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THE GENERALIZED SUM OF AN UPPER SEMILATTICE ORDERED SYSTEM OF ALGEBRAS

0. Preliminaries

In [10] a construction of algebras was defined called the sum of a direct system of algebras. This construction is called now the Płonka sum in literature. In [5] and [14] a different construction was considered based on the notion of a retraction of an algebra. In this paper this construction will be called the extension of an algebra by a retraction.

Let τ be a type of algebras. An identity $\phi=\psi$ is called regular (see 10) if the sets of variables in ϕ and ψ coincide. Regular varieties i.e. defined by regular identities were considered e.g. in [4], [6]-[11]. In particular, under some assumptions algebras from regular varieties can be represented by Płonka sums (see [11]). An identity $\phi=\psi$ is called non-trivializing if it is of the form x=x or none of ϕ and ψ is a single variable. It was proved in [10] that a nontrivial Płonka sum preserves all regular identities satisfied in every component and does not preserve any other. On the other hand the extension of an algebra $\mathfrak{A}$ by a retraction preserves all non-trivializing identities of $\mathfrak{A}$ (see [5]). For a class $\mathbb{K}$ of algebras of type τ we denote by Id($\mathbb{K}$) the set of all identities of type τ satisfied in each member of $\mathbb{K}$. If $\mathfrak{A}$ is an algebra we shall write Id($\mathfrak{A}$) instead of Id({$\mathfrak{A}$}).

J.Płonka asked the following question: let $\mathbb{K}$ be a variety of algebras and $\mathbb{K}^*$ be a variety of the same type defined by some regular and some non-trivializing identities from Id($\mathbb{K}$). Can algebras from $\mathbb{K}^*$ be represented by means of algebras from

$\mathbb{K}$. Note that if $\mathbb{K}$ satisfies an identity $f(x,y)=x$ for some term $f(x,y)$ containing x and y and $\mathbb{K}_R$ is the variety defined by all regular identities from $Id(\mathbb{K})$, then every algebra from $\mathbb{K}_R$ is the Płonka sum of algebras from $\mathbb{K}$ (see [11]). In this paper we answer the above question of Płonka for a class $\mathbb{K}^*$ defined by some natural choice of regular and non-trivializing identities from $Id(\mathbb{K})$. Namely, we define a construction of algebras, called the generalized sum of an upper $\{F_1,F_2\}$-semilattice ordered system of algebras that generalizes both the Płonka sum and the extension of an algebra by a retraction. Under some assumption on $\mathbb{K}$ and $\mathbb{K}^*$ we represent algebras from $\mathbb{K}^*$ by means of algebras from $\mathbb{K}$ using the above construction. In this way we also obtain some generalizations of theorems from [10], [11], [13] and [5].

1. Definition of the generalized sum of an upper (F_1,F_2)-semilattice ordered system of algebras

Let $\tau:F\longrightarrow N$ be a fixed type of algebras.

Definition 1. A quadruple

(1.1) $$\mathcal{A}=\langle (F_1,F_2),\ (I,\le),\ \{\mathfrak{U}_i\}_{i\in I},\ \{h_i^j\}_{i,j\in I,i\le j}\rangle$$

will be called an upper (F_1,F_2)-semilattice ordered system of algebras, or briefly an upper (F_1,F_2)-system of algebras, if it satisfies the following conditions:

(i) $F_1\cup F_2=F$, $F_1\cap F_2=\emptyset$, $\{f\in F:\ \tau(f)=0\}\subset F_2$.

(ii) $(I,\le)$ is a join-semilattice; if $F_2\ne\emptyset$ then $(I,\le)$ has the greatest element u.

(iii) $\mathfrak{U}_u$ is an algebra of type τ and $\mathfrak{U}_u=(A_u;F^u)$; for every $i\in I$, $i\ne u$, $\mathfrak{U}_i$ is an algebra of type $\tau|F_1$ and $\mathfrak{U}_i=(A_i;F_1^i)$, where $A_i\cap A_j=\emptyset$ if $i\ne j$.

(iv) The set $\{h_i^j\}_{i,j\in I,i\le j}$ satisfies the following:

(a_1) for every $i,j\in I$, $i\le j$, h_i^j is a mapping of A_i into A_j;

(a_2) for every $i\in I$, h_i^i is the identity map on A_i;

(a_3) for every $i,j\in I$ such that $i \le j\ne u$, h_i^j is a homomorphism of $\mathfrak{A}_i$ into $\mathfrak{A}_j$;

(a_4) for every $i\in I$, $i\ne u$, h_i^u is a homomorphism of $\mathfrak{A}_i$ into the reduct $(A_u;F_1^u)$ of $\mathfrak{A}_u$;

(a_5) for every $i,j,k\in I$ such that $i\le j\le k$ we have $h_j^k\circ h_i^j=h_i^k$.

For an upper (F_1,F_2)-system $\mathcal{A}$ of algebras we define a new algebra $\mathcal{S}(\mathcal{A})$ of type τ putting $\mathcal{S}(\mathcal{A})=\left(\bigcup_{i\in I}A_i;F^{\mathcal{S}}\right)$, where for $f\in F$, $a_j\in A_i$, $j=0,\dots,\tau(f)-1$ the operation $f^{\mathcal{S}}$ is defined by the formula:

$$f^{\mathcal{S}}(a_0,\dots a_{\tau(f)-1}) = \begin{cases} f^k(h_{i_0}^k(a_0),\dots,h_{i_{\tau(f)-1}}^k(a_{\tau(f)-1})) \text{ for} \\ f\in F_1 \text{ and } k=\sup\{i_0,\dots,i_{\tau(f)-1}\} \\ f^u(h_{i_0}^u(a_0),\dots,h_{i_{\tau(f)-1}}^u(a_{\tau(f)-1})), \text{ for} \\ f\in F_2 \end{cases}$$

The algebra $\mathcal{S}(\mathcal{A})$ will be called the generalized sum of the upper (F_1,F_2)-semilattice ordered system $\mathcal{A}$ of algebras or briefly, the sum of the upper (F_1,F_2)-system $\mathcal{A}$ of algebras.

The following may be checked easily.

(v) $\mathfrak{A}_u$ is a subalgebra of $\mathcal{S}(\mathcal{A})$.

(vi) If $i\ne u$ then $\mathfrak{A}_i$ is a subalgebra of the reduct $\left(\bigcup_{i\in I}A_i;F_1^{\mathcal{S}}\right)$ of $\mathcal{S}(\mathcal{A})$.

(vii) If $F_2=\tau^{-1}(\{0\})$ then the definition of $\mathcal{S}(\mathcal{A})$ is equivalent to that of the Płonka sum (see [13]).

