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CONGRUENCES ON MAXIMAL PARTIAL CLONES AND STRONG REGULAR VARIETIES GENERATED BY PREPRIMAL PARTIAL ALGEBRAS II

Primality and preprimality for finite partial algebras may be defined in a similar way as in the case of total algebras ([7]). A universal primality criterion for partial algebras reduces to finding the complete list of preprimal partial algebras. All preprimal partial algebras are given in [4] ([5], [6]).

In this paper we consider strong regular varieties generated by preprimal partial algebras and its subvarieties. Similarly as in the total case the subvarieties of the strong regular variety $V(A)$ generated by a partial algebra $A=(A;F)$ correspond to arity congruences of the clone $(T(A);*,\zeta,\tau,\Delta,e_1^2)$ generated by the set F of partial functions ([1]). If A is a primal partial algebra, then $V(A)$ has no nontrivial subvarieties ([2], [3]). For some classes of preprimal partial algebras A we determined all congruences of the clones of its term functions and in this way all nontrivial subvarieties of the strong regular variety $V(A)$ in the first part of this paper. In the second part we will give a complete survey on the subvarieties of $V(A)$ for any preprimal partial algebra A.

1. Preliminaries

Let $P_A^{(n)}$ be the set of all n-ary partial functions defined on the finite set A ($A=\{0,1,\dots,k-1\}$, $k>2$) and let $O_A^{(n)}$ be the set of all total n-ary functions on A. We set $P_A = \bigcup_{n=1}^{\infty} P_A^{(n)}$ and $O_A = \bigcup_{n=1}^{\infty} O_A^{(n)}$. An n-ary function is denoted by f^n. $*$, ζ, τ, Δ are symbols for a binary and three unary operations on P_A. For the definition of $*$, ζ, τ, Δ see [2].

Every subalgebra of the algebra $P_A = (P_A;*,\zeta,\tau,\Delta,e_1^2)$ is called a clone. (Remark that e_i^n denotes the n-ary total projection on the i-th component). For $f,g\in P_A^{(n)}$, g is called a subfunction of f, symbolically $g\le f$, if $D(g)\subseteq D(f)$ and if $f_{/D(g)}=g$, where $D(g)$ denotes the domain of g. A clone $C\subseteq P_A$ is strong if it is closed under taking subfunctions. Let $\rho\subseteq A^h$ be a h-ary relation ($h\ge 1$). A function f^n preserves ρ if for every $h\times n$ matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{h1} & \cdots & a_{hn} \end{pmatrix}$$

whose columns

$$\begin{pmatrix} a_{1j} \\ \vdots \\ a_{hj} \end{pmatrix} \in \rho \quad (j=1,\ldots,n)$$

and whose rows $(a_{i1},\ldots,a_{in})\in D(f)$ $(i=1,\ldots,h)$ we have

$$\begin{pmatrix} f(a_{11} & \cdots & a_{1n}) \\ & \vdots & \\ f(a_{h1} & \cdots & a_{hn}) \end{pmatrix} \in \rho .$$

The set POL ρ of all $f\in P_A$ which preserve ρ is a strong partial clone.

Let E_h be the set of all equivalence relations on the set $h = \{0,1,\ldots,h-1\}$. For $\varepsilon\subseteq E_h$ we set $\delta_\varepsilon = \{(x_0,\ldots,x_{h-1})\in \in A^h \mid (i,j)\in\varepsilon \Rightarrow x_i=x_j\}$. ρ is strongly reflexive if for each equivalence ε from E_h with $\varepsilon\neq\varepsilon_0$ (ε_0 is the zero equivalence) there exists an equivalence $\varepsilon'\supseteq\varepsilon$ such that $\rho\cap\delta_\varepsilon=\delta_{\varepsilon'}$, and if for $h=2$, $\rho\neq A^2$ and $\rho\neq\{(x,x) \mid x\in A\}$. The relation ρ is said to be areflexive if $\rho\cap\delta_\varepsilon=\emptyset$ for each $\varepsilon\in E_h$ with $\varepsilon\neq\varepsilon_0$. In [6] (see also [4]) all maximal partial clones of functions on a finite set A, $|A|\ge 3$ were determined. Let M be maximal partial clone on A. Then we have exactly the following cases:

(1) If M is not strong, then $M = O_A \cup \{o^n \mid n \in \mathbb{N}\}$, where o^n is the n-ary partial function with empty domain $(D(o^n)=\emptyset)$ and $\mathbb{N}$ is the set of all positive integers.

(2) If M is strong, then M=POL ρ, where ρ is a relation with the following property:

(i) a unary relation (h=1)

(ii) a h-ary strongly reflexive relation with $h \geq 2$.

(iii) a h-ary areflexive relation with $h \geq 2$.

Every clone $C \subseteq P_A$ has at least three congruences κ_0, κ_a, κ_1 defined by

$(f,g) \in \kappa_0 \;:\Longleftrightarrow\; \{f,g\} \subseteq C \wedge f=g$,

$(f,g) \in \kappa_a \;:\Longleftrightarrow\; \{f,g\} \subseteq C \wedge arf=arg$

(arf denotes the arity of the function f),

$(f,g) \in \kappa_1 \;:\Longleftrightarrow\; \{f,g\} \subseteq C$.

