last variable and therefore it is not a particular case of the algebraic sum. The both constructions coincide if we assume that $1\notin\tau(R)$ and $i_{\tau(r)}=q$ in (1.1).

(v) For every $r\in R$, r^S is a function in $S(\mathcal{A})$ iff for each $i\in I$, r^i is a function in A_i. If $\mathcal{A}$ is a semilattice ordered system of algebras then $S(\mathcal{A})$ is an algebra and the algebraic sum $S(\mathcal{A})$ is equivalent to that defined in [2] and [3].

2. Partition function

Let $\mathfrak{A} = (A;R^{\mathfrak{A}})$ be a relational system of type τ. A binary function $\circ : A\times A \longrightarrow A$ will be called a partition function of $\mathfrak{A}$ if it satisfies the following formulas:

(2.1) $a\circ a = a$, $(a\circ b)\circ c = a\circ(b\circ c)$, $a\circ b\circ c = a\circ c\circ b$ for every $a,b,c\in A$;

(2.2) if $\tau(r) = n>1$ for $r\in R$, $a_1,\dots,a_n\in A$ and $z\in A$, then $r^{\mathfrak{A}}(a_1,\dots,a_n)$ implies $r^{\mathfrak{A}}(a_1\circ z,\dots,a_n\circ z)$;

(2.3) for $r\in R$, $\tau(r) = n$, $a_1,\dots,a_n\in A$, we have $r^{\mathfrak{A}}(a_1,\dots,a_n)$ iff $a_n\circ a_i = a_n$ for $i=1,\dots,n$, $a_1\circ\dots\circ a_n = a_1\circ\dots\circ a_{n-1}$ and $r^{\mathfrak{A}}(a_1\circ a_n,\dots,a_{n-1}\circ a_n,a_n)$;

(2.4) if $\tau(r)=1$ for $r\in R$, $a_1\in A$ then $r^{\mathfrak{A}}(a_1)$ implies $z\circ a_1=z$ for every $z\in A$.

Remark 2. If $\mathfrak{A}$ is an algebra, then this notion of a partition function coincides with that defined in [3].

Theorem 1. To every partition function $\circ$ of a relational system $\mathfrak{A} = (A;R^{\mathfrak{A}})$ of type τ there corresponds a representation of $\mathfrak{A}$ in the form $\mathfrak{A} = S(\mathcal{A})$, where $\mathcal{A}$ is constructed as follows:

(c_1) Define in A a relation ~ putting for $a,b\in A$, $a\sim b$ iff $a\circ b=a$ and $b\circ a=b$. It turns out that ~ is an equivalence on A. Let $\{A_i\}_{i\in I}$ be the set of equivalence classes of ~.

(c_2) Define in I a relation $\le$ putting for $i,j\in I$, $i\le j$ iff for some $a\in A_i$, $b\in A_j$, $b\circ a=b$. It turns out that $(I;\le)$ is a join-semilattice. If $\tau(r)=1$ for some $r\in R$, then denote by ω the element of I such that $A_\omega\cap r^{\mathfrak{A}}\neq\emptyset$.

(c_3) For every $r\in R$, with $\tau(r)>1$ put $r^i=r^{\mathfrak{A}}|A_i$ for $i\in I$. If $\tau(r)=1$ put $r^i=\{x\circ b : x\in r^{\mathfrak{A}}$ and b is a fixed element of $A_i\}$. It turns out that for each $r\in R$ with $\tau(r)=1$, $r^i\subseteq A_i$; in particular, $r^\omega=r^{\mathfrak{A}}$. Put $R^i = \{r^i : r\in R\}$ and $\mathfrak{A}_i = (A_i;R^i)$. Then every $\mathfrak{A}_i$ is a relational subsystem of $\mathfrak{A}$.

(c_4) If $i\le j$ put for every $x\in A_i$ and fixed $b\in A_j$, $h_i^j(x)=x\circ b$. It turns out that (iii) is satisfied.

(c_5) It turns out that $\mathfrak{A}=S(\mathcal{A})$ for the semilattice ordered system $\mathcal{A}$ of relational systems constructed in this way.

(c_6) Conversely, if $\mathfrak{A} = S(\mathcal{A})$ for some semilattice ordered system $\mathcal{A}$ of relational systems, then put for $a\in A_i$, $b\in A_j$, $a\circ b = h_i^q(a)$ where $q=i\vee j$. Then $\circ$ satisfies formulas (2.1)-(2.4).

(c_7) The correspondence between partition functions of $\mathfrak{A}$ and representations of $\mathfrak{A}$ in the form $S(\mathcal{A})$ is one-to-one.

The proof is very similar to that in [3], so we do not present it here.

3. Example

Let τ_0: $\{r\} \longrightarrow N$ be a type of relational systems, where

$\tau_0(r)=3$. Denote by $\mathbb{R}$ the set of all reals and let $a\in\mathbb{R}$, $a>0$. Put $T = \{<x,y>\in\mathbb{R}\times\mathbb{R} : 0\le y<a,\ |x|+y<a\}$ (see Figure 1).