(viii) If $F_1=\emptyset$ then $\mathcal{S}(\mathcal{A})$ is an extension by a retraction of the algebra $\mathfrak{A}_u$ (see [5]). Conversely, if algebra $\mathfrak{A}$ is an extension by a retraction r of an algebra $\mathfrak{A}_0$ and $A\ne A_0$ then $\mathfrak{A}$ can represented as the generalized sum of the upper $(\emptyset,F)$-system:

$$\mathcal{A} = \langle(\emptyset,F),(-1,0),\{\mathfrak{A}_{-1},\mathfrak{A}_0\},\{h_{-1}^{-1},h_0^0,h_{-1}^0\}\rangle,$$

where $\mathfrak{A}_{-1}=(A-A_0;\emptyset)$, h_i^i is the identity map on A_i, $i\in\{-1,0\}$ and $h_{-1}^0=r|A-A_0$.

2. F_0-regular identities and F_0-symmetrical identities

In this section we consider identities of some special form which are preserved by the construction defined in section 1.

Let $\tau:F\longrightarrow T$ be a type of algebras. If ϕ is a term of type τ we denote by $F(\phi)$ the set of all fundamental operation symbols occuring in ϕ and by $Var(\phi)$ - the set of all variables occuring in ϕ. Let F_0 be a subset of F.

Definition 2. An identity $\phi=\psi$ of type τ will be called F_0-regular iff $F(\phi)\subset F_0$, $F(\psi)\subset F_0$ and $Var(\phi)=Var(\psi)$.

Every F_0-regular identity is regular in the sens of [10], [13]. Every regular identity is F-regular.

Definition 3. An identity $\phi=\psi$ of type τ will be called F_0-symmetrical iff $F(\phi)\cap F_0\neq\emptyset$ and $F(\psi)\cap F_0\neq\emptyset$.

Remark 1. Note that an identity $\phi=\psi$ of type τ is non-trivializing (see [12]) iff it is of the form x=x or it is F-symmetrical.

Example 1. Let $F=\{+,\circ,'\}$, $\tau(+)=\tau(\circ)=2$, $\tau(')=1$, $F_1=\{+,\circ\}$ and $F_2=\{'\}$. Then identity $x+y=x\circ y$ is F_1-regular, the identity $x+(x\circ x')=(x')'$ is F_2-symmetrical and the identities $x+(x\circ y)=x$, $x=x'$ are neither F_1-regular nor F_2-symmetrical.

Let F_1, F_2 be sets satisfying (i) from section 1. Denote by R_{F1} the set of all F_1-regular identities of type τ and by S_{F2} the set of all F_2-symmetrical identities of tyoe τ. Then

(ix) The set $R_{F1}\cup S_{F2}$ is an equational theory in the sense of Tarski (see [16]).

If ϕ is a term of type τ and $Var(\phi)=\{x_{j_0},\dots,x_{j_{n-1}}\}$ then we shall write $\phi(x_{j_0},\dots,x_{j_{n-1}})$ and we shall denote by $\phi^{\mathfrak{U}}(a_0,\dots,a_{n-1})$ the realization of ϕ in $\mathfrak{U}$ obtained by substituting elements $a_0,\dots,a_{n-1}$ of A for $x_{j_0},\dots,x_{j_{n-1}}$. We shall denote by ϕ^i the realization of ϕ in an algebra $\mathfrak{U}_i$ and by $\phi^{\mathcal{S}}$ - the realization of ϕ in $\mathcal{S}(\mathcal{A})$.

Lemma 1. Let $\mathcal{S}(\mathcal{A})$ by the sum of an upper (F_1,F_2)system

$\mathcal{A}$ defined by (1.1), $\phi(x_{j_0},\dots,x_{j_{n-1}})$ be a term of type τ and $a_m \in A_{i_m}$ for m=0,...,n-1. Then we have:

(b_1) if $F(\phi)\subset F_1$ then

$\phi^{\mathcal{S}}(a_0,\dots,a_{n-1})=\phi^k(h^k_{i_0}(a_0),\dots,h^k_{i_{n-1}}(a_{n-1}))$,

where $k=\sup\{i_0,\dots,i_{n-1}\}$;

(b_2) if $F_2\cap F(\phi)\neq\emptyset$ then

$\phi^{\mathcal{S}}(a_0,\dots,a_{n-1})=\phi^u(h^u_{i_0}(a_0),\dots,h^u_{i_{n-1}}(a_{n-1}))$.

Proof. The proof of (b_1) is analogous to that of Theorem 1 in [10]. The proof of (b_2) is by induction on the complexity of ϕ.

Since $F_2\cap F(\phi)\neq\emptyset$ so ϕ is not a single variable. Hence ϕ is of the form:

$f(\phi_0(x^0_0,\dots,x^0_{n_0-1}),\phi_1(x^1_0,\dots,x^0_{n_1-1}),\dots,\phi_{r-1}(x^{r-1}_0,\dots$
$\dots,x^{r-1}_{n_{r-1}-1}))$, where $r=\tau(f)$.

If ϕ is of the form $f(x_{j_0},\dots,x_{j_{\tau(f)-1}})$, where $f\in F_2$, then the statement follows from the definition of the operation in $\mathcal{S}(\mathcal{A})$.

Let ϕ be realized in p'th step, p>2, $F(\phi)\cap F_2\neq\emptyset$ and let (b_2) be true for each term obtained in q'th step, for every $2\le q<p$. Let:

$$a^0_0\in A_{i^0_0},\dots,a^0_{n_0-1}\in A_{i^0_{n_0-1}},\ a^1_0\in A_{i^1_0},\dots,a^1_{n_1-1}\in A_{i^1_{n_1-1}},\dots$$
$$\dots,a^{r-1}_0\in A^{r-1}_{i_0},\dots,a^{r-1}_{n_{r-1}-1}\in A^{r-1}_{i_{n_{r-1}-1}}.$$

Then

$$f^{\mathcal{S}}(\phi^{\mathcal{S}}_0(a^0_0,\dots,a^0_{n_0-1}),\phi^{\mathcal{S}}_1(a^1_0,\dots,a^1_{n_1-1}),\dots$$
$$\dots,\phi^{\mathcal{S}}_{r-1}(a^{r-1}_0,\dots,a^{r-1}_{n_{r-1}-1}))=$$
$$=f^{\mathcal{S}}(\phi^{k_0}_0(h^{k_0}_{i^0_0}(a^0_0),\dots,h^{k_0}_{i^0_{n_0-1}}(a^0_{n_0-1})),\ \phi^{k_1}_1(h^{k_1}_{i^1_0}(a^1_0),\dots$$

$$\ldots,h^{k_1}_{i^1_{n_1-1}}(a^1_{n_1-1})),\ldots,\phi^{k_{r-1}}_{r-1}(h^{k_{r-1}}_{i^{r-1}_0}(a^{r-1}_0),\ldots,h^{k_{r-1}}_{i^{r-1}_{n_{r-1}-1}}\quad))),$$

where

$$k_s = \begin{cases} u, \text{ if } F(\phi_s)\cap F_2\neq\emptyset \\ \sup\{i^s_0, i^s_1, \ldots, i^s_{n_s-1}\}, \text{ if } F(\phi_s)\subset F_1, \text{ for } s=0,\ldots,r-1. \end{cases}$$

We have two cases to consider:

(c_1) $f\in F_1$ and there exists $s_0\in\{0,\ldots,r-1\}$ such that $F(\phi_{s_0})\cap F_2\neq\emptyset$,

(c_2) $f\in F_2$.