Every congruence κ with $\kappa \subseteq \kappa_a$ is called arity congruence.
If $\{o^n \mid n \in N\} \subseteq C$, then $\kappa_\emptyset$ defined by

$$(f,g) \in \kappa_\emptyset \;:\Longleftrightarrow\; \{f,g\} \subseteq C \wedge (f=g \vee \{f,g\} \subseteq \{o^n \mid n \in \mathbb{N}\})$$

is a congruence of C.

Since every maximal clone contains the set $\{o^n \mid n \in \mathbb{N}\}$, every maximal clone has the four congruences κ_0, κ_a, κ_1, $\kappa_\emptyset$. In the first part we showed:

Lemma 1.1. Let C be a maximal clone. If C is not strong or if M=POL ρ where ρ is an h-ary strongly reflexive relation with $h \geq 2$, then C has exactly the four congruences κ_0, κ_a, κ_1, $\kappa_\emptyset$. ∎

Let ρ be an h-ary relation on A. Consider the following relations on POL ρ:

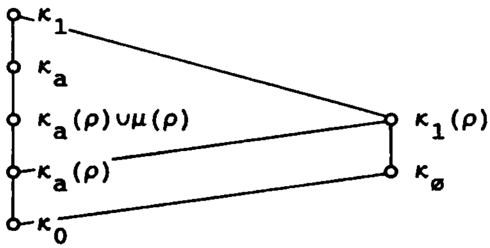
$$\kappa_1(\rho) := \{(f,g) \in \kappa_1 \mid f=g \vee \forall r_1,\ldots,r_{\max(arf,arg)} \in \rho : (r_1,\ldots,r_{arf}) \notin D(f) \wedge (r_1,\ldots,r_{arg}) \notin D(g)\},$$

$$\kappa_a(\rho) := \kappa_1(\rho) \cap \kappa_a,$$

$\kappa(\rho) \cup \mu(\rho)$ with
$$\mu(\rho) := \{(f,g) \in \kappa_a \mid \forall r_1,\ldots,r_{arf} \in \rho : ((r_1,\ldots,r_{arf}) \notin D(f) \wedge (r_1,\ldots,r_{arf}) \notin D(g)) \vee f(r_1,\ldots,r_{arf}) = g(r_1,\ldots,r_{arf})\}.$$

In [7] we proved:

Lemma 1.2. Let C be a maximal clone of the form C=POL ρ with $\emptyset\subset\rho\subset A$. Then C has exactly the following pairwise different congruences κ_0, κ_a, κ_1, $\kappa_\emptyset$, $\kappa_1(\rho)$, $\kappa_a(\rho)$, $\kappa_a(\rho)\cup\mu(\rho)$. Con POL ρ can be given by the following diagram



2. Some properties of maximal partial clones of the form C=POL ρ, where ρ is an h-ary areflexive relation with $h\geq2$

In this section let ρ be an arbitrary h-ary areflexive relation (i.e. $|\{a_0,\ldots,a_{h-1}\}| = h$ for every $(a_0,\ldots,a_{h-1})\in\rho$) with the property that POL ρ is a maximal partial clone and $h\geq2$. In [5] Haddad proved the following properties:

1.) Without loss of generality we can choose ρ such that $(0,1,\ldots,h-1)\in\rho$.

2.) Let S_h be the full symmetric group on $\{0,1,\ldots,h-1\}$. For every permutation $\pi\in S_h$, let

$$\rho^{(\pi)} := \{(a_{\pi(0)},\ldots,a_{\pi(h-1)}) \mid (a_0,\ldots,a_{h-1})\in\rho\} \quad \text{and}$$

$$G_\rho := \{\pi\in S_h \mid \rho\cap\rho^{(\pi)}\neq\emptyset\}.$$

We say that the relation ρ is symmetric under π if $\rho=\rho^{(\pi)}$ and asymmetric under π if $\rho\cap\rho^{(\pi)}=\emptyset$.

Then we have

(a) G_ρ is a subgroup of S_h.

(b) ρ is symmetric under each $\pi\in G_\rho (\forall\pi\in G_\rho : \rho=\rho^{(\pi)})$.

(c) ρ is asymmetric under each $\pi\in S_h\setminus G_\rho$

$$(\forall\pi\in S_h\setminus G_\rho : \rho\cap\rho^{(\pi)}=\emptyset).$$

3.) Under the model of ρ we understand the h-ary relation $\gamma_\rho := \{(\pi(0),\ldots,\pi(h-1)) \mid \pi\in G_\rho\}$. Note that $\gamma_\rho=\rho_{/h}$. Then there exists a surjective function $\varphi : A \longrightarrow h$, $\varphi\in O_A\cap\mathrm{POL}\,\rho$, which is a relational homomorphism from ρ to γ_ρ, i.e. for every $(a_0,\ldots,a_{h-1})\in\rho$ we have $(\varphi(a_0),\ldots,\varphi(a_{h-1}))\in\gamma_\rho$.