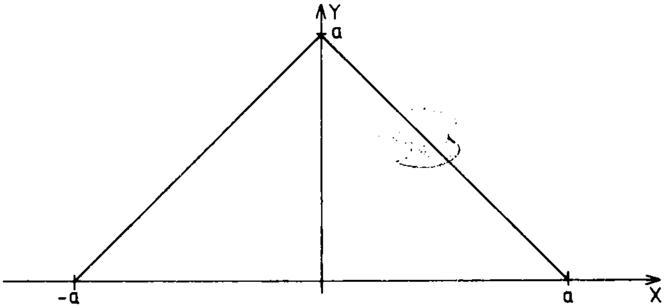

Figure 1.

For every b with $0\le b<a$ put $A_b=\{<x,b> : <x,b>\in T\}$. We define in A_b a ternary relation r^b by

$$r^b(<x_1,b>,<x_2,b>,<x_3,b>) \quad \text{iff} \quad x_1\le x_3\le x_2.$$

So r^b is the usual relation "to lie between" on an open interval. Let $\mathfrak{A}_b = (A_b;r^b)$. Denote $B = \{b : 0\le b<a\}$. We define a semilattice ordered system $\mathcal{A}$ of relational systems putting $\mathcal{A} = \big((B;\le),\ \{\mathfrak{A}_b\}_{b\in B},\ \{h_b^c\}_{b,c\in B;b\le c}\big)$, where $\le$ is the usual order on reals in B and for $b,c\in B$, $b\le c$ and $<x,b>\in A_b$ we put $h_b^c(<x,b>) = <\frac{c'}{b'}x,c>$, where $k'=a-k$ for $k\in B$ (see Figure 2).

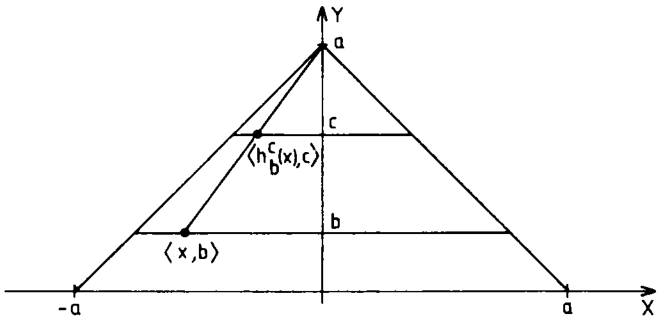

Figure 2.

Then $\mathcal{A}$ is a well defined semilattice ordered system of relational systems. In $S(\mathcal{A})$ we get a ternary relation r^S

satisfying: $r^S(\langle x_1,b\rangle,\langle x_2,c\rangle,\langle x_3,d\rangle)$ iff $d=\max\{b,c\}$ and $r^d(\langle\frac{d'}{b'}x_1,d\rangle,d\rangle,\langle\frac{d'}{c'}x_2,d\rangle,\langle x_3,d\rangle)$ (see Figure 3).

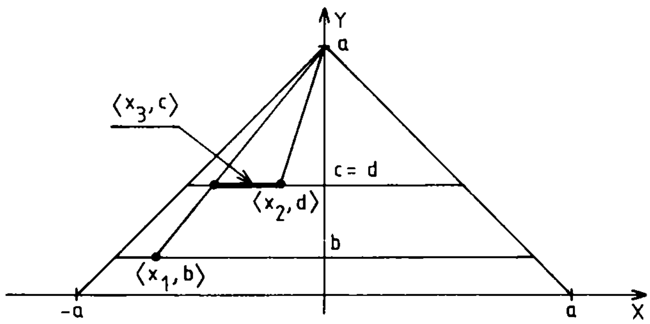

Figure 3.

One can see that $\langle x_3,d\rangle$ must lie inside the angle $\langle x_1,b\rangle$, $\langle 0,a\rangle$, $\langle x_2,c\rangle$, and on the level $\max\{b,c\}$.

Recall that an identity $\varphi=\psi$ is regular if the sets of variables occurring in φ and ψ are identical (see [2]). It was proved in [2] that if $\mathcal{A}$ is a non-trivial semilattice ordered system of algebras ($|I|>1$) then $S(\mathcal{A})$ preserves all regular identities satisfied in all $\mathfrak{A}_i$ and does not satisfy any other identities.

Problem. What formulas of the first order language are preserved by algebraic sums $S(\mathcal{A})$? Such problems were studied in [1].

REFERENCES

[1] **K. Hałkowska, J. Płonka:** On formulas preserved by the sum of a direct system of relational systems, Math. Nachrichten 86 (1978) 117-120

[2] **J. Płonka:** On a method of construction of abstract algebras, Fund. Math. 61(1967) 183-189

[3] **J. Płonka:** On the sum of a direct system of universal algebras with nullary polynomials, Algebra Universalis 19 (1984) 197-207

[4] J. **Płonka:** On the sum of direct system of relational systems, Bull. Acad., Polon. Sci. Ser. Math. 21(7) (1973) 595-597.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES,
KOPERNIKA 18, WROCŁAW, POLAND

Received December 22, 1989.