In case (c_1) we have $\sup\{k_0,\ldots,k_{r-1}\}=u$ so both in case (c_1) and (c_2) it follows that

$$f^{\mathcal{S}}(\phi^{\mathcal{S}}_0(a^0_0,\ldots,a^0_{n_0-1}),\phi^{\mathcal{S}}_1(a^1_0,\ldots,a^1_{n_1-1}),\ldots,\phi^{\mathcal{S}}_{r-1}(a^{r-1}_0,\ldots,a^{r-1}_{n_{r-1}-1}))=$$

$$=f^u(h^u_{k_0}(\phi^{k_0}_0(h^{k_0}_{i^0_0}(a^0_0),\ldots,h^{k_0}_{i^0_{n_0-1}}(a^0_{n_0-1}))),h^u_{k_1}(\phi^{k_1}_1(h^{k_1}_{i^1_0}(a^1_0),\ldots$$

$$\ldots,h^{k_1}_{i^1_{n_1-1}}(a^1_{n_1-1}))),\ldots,h^u_{k_{r-1}}(\phi^{k_{r-1}}_{r-1}(h^{k_{r-1}}_{i^{r-1}_0}(a^{r-1}_0),\ldots$$

$$\ldots,h^{k_{r-1}}_{i^{r-1}_{n_{r-1}-1}}(a^{r-1}_{n_{r-1}-1})))) = f^u(\phi^u_0(h^u_{k_0}(h^{k_0}_{i^0_0}(a^0_0)),\ldots$$

$$\ldots,h^u_{k_0}(h^{k_0}_{i^0_{n_0-1}}(a^0_{n_0-1}))),\phi^u_1(k^u_{k_1}(h^{k_1}_{i^1_0}(a^1_0)),\ldots$$

$$\ldots,h^u_{k_1}(h^{k_1}_{i^1_{n_1-1}}(a^1_{n_1-1}))),\ldots,\phi^u_{r-1}(h^u_{k_{r-1}}(h^{k_{r-1}}_{i^{r-1}_0}(a^{r-1}_0)),\ldots$$

$$\ldots,h^u_{k_{r-1}}(h^{k_{r-1}}_{i^{r-1}_{n_{r-1}-1}}(a^{r-1}_{n_{r-1}-1})))) = f^u(\phi^u_0(h^u_{i^0_0}(a^0_0),\ldots$$

$$\ldots,h^u_{i^0_{n_0-1}}(a^0_{n_0-1})),\phi^u_1(h^u_{i^1_0}(a^1_0),\ldots,h^u_{i^1_{n_1-1}}(a^1_{n_1-1})),\ldots$$

$$\ldots,\phi^u_{r-1}(h^u_{i^{r-1}_0}(a^{r-1}_0),\ldots,h^u_{i^{r-1}_{n_{r-1}-1}}(a^{r-1}_{n_{r-1}-1}))).$$

Theorem 1. Let $\mathcal{A}$ be an upper (F_1,F_2)-system defined in (1.1). Then $\mathcal{S}(\mathcal{A})$ satisfies all F_1-regular identities which are satisfied in every $\mathfrak{U}_i$, satisfies all F_2-symmetrical identities which are satisfied in $\mathfrak{U}_u$ and does not satisfy any other.

Proof. The proof of the first statement is analogous to the proof of Theorem 1 in [10]. To prove the second, let $\phi(x_{j_0},\dots,x_{j_{n-1}})=\psi(x_{k_0},\dots,x_{k_{m-1}})$ be an F_2-symmetrical identity satisfied in $\mathfrak{U}_u$. Then $F(\phi)\cap F_2\neq\emptyset$ and $F(\psi)\cap F_2\neq\emptyset$. Let $a_0\in A_{i_0},\dots,a_{n-1}\in A_{i_{n-1}},b_0\in A_{t_0},\dots,b_{m-1}\in A_{t_{m-1}}$. By Lemma 1 we have $\phi^{\mathcal{S}}(a_0,\dots,a_{n-1})=\phi^u(h^u_{i_0}(a_0),\dots,h^u_{i_{n-1}}(a_{n-1}))$ and $\psi^{\mathcal{S}}(b_0,\dots,b_{m-1})=\psi^u(h^u_{t_0}(b_0),\dots,h^u_{t_{m-1}}(b_{m-1}))$. Then $\phi^u(h^u_{i_0}(a_0),\dots,h^u_{i_{n-1}}(a_{n-1}))=\psi^u(h^u_{t_0}(b_0),\dots,h^u_{t_{m-1}}(b_{m-1}))$. Hence $\phi^{\mathcal{S}}(a_0,\dots,a_{n-1})=\psi^{\mathcal{S}}(b_0,\dots,b_{m-1})$.

By (v), $Id(\mathcal{S}(\mathcal{A}))\subset Id(\mathfrak{U}_u)$, so the second statement holds. Now let $(\phi=\psi)\in Id(\mathfrak{U}_u)$ be such that $F(\phi)\cap F_2\neq\emptyset$ and $F(\psi)\subset F_1$. Since $|I|>1$, there exists $i_0\in I$ with $i_0\neq u$. If $a_0\in A_{i_0}$, then

$$\phi^{\mathcal{S}}(a_0,\dots,a_0)=\phi^u(h^u_{i_0}(a_0),\dots,h^u_{i_0}(a_0))\neq\psi^{i_0}(a_0,\dots,a_0)=\psi^{\mathcal{S}}(a_0,\dots,a_0).$$ Thus $(\phi=\psi)\notin Id(\mathcal{S}(\mathcal{A}))$.

3. Varieties defined by F_1-regular and F_2-symmetrical identities

Let a type $\tau:F\longrightarrow N$ be given and let F_1, F_2 satisfy condition (i) of (1.1). In this section we generalize some of results from [7].

Consider an algebra $\mathfrak{U}(F_1,F_2)=(\{a,b\};F^{\mathfrak{U}})$ of type τ, where

$$f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1})=\begin{cases}a_0 & \text{if } a_0=\dots=a_{\tau(f)-1}\\ b & \text{otherwise}\end{cases}\quad\text{for } f\in F_1,$$

$$f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1})=b\quad\text{for } f\in F_2.$$

(x) The algebra $\mathfrak{U}(F_1,F_2)$ is the sum of an upper (F_1,F_2)-system of 1-element algebras.

Indeed, consider 1-element algebras $\mathfrak{U}_1=(\{a\};F^1_1)$,

$\mathfrak{U}_2=(\{b\};F^2)$ and define a mapping $h_1^2\colon \{a\}\to\{b\}$, $h_1^2(a)=b$. Then $\mathfrak{U}(F_1,F_2)=\mathcal{S}(\mathcal{A})$, where

$\mathcal{A}=\langle(F_1,F_2),(\{1,2\},\le),\{\mathfrak{U}_1,\mathfrak{U}_2\},\{h_i^j\}_{\substack{i,j\in\{1,2\}\\ i\le j}}\rangle$, h_i^i are identity maps.

(xi) Algebra $\mathfrak{U}(F_1,F_2)$ satisfies all F_1-regular identities, all F_2-symmetrical identities of type τ and does not satisfy any other.