Then we obtain the following properties of the relation ρ:

Lemma 2.1. Let $\iota_h := \{(a_0,\ldots,a_{h-1})\in A^h \mid |\{a_0,\ldots,a_{h-1}\}|\leq \leq h-1\}$. Then we have $\rho\setminus\iota_h\neq\emptyset$ and for all $(a_0,\ldots,a_{h-1})\in\rho$ there exists a function $h_a\in\mathrm{POL}(\rho\cap O_A)$ such that $\mathrm{Im}\,h_a\subseteq\{a_0,\ldots \ldots,a_{h-1}\}$ (Im h_a denotes the image of h_a).

Proof. For an arbitrary $a := (a_0,\ldots,a_{h-1})\in\rho$ we consider the function $\varphi_a\colon h \longrightarrow \{a_0,\ldots,a_{h-1}\}$ with $\varphi_a(i) = a_i$ for all $i\in h$.

We define $h_a := \varphi_a*\varphi$, where $\varphi\colon A \longrightarrow h$ is the relational homomorphism considered in 3.). Then we have $\mathrm{Im}\,h_a = \{a_0,\ldots \ldots,a_{h-1}\}$ and $h_a\in\mathrm{POL}\,\rho$ since by 3.) for every $b := (b_0,\ldots \ldots,b_{h-1})\in\rho$ there exists a permutation $\pi_b\in G_\rho$ with $\varphi(b_0,\ldots \ldots b_{n-1}) = (\pi_b(0),\ldots,\pi_b(h-1))$ and therefore we get $h_a(b_0,\ldots \ldots,b_{h-1}) = \varphi_a(\varphi(b_0,\ldots,b_{h-1})) = \varphi_a(\pi_b(0),\ldots,\pi_b(h-1)) = = (a_{\pi_b(0)},\ldots,a_{\pi_b(h-1)})\in\rho$ by 2.), (b). ∎

Lemma 2.2. For all h-tuples $r_i=(r_{1i},\ldots,r_{hi})$ $(i=1,\ldots,n)$ there exist unary functions $h_1,\ldots,h_n$ on A with $h_1,\ldots,h_n\in \in\mathrm{POL}\,\rho\cap O_A$ such that

$$\{(h_1(j),\ldots,h_n(j)) \mid j\in A\} = \{(r_{j1},\ldots,r_{jn}) \mid j\in\{1,2,\ldots,h\}\}.$$

Proof. We define $h_i := h_{r_i}$ with $r_i = (r_{1i},\ldots,r_{hi})$ $(i=1,\ldots,n)$ and obtain our result. ∎

3. Congruences on maximal clones of the form C=POL ρ, where ρ is an areflexive at least binary relation

Consider the following relations: κ_0, κ_1, κ_a, $\kappa_\emptyset$, $\kappa_1(\rho)$, $\kappa_a(\rho)$, $\kappa_1(\rho)\cup\mu(\rho)$, $\kappa_a(\rho)\cup\mu(\rho)$, $\kappa_1(\rho)\cup\mu_0(\rho)$ with

$\mu_0(\rho) := \{(f,g)\in\kappa_a \mid \forall a\in(A\setminus U)^{\mathrm{arf}} : f(a)=g(a)\}$ and

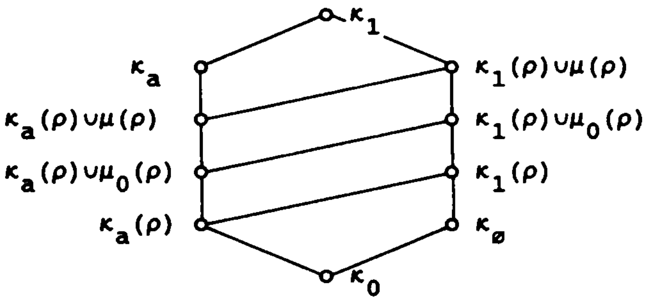
$U := \{\alpha\in A \mid \forall(c_1,\ldots,c_h)\in\rho : \alpha\notin\{c_1,\ldots,c_h\}\}$ and

$\kappa_a(\rho) \cup \mu_0(\rho)$.

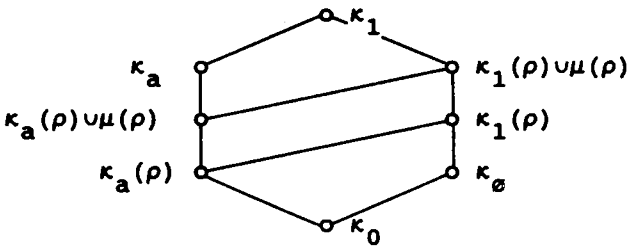
(Remark that for $\emptyset \subset \rho \subset A$ we have $A \setminus U = \rho$, therefore $\kappa_a(\rho) \cup \cup \mu(\rho) = \kappa_a(\rho) \cup \mu_0(\rho)$ and further in this case we have $\kappa_1(\rho) \cup \cup \mu(\rho) = \kappa_1(\rho) \cup \mu_0(\rho) = \kappa_1(\rho)$.)

The main theorem of this section is the following

Theorem 3.1. If ρ be an areflexive h-ary relation on A with $h \geq 2$, then the congruence lattice of the clone $C = POL\ \rho$ is given by



if $U \neq \emptyset$ and by



if $U = \emptyset$.