This follows by (x) and Theorem 1 from the fact that in a 1-element algebra all identities of its type are satisfied,

(xii) For every variety $\mathbb{K}$ of type τ we have $Id(\mathbb{K})\subset R_{F_1}\cup S_{F_2}$ iff $\mathfrak{U}(F_1,F_2)\in\mathbb{K}$.

Note that property (xii) was proved in [7] in the case $F_1=F$.

Let E be a set of identities of type τ. Denote by V(E) the variety of type τ defined by E.

We accept the following notation: if an identity $\phi=\psi$ is F_1-regular we shall write $\phi \underset{F_1}{=} \psi$, if $\phi=\psi$ is F_2-symmetrical we shall write $\phi \overset{F_2}{=} \psi$.

Theorem 2. The variety $V(R_{F1}\cup S_{F2})$ is equationally complete.

Proof. $V(R_{F1}\cup S_{F2})$ contains $\mathfrak{U}(F1,F2)$ so it is non-degenerated.

Let $(\phi=\psi)\notin Id(V(R_{F1}\cup S_{F2}))$, let us add it to $Id(V(R_{F1}\cup S_{F2}))$. Without least of generality it is enough to consider two cases:

(1) $F(\phi)\cap F_2\neq\emptyset$ and $F(\psi)\subset F_1$,

(2) $F(\phi)$, $F(\psi)\subset F_1$ and $x\in(Var(\phi)-Var(\psi))$.

In case (1) the identity $\phi=\psi$ must be of the form:

(3.1) $\phi=\psi(x_{j_1},\ldots,x_{j_m})$

where ϕ is a nullary term or

(3.2) $\phi(x_{i_1},\ldots,x_{i_n})=\psi(x_{j_1},\ldots,x_{j_m})$.

If we have (3.1) then substituting x for all variables in (3.1) we get

(3.3) $\phi=\psi(x,\ldots,x)$.

So $x \underset{F_1}{=} \psi(x,\ldots,x)=\phi=\psi(y,\ldots,y) \underset{F_1}{=} y$.

If we have (3.2) then substituting x for all variables in (3.2) we get

(3.4) $\phi(x,\ldots,x)=\psi(x,\ldots,x)$.

So we have

$$x \underset{F_1}{=} \psi(x,\ldots,x) = \phi(x,\ldots,x) \overset{F_2}{=} \phi(y,\ldots,y) \underset{F_1}{=} y.$$

Now we consider (2). Then $\phi=\psi$ must be of the form:

(3.5) $\phi(x)=\psi(x_{i_1},\ldots,x_{i_n})$

or

(3.6) $\phi(x,x_{i_1},\ldots,x_{i_n})=\psi(x_{j_1},\ldots,x_{j_m})$.

If we have (3.5) then substituting y for all variables different from x in (3.5) we get

(3.7) $\phi(x)=\psi(y,\ldots,y)$.

So we have $x \underset{F_1}{=} \phi(x)=\psi(y,\ldots,y) \underset{F_1}{=} y$. If we have (3.6) then substituting y for all variables different from x in (3.6) we get

(3.8) $\phi(x,y,\ldots,y)=\psi(y,\ldots,y)$.

Substituting x for y and y for x in (3.8) we get

(3.9) $\phi(y,x,\ldots,x)=\psi(x,\ldots,x)$.

So we have

$$x \underset{F_1}{=} \psi(x,\ldots,x) = \phi(y,x,\ldots,x) \underset{F_1}{=} \phi(x,y,\ldots,y) = \psi(y,\ldots,y) \underset{F_1}{=} y.$$

Let $\mathbb{K}$ be a variety of algebras of type τ.

Denote by $\mathbb{K}_{F1}^{F2}$ the variety defined by all F_1-regular identities and all F-symmetrical identities satisfied in $\mathbb{K}$, denote by $\mathbb{K}_{F_1}$ the variety of algebras of type $\tau|F_1$ defined by all identities of type $\tau|F_1$ belonging to $Id(\mathbb{K})$. Let $\mathbb{D}(\tau)$ denote the variety of type τ defined by the identity $x=y$. We have:

(xiii) $V\left(R_{F_1} \cup S_{F_2}\right) = \mathbb{D}(\tau)_{F_1}^{F_2}$

(xiv) $\mathbb{K}_{F_1}^{F_2} = \mathbb{K} \vee V\left(R_{F_1} \cup S_{F_2}\right)$

(xv) $\mathbb{K}$ is defined only by F_1-regular identities and F_2-symmetrical identities iff $\mathbb{K}$ contains every sum of an upper (F_1,F_2) system $\mathcal{A}$ of algebras $\{\mathfrak{A}_i\}_{i\in I}$, where $\mathfrak{A}_u\in\mathbb{K}$ and $\mathfrak{A}_i\in\mathbb{K}_{F_1}$, for $i\neq u$.

The necessity follows from Theorem 1.
The sufficency follows from the fact that $\mathfrak{A}(F_1,F_2)$ belongs to $\mathbb{K}$ and from (x), we have $Id(\mathbb{K})\subset R_{F_1}\cup S_{F_2}$.

4. The upper (F_1,F_2)-partition function

In this section we give some generalization of the notion of a partition function defined in [10].

Let us fix a type $\tau:F\longrightarrow N$ and sets F_1, F_2 satisfying (i) in (1,1).
Let $\mathfrak{A}=(A;F^{\mathfrak{A}})$ be an algebra of type τ.

Definition 4. A mapping $\circ:A^2\longrightarrow A$ will be called an upper (F_1,F_2)-partition function of $\mathfrak{A}$ if $\circ$ satisfies the following formulas for all $a,b,c,a_0,\dots,a_{\tau(f)-1}\in A$.

(4.1) $a\circ a=a$;

(4.2) $(a\circ b)\circ c=a\circ(b\circ c)$;

(4.3) $a\circ(b\circ c)=a\circ(c\circ b)$;

(4.4) $f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})\circ b = f^{\mathfrak{A}}(a_0\circ b,\dots,a_{\tau(f)-1}\circ b)$, for all $f\in F$;

(4.5) $f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})\circ b = f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})$, for all $f\in F_2$;

(4.6) $f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})\circ a_i = f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})$, for all $f\in F_1$ and $i\in\{0,\dots,\tau(f)-1\}$;

(4.7) $b\circ f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1}) = b\circ f^{\mathfrak{A}}(b\circ a_0,\dots,b\circ a_{\tau(f)-1})$, for $f\in F_1$;

(4.8) $a\circ f^{\mathfrak{A}}(a,\dots,a) = a$, for $f\in F_1$.

In [10] J.Płonka defined the notion of partition function by means of which, he decomposed an algebra into the sum of a direct system of subalgebras in the following way:

(1^{o}) A relation R defined by

(4.9) aRb iff $a\circ b=a$ and $b\circ a=a$

is an equivalence on A. Let $A/R=\{A_i\}_{i\in I}$

(2°) A relation $\le$ defined by

(4.10) $i\le j$ iff $\forall_{a\in A_i} \forall_{b\in A_j}$ $b\circ a=b$

is a join-semilattice order in I.

(3°) Each $\mathfrak{U}_i=(A_i;\{f^{\mathfrak{U}}|A_i\}_{f\in F})$ is a subalgebra of $\mathfrak{U}$.

(4°) Each mapping $h_i^j:A_i\longrightarrow A_j$ for $i,j\in I$, $i\le j$ defined by

(4.11) $h_i^j(x)=x\circ b$, $b\in A_j$

is a homomorphism of $\mathfrak{U}_i$ into $\mathfrak{U}_j$.

(5°) $\mathfrak{U}=\mathcal{G}(\mathcal{A})$, where $\mathcal{A}=<(I,\le),\{\mathfrak{U}_i\}_{i\in I},\{h_i^j\}_{i,j\in I,i\le j}>$.

Theorem 3. Every upper (F_1,F_2)-partition function of $\mathfrak{U}=(A;F^{\mathfrak{U}})$ establishes a decomposition of this algebra into the sum of an upper (F_1,F_2)-semilattice ordered system of algebras. More precisely, $\mathfrak{U}$ is the sum of an upper (F_1,F_2)-system.

(4.12) $\mathcal{A}(\circ)=<(F_1,F_2),(I,\le),\{\mathfrak{U}_i\}_{i\in I},\{h_i^j\}_{i,j\in I,i\le j}>$,

where $A/R=\{A_i\}_{i\in I}$: R is defined as in (4.9);

$\mathfrak{U}_i=(A_i;\{f^{\mathfrak{U}}|A_i\}_{f\in F_1})$, for $i\in I-\{u\}$, $\mathfrak{U}_u=(A_u;\{f^{\mathfrak{U}}|A_u\}_{f\in F})$; the relation $\le$ and the mappings h_i^j are defined as in (4.10) and (4.11).

Proof. Analogously as in [10] one proves the following:

(α_1) the relation defined as in (4.9) is an equivalence;

(α_2) the relation $\le$ defined in (4.10) is a join-semilattice order;

(α_3) $(A_i;\{f^{\mathfrak{U}}|A_i\}_{f\in F_1})$ is a subalgebra of $(A;F_1^{\mathfrak{U}})$ for each $i\in I$;

(α_4) for $i,j\in I$, $i\le j$, h_i^j defined as in (4.10) is a homomorphism of $(A_i;\{f^{\mathfrak{U}}|A_i\}_{f\in F_1})$ into $(A_j;\{f^{\mathfrak{U}}|A_j\}_{f\in F_1})$, each h_i^i is identity map and $h_j^k\circ h_i^j=h_i^k$ for every $i,j,k\in I$, $i\le j\le k$.

We have to prove, that if $F_2\neq\emptyset$ then there exists the greatest element u in I; $\mathfrak{A}_u$ is a subalgebra of $\mathfrak{A}$ and $\mathfrak{A}=\varphi(\mathcal{A}(\circ))$.

Let $f\in F_2$ and let $a_0,\dots,a_{\tau(f)-1}\in A$. Then $f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})\in A_{i_0}$ for some $i_0\in I$. By (4.5), for every $a\in A$ we have $f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})\circ a=f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})$. Thus i_0 is the greatest element in I and we can put $u=i_0$. Moreover, the values of all $f^{\mathfrak{A}}$ belong to A_u for $f\in F_2$ and consequently, $\mathfrak{A}_u$ is a subalgebra of $\mathfrak{A}$. Let $f\in F$, $a_0\in A_{i_0},\dots,a_{\tau(f)-1}\in A_{i_{\tau(f)-1}}$. Analogously as in [10] we show that $f^{\varphi}(a_0,\dots,a_{\tau(f)-1})=$ $=f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})$ for $f\in F_1$. If $f\in F_2$ then by (4.4), (4.5) we have

$$f^{\varphi}(a_0,\dots,a_{\tau(f)-1}) = f^{u}(h^{u}_{i_0}(a_0),\dots,h^{u}_{i_{\tau(f)-1}}(a_{\tau(f)-1})) =$$
$$= f^{\mathfrak{A}}(a_0\circ z,\dots,a_{\tau(f)-1}\circ z) = f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})\circ z =$$
$$= f^{\mathfrak{A}}(a_0,\dots,a_{\tau(f)-1})$$

for some $z\in A_u$. Thus the operations in $\mathfrak{A}$ and $\varphi(\mathcal{A}(\circ))$ coincide.

5. Representation of algebras in the variety $\mathbb{K}^{F2}_{F1}$.

Let $\mathbb{K}$ be a variety of algebras of type τ. Consider the following condition.

(c_1) There exists a term $\phi(x,y)$ of type τ such that, $F(\phi(x,y))\subset F_1$ and the identity $\phi(x,y)=x$ belongs to $Id(\mathbb{K})$.

Theorem 4. If a variety $\mathbb{K}$ of type τ satisfies (c_1), then algebra $\mathfrak{A}$ belongs to $\mathbb{K}^{F2}_{F1}$ iff $\mathfrak{A}$ is the sum of an upper (F_1,F_2)-semilattice ordered system of algebras $\{\mathfrak{A}_i\}_{i\in I}$, where $\mathfrak{A}_i\in\mathbb{K}_{F_1}$ for $i\neq u$ and $\mathfrak{A}_u\in\mathbb{K}$.

Proof. Let $\mathfrak{A}\in\mathbb{K}^{F2}_{F1}$ and let $\phi(x,y)$ be a fixed binary term in (c_1). Define a binary operation $\circ: A^2\longrightarrow A$ by:

(5.1) $\quad a\circ b = \phi^{\mathfrak{A}}(a,b),\quad$ for $a,b\in A$.

Then $\circ$ defined by (5.1) satisfies conditions (4.1)-(4.8).

Then by Theorem 3 the function $\circ$ induces a decomposition of $\mathfrak{U}$ into the sum of an upper (F_1,F_2)-semilattice ordered system of algebras $\mathfrak{U}_i$, $i\in I$.

Let $\phi_1=\phi_2$ be an identity of type $\tau|F_1$ belonging to $Id(\mathbb{K})$. Then the identity $\phi(\phi_1,\phi_2)=\phi(\phi_2,\phi_1)$ belongs to $Id(\mathbb{K}_{F1}^{F2})$. Moreover, by the definition of A_i, the identities $\phi(\phi_1,\phi_2)=\phi_1$, $\phi(\phi_2,\phi_1)=\phi_2$ are satisfied in $\mathfrak{U}_i$. Hence the identity $\phi_1=\phi_2$ is satisfied in $\mathfrak{U}_i$.

Let $\phi_1=\phi_2$ be an identity of type τ but not of type $\tau|F_1$ and $(\phi_1=\phi_2)\in Id(\mathbb{K})$. Then the identity $\phi(\phi_1,\phi_2) \overset{F_2}{=} \phi(\phi_2,\phi_1)$ belongs to $Id(\mathbb{K}_{F1}^{F2})$, so it is satisfied in $\mathfrak{U}_u$ by (v), since the identities $\phi(\phi_1,\phi_2)=\phi_1$, $\phi(\phi_2,\phi_1)=\phi_2$ are satisfied in $\mathfrak{U}_u$. Thus $\mathfrak{U}_u$ belongs to $\mathbb{K}$. The necessity follows from Theorem 1.