The proof of Theorem 3.1 will be given in the form of five lemmas:

Lemma 3.2. Let κ be a congruence on POL ρ. Then we have:

a) $\kappa_0 \subset \kappa \leq \kappa_a \Longrightarrow \kappa_a(\rho) \leq \kappa$,

b) $\kappa \not\leq \kappa_a \Longrightarrow \kappa_\emptyset \leq \kappa$,

c) $\kappa_\emptyset \subset \kappa \Longrightarrow \kappa_1(\rho) \leq \kappa$.

Proof. a) If $\kappa_0 \subset \kappa \leq \kappa_a$, then there are functions f^n, g^n with $(f^n, g^n) \in \kappa$ and elements $\tilde{a} := (a_1, \ldots, a_n) \in A^n$, $a, b \in A$, $a \neq b$, such that $f(\tilde{a}) := a \neq b := g(\tilde{a})$. POL ρ contains functions t_α and $h_{\beta\gamma}$ with

$$t_\alpha(x,y) := \begin{cases} y \text{ if } x=\alpha \\ \text{not defined otherwise} \end{cases} \quad (\alpha\in A) \text{ and}$$

$$h_{\beta\gamma}(x) := \begin{cases} \gamma \text{ if } x=\beta \\ \text{not defined otherwise.} \end{cases}$$

If ρ is areflexive or $\{c_0,\dots,c_{k-1}\}\subseteq \mathrm{POL}\ \rho$, where c_i is the constant unary function with value $i\in A$, β and γ are arbitrary elements of A, otherwise $\beta\notin\{a \mid (a,a,\dots,a)\notin\rho\}$.

Therefore we get

$$f(h_{\beta a_1}(x),\dots,h_{\beta a_n}(x)) = h_{\beta a}(x)$$
$$\sim g(h_{\beta a_1}(x),\dots,h_{\beta a_n}(x) = h_{\beta b}(x) \quad (\kappa)$$

and further we obtain

$$t_a(h_{\beta a}(x),y) = t_\beta(x,y) \sim t_a(h_{\beta b}(x),y) = o^2(x,y) \quad (\kappa).$$

If u^m is an arbitrary function from POL ρ with the property:

$$\forall\ r_1,\dots,r_m\in\rho : (r_1,\dots,r_m)\notin D(u^m),$$

then the following function v^m is an element of POL ρ:

$$v^m(\tilde{x}) := \begin{cases} \beta \text{ if } u(\tilde{x})\in A \\ \text{not defined otherwise.} \end{cases}$$

Then we have

$$t_\beta(v(\tilde{x}),u(\tilde{x})) = u(\tilde{x}) \sim o^2(v(\tilde{x}),u(\tilde{x})) = o^m(\tilde{x}) \quad (\kappa),$$

i.e. $\kappa_a(\rho)\subseteq\kappa$.

b) If $\kappa\nsubseteq\kappa_a$ then there are two functions f^n, g^m, $n>m$ in POL ρ with $(f,g)\in\kappa$. Then we get

$$(\tau((\Delta^{n-2}f)*o^1))*o^1 = o^2 \sim (\tau((\Delta^{n-2}g)*o^1))*o^1 = o^1 \quad (\kappa).$$

It follows that $o^2*e_1^r = o^{r+1} \sim o^1*e_1^r = o^r$ (κ) for arbitrary $r\in\mathbb{N}$. Therefore, we have $\kappa_\emptyset\subseteq\kappa$.

c) For $\kappa_\emptyset\subset\kappa$ and $\kappa\cap\kappa_a\neq\kappa_0$ we have $\kappa_a(\rho)\subseteq\kappa$ by a) (since $\kappa\cap\kappa_a = \kappa' \neq \kappa_0$, $\kappa'\subseteq\kappa_a$ implies $\kappa_a(\rho)\subseteq\kappa'$ and because of $\kappa'\subseteq\kappa$ we get $\kappa_a(\rho)\subseteq\kappa$).

From $\kappa_a(\rho)\subseteq\kappa$ and $\kappa_\emptyset\subseteq\kappa$ it follows $\kappa_1(\rho)\subseteq\rho$, since, if $(f^n,g^m)\in\kappa_1(\rho)$ $(n\neq m)$, then we have $(f^n,o^n)\in\kappa$ and $(o^m,g^m)\in\kappa$ because of $\kappa_a(\rho)\subseteq\kappa$ and $(o^n,o^m)\in\kappa$ because of $\kappa_\emptyset\subseteq\kappa$. Transitivity gives $(f^n,g^m)\in\kappa$. If $\kappa_\emptyset\subset\kappa$ and $\kappa\cap\kappa_a = \kappa_0$, then there exist two functions s^l, t^r $(l\neq r)$ in POL ρ with $(s,t)\in\kappa$. We can assume that $l>r$ and that $s^l\neq o^l$, i.e. there exists an element $\tilde{a} := (a_1,\dots,a_l)$ with $s(\tilde{a})\in A$. We consider the following two cases:

Case 1. For all $a_i\in A$ we have $(a_i,\dots,a_i)\in\rho$. In this case $\{c_{a_1},\dots,c_{a_n}\}\subseteq$POL ρ. From $(s,t)\in\kappa$ and $\kappa_\emptyset\subseteq\kappa$ we get that two constant functions of different arity are congruent under κ. It follows that (POL $\rho\cap O_A)\times($POL $\rho\cap O_A)\subseteq\kappa$. This contradicts $\kappa\cap\kappa_a=\kappa_0$.