Algebras from $\mathbb{K}_{F1}^{F2}$ can be represented in another way as it is shown below.

For a variety $\mathbb{K}$ we denote by $\mathbb{K}(F_2)$ the variety defined by all F_1-regular identities belonging to $Id(\mathbb{K})$ and all F_2-symmetrical identities of type τ.

For two varieties $\mathbb{K}_1$, $\mathbb{K}_2$ of type τ we denote by $\mathbb{K}_1\otimes\mathbb{K}_2$ the class of all algebras $\mathfrak{U}$ isomorphic to a subdirect product of algebras $\mathfrak{U}_1$ and $\mathfrak{U}_2$, where $\mathfrak{U}_1\in\mathbb{K}_1$ and $\mathfrak{U}_2\in\mathbb{K}_2$.

Consider the following condition.

(c_2) There exists a term $\psi(x)$ such that $F(\psi(x))\cap F_2\neq\emptyset$ and the identity $\psi(x)=x$ belongs to $Id(\mathbb{K})$.

Theorem 5. If a variety $\mathbb{K}$ satisfies condition (c_2) then $\mathbb{K}_{F1}^{F2}=\mathbb{K}\otimes\mathbb{K}(F_2)$.

Proof. Let a variety $\mathbb{K}$ satisfies condition (c_2) for a fixed term $\psi(x)$. Let $\mathfrak{U}=(A;F^{\mathfrak{U}})$ and $\mathfrak{U}\in\mathbb{K}_{F1}^{F2}$. Consider two relations R1, R2 on A defined as follows:

(5.2) $\quad aR1b \quad \text{iff} \quad \psi^{\mathfrak{U}}(a)=\psi^{\mathfrak{U}}(b)$

(5.3) $\quad aR2b \quad \text{iff} \quad a=b \ \text{or} \ \psi^{\mathfrak{U}}(a)=a \ \text{and} \ \psi^{\mathfrak{U}}(b)=b.$

Obviously R1, R2 are equivalence relations on A and $R1\cap R2=\omega$, where $\omega=\{(a,a)\colon a\in A\}$. We prove that R1 and R2 are congruences of $\mathfrak{U}$. Let $f\in F$ and $a_0R1b_0,\dots,a_{\tau(f)-1}R1b_{\tau(f)-1}$. Then:

$\psi^{\mathfrak{U}}(a_0)=\psi^{\mathfrak{U}}(b_0),\dots,\psi^{\mathfrak{U}}(a_{\tau(f)-1}) = \psi^{\mathfrak{U}}(b_{\tau(f)-1})$. Hence $f^{\mathfrak{U}}(\psi^{\mathfrak{U}}(a_0),\dots,\psi^{\mathfrak{U}}(a_{\tau(f)-1})) = f^{\mathfrak{U}}(\psi^{\mathfrak{U}}(b_0),\dots,\psi^{\mathfrak{U}}(b_{\tau(f)-1}))$. But the identity $\psi(f(x_0,\dots,x_{\tau(f)-1}))=f(\psi(x_0),\dots\psi(x_{\tau(f)-1}))$ belongs to $Id(\mathbb{K}^{F2}_{F1})$. Hence

$\psi^{\mathfrak{U}}(f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1})) = f^{\mathfrak{U}}(\psi^{\mathfrak{U}}(a_0),\dots,\psi^{\mathfrak{U}}(a_{\tau(f)-1}))$ and

$\psi^{\mathfrak{U}}(f^{\mathfrak{U}}(b_0,\dots,b_{\tau(f)-1})) = f^{\mathfrak{U}}(\psi^{\mathfrak{U}}(b_0),\dots,\psi^{\mathfrak{U}}(b_{\tau(f)-1}))$, thus

$\psi^{\mathfrak{U}}(f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1})) = \psi^{\mathfrak{U}}(f^{\mathfrak{U}}(b_0,\dots,b_{\tau(f)-1}))$ and R1 is a congruence of $\mathfrak{U}$.

Let $a_0 R2 b_0,\dots,a_{\tau(f)-1} R2 b_{\tau(f)-1}$. If for some $i\in\{0,\dots,\tau(f)-1\}$ we have $\psi^{\mathfrak{U}}(a_i)=a_i$ and $\psi^{\mathfrak{U}}(b_i)=b_i$ then

$f^{\mathfrak{U}}(a_0,\dots,a_i,\dots,a_{\tau(f)-1})=f^{\mathfrak{U}}(a_0,\dots,\psi^{\mathfrak{U}}(a_i),\dots,a_{\tau(f)-1})$ and

$f^{\mathfrak{U}}(b_0,\dots,b_i,\dots,b_{\tau(f)-1})=f^{\mathfrak{U}}(b_0,\dots,\psi^{\mathfrak{U}}(b_i),\dots,b_{\tau(f)-1})$. But the identity $\psi(f(x_0,\dots,x_{\tau(f)-1}))=f(x_0,\dots,\psi(x_i),\dots,x_{\tau(f)-1})$ belongs to $Id(\mathbb{K}^{F2}_{F1})$, so we obtain:

$\psi^{\mathfrak{U}}(f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1})) = f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1})$ and

$\psi^{\mathfrak{U}}(f^{\mathfrak{U}}(b_0,\dots,b_{\tau(f)-1})) = f^{\mathfrak{U}}(b_0,\dots,b_{\tau(f)-1}))$. Thus

$f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1}) R2 f^{\mathfrak{U}}(b_0,\dots,b_{\tau(f)-1})$. If $a_0=b_0,\dots,a_{\tau(f)-1}=b_{\tau(f)-1}$ then $f^{\mathfrak{U}}(a_0,\dots,a_{\tau(f)-1}) = f^{\mathfrak{U}}(b_0,\dots,b_{\tau(f)-1})$. By Birkhoff's theorem (see [1]), $\mathfrak{U}$ is isomorphic to a subdirect product of $\mathfrak{U}/R1$ and $\mathfrak{U}/R2$. We shall show that $\mathfrak{U}/R1\in\mathbb{K}$ and $\mathfrak{U}/R2\in\mathbb{K}(F2)$. Let $\phi_1(x_{i_0},\dots,x_{i_{n-1}}) = \phi_2(x_{j_0},\dots,x_{j_{m-1}})$ belongs to $Id(\mathbb{K})$ and $a_0,\dots,a_{n-1},\dots,b_0,\dots,b_{m-1}\in A$. Then

$\phi_1^{\mathfrak{U}/R1}([a_0]_{R1},\dots,[a_{n-1}]_{R1}) = [\phi_1^{\mathfrak{U}}(a_1,\dots,a_{n-1})]_{R1} =$

$= [\phi_2^{\mathfrak{U}}(b_0,\dots,b_{m-1})]_{R1} = \phi_2^{\mathfrak{U}/R1}([b_0]_{R1},\dots,[b_{m-1}]_{R1})$ since

$(\psi(\phi_1(x_{i_0},\dots,x_{i_{n-1}})) = \psi(\phi_2(x_{j_0},\dots,x_{j_{m-1}})))\in Id(\mathbb{K}^{F2}_{F1})$. Thus $\mathfrak{U}/R1$ belongs to $\mathbb{K}$. Let $\phi_1=\phi_2$ belongs to $Id(\mathbb{K})$ and $\phi_1=\phi_2$ be F_1-regular. Hence $\phi_1=\phi_2$ is of the form $\phi_1(x_{j_1},\dots,x_{j_n}) =$

$= \phi_2(x_{j_1},\dots,x_{j_n})$. Let $a_0,\dots,a_n \in A$. Then

$\phi_1^{\mathfrak{U}/R2}([a_1]_{R2},\dots,[a_n]_{R2}) = [\phi_1^{\mathfrak{U}}(a_1,\dots,a_n)]_{R2} =$

$= [\phi_2^{\mathfrak{U}}(a_1,\dots,a_n)]_{R2} = \phi_2^{\mathfrak{U}/R2}([a_1]_{R2},\dots,[a_n]_{R2})$.

Thus the identity $\phi_1=\phi_2$ is satisfied in $\mathfrak{U}/R2$. Let $\phi_1(x_{i_0},\dots$ $\dots,x_{i_{n-1}}) = \phi_2(x_{j_0},\dots,x_{j_{m-1}})$ be an F_2-symmetrical identity of type τ. Then identities:

$\psi(\phi_1(x_{i_0},\dots,x_{i_{n-1}})) = \phi_1(x_{i_0},\dots,x_{i_{n-1}})$,

$\psi(\phi_2(x_{j_0},\dots,x_{j_{n-1}})) = \phi_2(x_{j_0},\dots,x_{j_{m-1}})$ belong to $Id(\mathbb{K}_{F1}^{F2})$.

Hence $\phi_1^{\mathfrak{U}}(a_0,\dots,a_{n-1})R2\phi_2^{\mathfrak{U}}(b_0,\dots,b_{m-1})$. We have

$\phi_1^{\mathfrak{U}/R2}([a_0]_{R2},\dots,[a_{n-1}]_{R2}) = [\phi_1^{\mathfrak{U}}(a_0,\dots,a_{n-1})]_{R2} =$

$= [\phi_2^{\mathfrak{U}}(b_0,\dots,b_{m-1})]_{R2} = \phi_2^{\mathfrak{U}/R2}([b_0]_{R2},\dots,[b_{m-1}]_{R2})$.

Thus $\mathfrak{U}/R2 \in \mathbb{K}(F_2)$, what completes the proof.

Corollary 1. If a variety $\mathbb{K}$ satisfies condition (c_1) and $F_2 \neq \emptyset$ then the variety $\mathbb{K}$ satisfies condition (c_2), as well.

The converse implication need not hold as it shows the following example:

Example 2. Let $\mathbb{G}$ be the variety of all groups with the fundamental operations $\circ$, $^{-1}$. Let $F_1=\{\circ\}$, $F_2=\{^{-1}\}$. Then the term $(x^{-1})^{-1}$ satisfies condition (c_2), but (c_1) is not satisfied. In fact, if there exists a term $\phi(x,y)$ such that $F(\phi(x,y))=\{\circ\}$ and $\phi(x,y)=x$ belongs to $Id(\mathbb{G})$, then we put $y\circ y^{-1}$ for x and we get $y^k=y\circ y^{-1}$. However the last identity is not satisfied in a cyclic group of order k+1.

6. Equational bases for varieties $\mathbb{K}_{F1}^{F2}$.

Theorems 4, 5 gives a method of constructing an equational base of $\mathbb{K}_{F1}^{F2}$. Let B be an equational base of variety $\mathbb{K}$ and B_{F1} be an equational base of $\mathbb{K}_{F1}$, let $x\circ y$ be a fixed term in (c_1). We define a set B′ of identities as follows:

(6.1) identities (4.1),...,(4.8) belong to B′;

(6.2) if $\phi_1=\phi_2$ belongs to $B\cup B_{F1}$ and it is F_1-regular then it belongs to B′;

(6.3) if $\phi_1=\phi_2$ belongs to $B\cup B_{F1}$ and it is not F_1-regular then the identity $\phi_1\circ\phi_2=\phi_2\circ\phi_1$ belongs to B′;

(6.4) if $\phi_1=\phi_2$ belongs to B, $F(\phi_1)\cap F_2\neq\emptyset$ and $F(\phi_1)\subset F_1$ then the identity $\phi_1=\phi_2\circ\phi_1$ belongs to B′;

(6.5) if $\phi_1=\phi_2$ belongs to B and $F(\phi_1)\subset F_1$, $F(\phi_2)\cap F_2\neq\emptyset$ then the identity $\phi_1\circ\phi_2=\phi_2$ belongs to B′;

(6.6) if $\phi_1=\phi_2$ belongs to B and $F(\phi_1)\cap F_2\neq\emptyset$, $F(\phi_2)\cap F_2\neq\emptyset$ then the identity $\phi_1=\phi_2$ belongs to B′;

(6.7) B′ contains only identities defined by (6.1)-(6.6).

Theorem 6. If $\mathbb{K}$ satisfies (c_1) then set B′ is an equational base of $\mathbb{K}_{F1}^{F2}$.

Proof. Let V(B′) be the variety defined by B′. Since all identities from B′ are either F_1-regular or F_2-symmetrical and belong to Id($\mathbb{K}$), so $\mathbb{K}_{F1}^{F2}\subset V(B')$. Let $\mathfrak{U}$ belongs to V(B′). The function $\circ$ defined by (5.1) in $\mathfrak{U}$ satisfies (4.1)-(4.8) by (6.1), so by Theorem 3 $\mathfrak{U}$ is the sum of an upper system $\mathcal{A}(\circ)$ of algebras constructed as in (4.12). By (6.2), (6.3), $B_{F1}\subset\subset Id(\mathfrak{U}_i)$, for each $i\in I$ and by (6.2)-(6.6), $B\subset Id(\mathfrak{U}_u)$. So $\mathfrak{U}_i\in\mathbb{K}_{F1}$ for $i\neq u$ and $\mathfrak{U}_u\in\mathbb{K}$. By Theorem 4, $\mathfrak{U}\in\mathbb{K}_{F1}^{F2}$. Thus $V(B')\subset\mathbb{K}_{F1}^{F2}$, what completes the proof.

Corollary 2. If a variety $\mathbb{K}$ of type τ satisfies (c_1), $|F|<\chi_0$, $|B|<\chi_0$ and $|B_{F1}|<\chi_0$ then $\mathbb{K}_{F1}^{F2}$ is finitely based.