Case 2. There is an element $a_i\in A$ with $(a_i,\dots,a_i)\notin\rho$. Without loss of generality we assume that $(a_1,\dots,a_1)\notin\rho$. Then the function $h^1_{a_1a_j}$ belongs to POL ρ for any $a_j\in A$, $j=1,\dots n$. Moreover we can assume that $a_1\neq a_{l-r}$. (From $a_1=a_{l-r}$ we consider $s' := s*e^2_2$ and $t' := t*e^2_2$. Then we apply the operations ζ and τ and obtain functions s'', t'' with $(s'',t'')\in\kappa$, $s''(\tilde{a})\in A$, $(a''_1,\dots,a''_1)\notin\rho$ and $a''_1\neq a''_{l-r}$). Then we get

$$s_1:=(\dots(\zeta((\zeta((\zeta s)*h_{a_1a_l}))*h_{a_1a_{l-1}}))*h_{a_1a_{l-2}}))*\dots)*h_{a_1a_{l-r}}) \sim$$

$$t_1:=(\dots(\zeta((\zeta((\zeta t)*h_{a_1a_l}))*h_{a_1a_{l-1}}))*h_{a_1a_{l-2}}))*..)*h_{a_1a_{l-r}})(\kappa)$$

with $s_1(x_1,\dots,x_l) =$

$$s(x_{l-r+1},\dots,x_l,h_{a_1a_{l-r}}(x_1),\dots,h_{a_1a_l}(x_{l-r}))$$

and $t_1(x_1,\dots,x_r) =$

$$t(h_{a_1a_{l-r+1}}(x_2),\dots,h_{a_1a_{l-1}}(x_r),h_{a_1a_l}(h_{a_1a_{l-r}}(x_1))) = o^r(\tilde{x})$$

since $h_{a_1a_l}*h_{a_1a_{l-r}} = o^1$ and because of $a_1\neq a_{l-r}$.

From this argumentation and from $\kappa_\emptyset\subseteq\kappa$ it follows that $\kappa\cap\kappa_a\neq\kappa_0$ which contradicts the presumption. Therefore we have c). ■

Lemma 3.3. Let POL ρ be a maximal clone where ρ is an h-ary areflexive relation ($h\geq 2$). Then for every congruence κ on POL ρ we have

a) $\kappa\not\subseteq\kappa_a(\rho)\cup\mu(\rho) \wedge \kappa\subseteq\kappa_a \Longrightarrow \kappa=\kappa_a$,

b) $\kappa\not\subseteq\kappa_1(\rho)\cup\mu(\rho) \wedge \kappa\not\subseteq\kappa_a \Longrightarrow \kappa=\kappa_1$.

Proof. a) $\kappa\not\subseteq\kappa_a(\rho)\cup\mu(\rho)$ and $\kappa\subseteq\kappa_a$ means that there exist two functions f^n, g^n with $(f^n,g^n)\in\kappa$ and an n-tuple $\tilde{a} = (r_1,\dots,r_n)\in\rho^n$ with

$\tilde{a}\subseteq D(f)$, $\tilde{a}\subseteq D(g)$, $f(\tilde{a})\neq g(\tilde{a})$ or

$\tilde{a}\subseteq D(f)$ and $\tilde{a}\not\subseteq D(g)$.

The first case can be reduced to the second case. Let $\hat{e}_3^3$ be the ternary function from POL ρ defined by

$$\hat{e}_3^3(x,y,z) = \begin{cases} z & \text{if } x=y \\ \text{not defined otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} &\hat{e}_3^3(f(\tilde{x}),f(\tilde{x}),f(\tilde{x})) = f(\tilde{x}) \\ \sim\ &\hat{e}_3^3(f(\tilde{x}),g(\tilde{x}),f(\tilde{x})) = g'(\tilde{x}) \quad (\kappa) \end{aligned}$$

with $\tilde{a}\not\subseteq D(g')$.

Therefore, we can assume that $\tilde{a}\subseteq D(f)$ and $\tilde{a}\not\subseteq D(g)$. Let $(a_1,\dots,a_h)\in\rho\setminus\iota_h$. Then the functions $t_1,\dots,t_n$ with

$$t_i \begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} = r_i$$

and $t_i(x)$ is not defined if $x\notin\{a_1,\dots,a_h\}$ $(i=1,\dots,n)$ belongs to POL ρ. $(f,g)\in\kappa$ implies

$f(t_1(x),\dots,t_n(x)) =: f'(x)\sim g(t_1(x),\dots,t_n(x)) =: g'(x) \quad (\kappa)$,

where for every $r\in\rho$ $r\not\subseteq D(g')$ and $(f'(a_1),\dots,f'(a_h))\in\rho$. By Lemma 3.2 we have $\kappa_a(\rho)\subseteq\kappa$ and consequently $(g',o^1)\in\kappa$, and $(f',o^1)\in\kappa$. By Lemma 2.1 there exists a function $t^1\in O_A\cap\text{POL }\rho$ with $\text{Im}(t^1)\subseteq\{a_1,\dots,a_h\}$. Using this function and the fact that $f''=f'*t$ we obtain a further function from POL $\rho\cap O_A$

with $(f'',o^1)\in\kappa$. Then we get $(\Delta(e_2^2*f''),\Delta(e_2^2*o^1)) = (e_1^1,o^1)\in\kappa$ and $\kappa=\kappa_a$.

b) One can show that from $\kappa\not\subseteq\kappa_1(\rho)\cup\mu(\rho)$ and $\kappa\not\subseteq\kappa_a$ it follows $\kappa\cap\kappa_a \not\subseteq \kappa_a(\rho)\cup\mu(\rho)$. Then from a) and $\kappa_\emptyset\subseteq\kappa$ we obtain b).■

Lemma 3.4. Let $\mathrm{POL}\,\rho$ be a maximal clone where ρ is an areflexive relation and suppose that there exists an element $a\in U := \{\alpha\in A \mid \forall(a_1,\ldots,a_h)\in\rho : \alpha\notin\{a_1,\ldots,a_h\}\}$ $(U\neq\emptyset)$. Then for every congruence κ on $\mathrm{POL}\,\rho$ we have

$$\kappa \not\subseteq \kappa_a(\rho) \wedge \kappa\subseteq\kappa_a \Longrightarrow \kappa_a(\rho)\cup\mu_0(\rho)\subseteq\kappa.$$

Proof. Because of Lemma 3.2, a) we have to show that $\mu_0(\rho)\subseteq\kappa$. Let κ be a congruence on $\mathrm{POL}\,\rho$ with $\kappa\subseteq\kappa_a$ and $\kappa\not\subseteq\kappa_a(\rho)$. Then there are two different n-ary functions f, g with $(f,g)\in\kappa$ and a n-tuple $(r_1,\ldots,r_n)\in\rho^n$ such that $(r_1,\ldots,r_n)\subseteq D(f)$ or $(r_1,\ldots,r_n)\subseteq D(g)$.

Further we can assume that $\kappa\neq\kappa_a$. By Lemma 3.3 we have $\kappa\subseteq\kappa_a(\rho)\cup\mu(\rho)$, i.e. we have $f(r_1,\ldots,r_n) = g(r_1,\ldots,r_n)$. Since $f\neq g$ there exists an n-tuple $\tilde{a} := (a_1,\ldots,a_n)\in A^n$ with $f(\tilde{a})\neq g(\tilde{a})$. As shown in the proof of Lemma 3 3 we can assume that $\tilde{a}\notin D(g)$ and $\tilde{a}\in D(f)$.

Let $r := (\alpha_1,\ldots,\alpha_h)\in\rho\setminus\iota_h$ and $a\in U$. Then the functions $h_1,\ldots,h_n$ defined by $h_i(r)=r_i$, $h_i(a)=a_i$ and h_i not defined otherwise, belong to $\mathrm{POL}\,\rho$.

Then we get

$f(h_1(x),\ldots,h_n(x)) =: f'(x) \sim g(h_1(x),\ldots,h_n(x)) =: g'(x)\ (\kappa)$, where $f'(r) = f(h_1(r),\ldots,h_n(r)) = f(r_1,\ldots,r_n) = g(h_1(r),\ldots$ $\ldots,h_n(r)) = g'(r)$, $f'(a) = f(h_1(a),\ldots,h_n(a)) = f(a_1,\ldots,a_n)$ and $a\notin D(g')$. By Lemma 2.1, $\mathrm{POL}\,\rho$ does contain a function $t\in O_A$ with $\mathrm{Im}(t)\subseteq\{\alpha_1,\ldots,\alpha_h\}$. Then the function t' defined by

$$t'(x) := \begin{cases} t(x) & \text{for } x\notin U \\ a & \text{otherwise} \end{cases}$$

belongs to $\mathrm{POL}\,\rho$ and we get

$f'*t' =: f'' \sim g'' := g'*t'\ (\kappa)$ and

$e_2^2(f''(x),x) = e_1^1(x) \sim \check{e}_1^1(x) := e_2^2(g''(x),x)\ (\kappa)$ with

$$\check{e}_1^1(x) := \begin{cases} x & \text{if } x \notin U \\ \text{not defined if} & x \in U. \end{cases}$$

Let p^m be an arbitrary function from POL ρ and let

$$p_a(\tilde{x}) := \begin{cases} p(\tilde{x}) & \text{if } \tilde{x} \in (A\setminus U)^m \\ a & \text{otherwise} \end{cases} \quad \text{and}$$

$$p_\emptyset(\tilde{x}) := \begin{cases} p(\tilde{x}) & \text{if } \tilde{x} \in (A\setminus U)^m \\ \text{not defined otherwise.} \end{cases}$$

From $(e_1^1, \check{e}_1^1) \in \kappa$ we get $e_1^1 * p_a = p_a \sim \check{e}_1^1 * p_a = p_\emptyset(\kappa)$ and

$e_2^2(p_a(\tilde{x}), p(\tilde{x})) = p(\tilde{x}) \sim e_2^2(p_\emptyset(\tilde{x}), p(\tilde{x})) = p_\emptyset(\tilde{x})\ (\kappa)$.