Example 3. Consider the variety $\mathbb{B}$ of all Boolean algebras of type $\tau:F\longrightarrow N$, where $F=\{+,\circ,'\}$, $\tau(+)=\tau(\circ)=2$ and $\tau(')=1$. Put $F_1=\{+,\circ\}$, $F_2=\{'\}$. Then $\mathbb{B}_{F1}$ is the variety of distributive lattices. Denote $\phi(x,y)=x+(x\circ y)$. By Theorem 4, each algebra from $\mathbb{B}_{F1}^{F2}$ is the sum of an upper (F_1,F_2)-system of algebras, where $\mathfrak{U}_i$ are distributive lattices for $i\neq u$ and $\mathfrak{U}$ is Boolean algebra. By Corollary 2. the variety $\mathbb{B}_{F1}^{F2}$ is finitely based.

Example 4. Consider the variety $\mathbb{P}$ of rings of type $\tau:F\longrightarrow N$ where $F=\{+,\circ,-\}$, $\tau(\circ)=\tau(+)=2$ and $\tau(-)=1$. Put $F_1=\{+,-\}$, $F_2=\{\circ\}$. Then $\mathbb{P}_{F1}$ is the variety of abelian groups. We can accept $\phi(x,y)=x+(y+(-y))$. So each $\mathfrak{U}\in\mathbb{P}_{F1}^{F2}$ is the sum of an upper (F_1,F_2)-system group and $\mathfrak{U}_u$ belongs to $\mathbb{P}$. If $\mathbb{P}$ is finitely based then $\mathbb{P}_{F1}^{F2}$ is finitely based.

Let $\mathbb{K}$ be a variety of type τ satisfying (c_2) for a fixed term $\psi(x)$. Let sets B, $B(F_2)$ be egational bases of $\mathbb{K}$ and $\mathbb{K}(F_2)$, respectively. We define a set B'' of identities as follows:

(6.8) the identity $\psi(f(x_0,..,x_{\tau(f)-1}))=f(\psi(x_0),...,\psi(x_{\tau(f)-1}))$ belongs to B'' for each $f\in F$;

(6.9) the identity $\psi(f(x_0,..,x_{\tau(f)-1})) = f(x_0,..,\psi(x_i),...$ $...,x_{\tau(f)-1})$ belongs to B'', for $f\in F$, $i\in\{0,...,\tau(f)-1\}$;

(6.10) if $\phi_1=\phi_2$ belongs to B then the identity $\psi(\phi_1)=\psi(\phi_2)$ belongs to B'';

(6.11) if $\phi_1=\phi_2$ belongs to $B(F_2)$ and $F(\phi_1),F(\phi_2)\subset F_1$, then $\phi_1=\phi_2$ belongs to B'';

(6.12) if $\phi_1=\phi_2$ belongs to $\mathbb{B}(F_2)$, $F(\phi_1)\cap F_2\neq\emptyset$, $F(\phi_2)\cap F_2\neq\emptyset$ then $\psi(\phi_1)=\phi_1$, $\psi(\phi_2)=\phi_2$ belong B'';

(6.13) B'' contains only identities defined by (6.8)-(6.12).

Theorem 7. If a variety $\mathbb{K}$ satisfies (c_2) then the set B'' is an equational base of $\mathbb{K}_{F1}^{F2}$.

Proof. Let $V(B'')$ be the variety defined by B''. Since the identities from B'' are either F_1-regular or F_2-symmetrical and belong to $Id(\mathbb{K})$, so $\mathbb{K}_{F1}^{F2}\subset V(B')$. Let $\mathfrak{U}\in V(B'')$. Then the relations R1, R2 defined as in (5.2), (5.3) are congruences of $\mathfrak{U}$ and $R1\cap R2=\omega$ by (6.8), (6.9). Hence $\mathfrak{U}$ is isomorphic to a subdirect product of $\mathfrak{U}/R1$ and $\mathfrak{U}/R2$. By (6.10)-(6.12), $(\mathfrak{U}/R1)\in\mathbb{K}$ and $(\mathfrak{U}/R2)\in\mathbb{K}(F_2)$, so $\mathfrak{U}\in\mathbb{K}_{F1}^{F2}$ by Theorem 5. Thus $V(B'')=\mathbb{K}_{F1}^{F2}$.

Corollary 3. If $\mathbb{K}$ satisfies (c_1), $|F|<\chi_0$, $|B|<\chi_0$, $|B(F_2)|<\chi_0$ then $\mathbb{K}_{F1}^{F2}$ is finitely based.

Corollary 4. If $\mathbb{K}$ satisfies (c_1) then $\mathbb{K}_{F1}^{F2}$ covers $\mathbb{K}$.

In fact, the variety $V(R_{F1}\cup S_{F2})$ defined by all F_1-regular and all F_2-symmetrical identities of type τ is equationally complete by Theorem 2. If $\mathfrak{U}=\mathcal{G}(\mathcal{A}(\circ))$ belongs to $\mathbb{K}_{F1}^{F2}$ and $|I|=1$ then $\mathfrak{U}\in\mathbb{K}$. If $|I|\geq 2$ then the relation R from (4.9) satisfies assumptions from [15]. Thus by [15] we get the statement.

REFERENCES

[1] **G. Birkhoff:** On the structure of abstract algebras, Proc. Cambr. Soc. 31(1935), 433-454.

[2] **G. Birkhoff:** Subdirect unions in universal algebra, Bull. Amer. Math. Soc. 50(1944), 764-765.

[3] **S. Bunis, H.P. Sankappanavar:** A course in universal algebra, Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[4] **E. Graczyńska:** On regular identities, Algebra Universalis, 17(1983), 369-375.

[5] **K. Halkowska:** On some operator defined on equational classes of algebras, Arch. Matech., 4. Scripta Fal. Sci. Nat. Ujeb. Brunenisis (1976), 209-212.

[6] **R. John:** On classes of algebras definable by regular equations, Colloq. Math. 36(1976), 17-21.

[7] **B. Johnsson, E. Nelson:** Relatively free products in regular varieties, Algebra Universalis 4(1974), 14-19.

[8] **H. Lakser, R. Padmanabhan, C.R. Platt:** Subdirect decompositions of Płonka Sum, Duke Math. I, vol. 39, 3(1972), 485-488.

[9] **R. Padmanabhan:** Regular identities in lattices, Trans. Amer. Math. Soc. 158(1974), 179-188.

[10] **J. Płonka:** On a method of construction of abstract algebras, Fund. Math. 61(1967), 183-191.

[11] **J. Płonka:** On equational classes of abstract algebras defined by regular equations, Fund. Math. 64(1969), 241.

[12] **J. Płonka:** On the subdirect product of some equational classes of algebras, Math. Nachri. 63(1974), 303-305.

[13] **J. Płonka:** On the sum of ι-semilattice ordered system of algebras, Studia Scient. Math., Hungarica 20(1985), 301-307.

[14] **J. Płonka:** On the methods of representation of some algebras, Colloq. Math. 16(1976), 249-251.

[15] **J. Płonka, J. Dudek:** On covering in lattices of varieties of algebras, Bull. Acad. Pol. Sci. (Ser. Matech.), vol. 31, No 1-2, (1983), 1-4.

[16] **A. Tarski:** Equational logic and equational theories of algebras in constructions to mathematical logic, Coll. Hannover, 1966, North-Holland, Amsterdam, (1968), 275-288.

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