We obtain that two m-ary functions p, q with $p(\tilde{x}) = q(\tilde{x})$ for $\tilde{x} \in (A\setminus U)^m$ are congruent under κ, i.e. $\mu_0(\rho) \subseteq \kappa$. ∎

Lemma 3.5. Let POL ρ be a maximal clone where ρ is an areflexive relation. Then for every congruence κ on POL ρ we have:

$$\kappa_1(\rho) \subseteq \kappa_1(\rho) \cup \mu_0(\rho) \subset \kappa \subseteq \kappa_1(\rho) \cup \mu(\rho) \Longrightarrow \kappa = \kappa_1(\rho) \cup \mu(\rho).$$

Proof. If for the congruence κ on POL ρ we have the relation $\mu_0(\rho) \subset \kappa \setminus \kappa_1(\rho) \subseteq \mu(\rho)$ then there are two functions f^n, g^n, $(f^n, g^n) \in \kappa$ such that for certain h-tuples $(r_{11}, \dots, r_{h1}), \dots, (r_{1n}, \dots, r_{hn}) \in \rho$, $(a_1, \dots, a_h) \in \rho$ there exist an n-tuple $(\alpha_1, \dots, \alpha_n) \in A^n$ and an element $\alpha \in A$ with $f(\alpha_1, \dots, \alpha_n) = \alpha$, $(\alpha_1, \dots, \alpha_n) \notin D(g)$ and

$$f\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ r_{h1} & \cdots & r_{hn} \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ a_h \end{pmatrix} = g\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ r_{h1} & \cdots & r_{hn} \end{pmatrix}.$$

By Lemma 2.2 there exist functions $h_1, \dots, h_n \in O_A \cap \mathrm{POL}\ \rho$ such that $\{(h_1(i), \dots, h_n(i)) \mid i \in A\} = \{(r_{j1}, r_{j2}, \dots, r_{jn}) \mid j \in \{1, 2, \dots, h\}\}$. Let t^m be an arbitrary function from POL ρ. Then for

$$D_t := \Big\{ \tilde{x} \in A^m \mid \exists\ \tilde{a}_1, \dots, \tilde{a}_h : \tilde{x} \in \{\tilde{a}_1, \dots, \tilde{a}_h\} \wedge$$

$$\wedge \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_h \end{pmatrix} \subseteq \rho \wedge t \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_h \end{pmatrix} \in \rho \Big\},$$

the functions t_i defined by

$$t_i(\tilde{x}) := \begin{cases} h_i(t(\tilde{x})) & \text{if } \tilde{x}\in D_t \\ \alpha_i \text{ if } \tilde{x}\notin D_t \wedge \tilde{x}\in D(t) & \\ \text{not defined, otherwise} & \end{cases}$$

$(i=1,2,\dots,n)$ belong to POL ρ and we have:

$u(f(t_1(\tilde{x}),\dots,t_n(\tilde{x})) = t(\tilde{x}) \sim t'(\tilde{x}) = u(g(t_1(\tilde{x}),\dots,t_n(\tilde{x}))$ (κ) with

$$u(x,y) := \begin{cases} y \text{ if } x\in\{a_1,\dots,a_h,\alpha\} \\ \text{not defined, otherwise,} \end{cases}$$

and

$$t'(x) := \begin{cases} t(\tilde{x}) & \text{if } \tilde{x}\in D_t \\ \text{not defined, otherwise.} & \end{cases}$$

Now it follows that $\mu(\rho)\subseteq\kappa$. ∎

Lemma 3.6. Let POL ρ be a maximal clone where ρ is an areflexive relation. Then for every congruence κ on POL ρ we have:

a) $\kappa_1(\rho) \subset \kappa \subseteq \kappa_1(\rho)\cup\mu_0(\rho) \Rightarrow \kappa = \kappa_1(\rho)\cup\mu_0(\rho)$

b) $\kappa_a(\rho)\cup\mu_0(\rho) \subset \kappa \subseteq \kappa_a(\rho)\cup\mu(\rho) \Rightarrow \kappa = \kappa_a(\rho)\cup\mu(\rho)$

c) $\kappa_a(\rho) \subset \kappa \subseteq \kappa_1(\rho) \Rightarrow \kappa = \kappa_1(\rho)$

$\kappa_a(\rho)\cup\mu_0(\rho) \subset \kappa \subseteq \kappa_1(\rho)\cup\mu_0(\rho) \Rightarrow \kappa = \kappa_1(\rho)\cup\mu_0(\rho)$

$\kappa_a(\rho)\cup\mu(\rho) \subset \kappa \subseteq \kappa_1(\rho)\cup\mu(\rho) \Rightarrow \kappa = \kappa_1(\rho)\cup\mu(\rho)$.

Proof. a) $\kappa_1(\rho) \subset \kappa \subseteq \kappa_1(\rho)\cup\mu_0(\rho)$ implies $\kappa_1(\rho)\cap\kappa_a \subseteq \kappa\cap\kappa_a \subseteq (\kappa_1(\rho)\cup\mu_0(\rho))\cap\kappa_a$ and further $\kappa_a(\rho) \subseteq \kappa\cap\kappa_a \subseteq \kappa_a(\rho)\cup\mu_0(\rho)$ because $\kappa_a(\rho) = \kappa_1(\rho)\cap\kappa_a$ and $\mu_0(\rho) \subset \kappa_a$.

Lemma 3.4 gives $\mu_0(\rho)\cup\kappa_a(\rho) \subseteq \kappa\cap\kappa_a$ and therefore $\kappa\cap\kappa_a = \kappa_a(\rho)\cup\mu_0(\rho)$. κ can be written as $\kappa = \kappa_1(\rho)\cup\mu_0'(\rho)$ with $\mu_0'(\rho) \subseteq \mu_0(\rho)$. Then we have $\kappa\cap\kappa_a = (\kappa_1(\rho)\cup\mu_0'(\rho))\cap\kappa_a = \kappa_a(\rho)\cup(\mu_0'(\rho)\cap\kappa_a) = \kappa_a(\rho)\cup\mu_0'(\rho) = \kappa_a(\rho)\cup\mu_0(\rho)$. Therefore $\mu_0'(\rho) = \mu_0(\rho)$ and $\kappa = \kappa_1(\rho)\cup\mu_0(\rho)$.

b) κ can be written in the form $\kappa = \kappa_a(\rho)\cup\mu'$ with $\mu_0(\rho) \subset \mu' \le \mu(\rho)$. Then we have $\kappa\cup\kappa_\emptyset = \kappa_a(\rho)\cup\mu'\cup\kappa_\emptyset = \kappa_1(\rho)\cup\mu'$ by Lemma 3.2 and $\kappa_1(\rho)\cup\mu_0(\rho) \le \kappa_1(\rho)\cup\mu' \le \kappa_1(\rho)\cup\mu(\rho)$. By Lemma 3.5 it follows $\kappa_1(\rho)\cup\mu' = \kappa_1(\rho)\cup\mu(\rho)$ or $\kappa_1(\rho)\cup\mu' = \kappa_1(\rho)\cup\mu_0(\rho)$. In the first case we get $\mu' = \mu(\rho)$, in the second case $\mu' = \mu_0(\rho)$. It follows $\kappa = \kappa_a(\rho)\cup\mu_0(\rho)$.

c) $\kappa_a(\rho)\subset\kappa\le\kappa_1(\rho)$ implies $\kappa\not\le\kappa_a$ and by Lemma 3.2 b) $\kappa_\emptyset\le\kappa$. $\kappa_\emptyset\le\kappa\le\kappa_1(\rho)$ implies $\kappa = \kappa_\emptyset$ or $\kappa = \kappa_1(\rho)$. The first case is impossible. In a similar way we obtain the other propositions. ■

4. Subvarieties of the strong regular varieties generated by a preprimal partial algebra

Let $A=(A;F)$ be a partial algebra of type τ and let $W(X)$ be the total term algebra of this type. Every term w induces an n-ary term function w_A of A ([2]).

We need the concept of a strong regular identity. Let w_1, w_2 be two terms which contain the same free variables. Then an equation $w_1=w_2$ is called a strong regular identity of A iff whenever one of the term functions w_{1A}, w_{2A} induced by w_1, w_2 is defined, then the other is defined, and, when both are defined, then they have the same values ([2]). Let Id A be the set of all strong regular identities of A. In [2] it was proved:

Lemma 4.1. Id A is a fully invariant congruence relation of $W(X)$. ■

We define the concept of a strong regular variety $V(A)$ defined by A as the class of all algebras B of the same type as A with Id B $\ge$ Id A.

Then we obtain

Lemma 4.2. If V is a strong regular subvariety of $V(A)$, then Id V is a fully invariant congruence relation of $F(X) := W(X)_{/\mathrm{Id}\ A}$.

Proof. Id V = $\bigcap_{B\in V}$Id B is a fully invariant congruence relation on $W(X)$. By the second isomorphism theorem $W(X)_{/\mathrm{Id}\ V}$ is a homomorphic image of $F(X) = W(X)_{/\mathrm{Id}\ A}$ and there exists a congruence relation θ on $F(X)$ such that $F(X)_{/\theta}$ is isomorphic to $W(X)_{/\mathrm{Id}\ V}$, namely $\theta = \mathrm{Id}\ V_{/\mathrm{Id}\ A}$ and θ is fully invariant. ∎

Further we have

Lemma 4.3 **([2]).** To every fully invariant congruence κ_F of $F(X)$ corresponds a congruence $\kappa \le \kappa_a$ of $T(A)$. ∎

From Lemma 4.2 and Lemma 4.3 it follows that an arity congruence of $T(A)$ is assigned to every strong regular subvariety of $V(A)$. Therefore for a preprimal algebra A, $V(A)$ has at most three strong regular subvarieties. Considering the congruences $\kappa_a(\rho)$, $\kappa_a(\rho) \cup \mu_0(\rho)$, $\kappa_a(\rho) \cup \mu(\rho)$ which lead to three fully invariant congruences κ_F of $F(X)$ we see that each of these fully invariant congruences corresponds to a strong regular subvariety of $V(A)$. Therefore we have:

Theorem 4.4. Let A be a preprimal partial algebra and let $V(A)$ be the strong regular variety generated by A. Then we have exactly the following cases for subvarieties of $V(A)$:

1. $V(A)$ has no nontrivial strong regular subvariety,
2. $V(A)$ has exactly two nontrivial strong regular subvarieties,
3. $V(A)$ has exactly three nontrivial strong regular subvarieties. ∎

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