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GROUP RELATED SYMMETRIC GROUPOIDS

Let R be a subset of an abelian group $(G,+)$, furthermore let $\cdot$ be a binary operation on R, such that $(R,\cdot)$ is a symmetric groupoid, i.e. the following identities are assumed to be satisfied:

(i) $$a\cdot a=a \quad \forall a\in R,$$

(ii) $$a\cdot(a\cdot b)=b \quad \forall a,b\in R,$$

(iii) $$(a\cdot a)\cdot c=a\cdot(b\cdot(a\cdot c)) \quad \forall a,b,c\in R.$$

In addition, we suppose a relationship between the binary operation of the group and the symmetric groupoid, given by $a\cdot b=\sigma(a)+\tau(b)$, $a,b\in R$, where $\sigma,\tau:R\longrightarrow G$ are mappings. Whenever there is such a relationship between the group structure $(G,+)$ and the structure of the symmetric groupoid $(R,\cdot)$, we shall write $(R,\cdot)\le(G,+)$, and call R a group related symmetric groupoid.

Group related symmetric groupoids arise in a natural way in algebraic topology. Let $S^k, k\in\mathbb{N}$, denote the k-sphere, $[S^m;S^n]$ the set of homotopy classes of continuous, basepoint preserving mappings $S^m\longrightarrow S^n$. It is well known that $[S^m;S^n]$ is the underlying set for an abelian group, mostly written by $\pi_m(S^n)$ and called the m-th homotopy group of S^n. For $x,y\in S^n$, we put

$$x\cdot y:=-y+2\langle x;y\rangle x\in S^n,$$

where $\langle\cdot;\cdot\rangle$ stands for the ordinary scalar product in $\mathbb{R}^n$. This binary operation on S^n is called reflecting product and

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makes S^n a symmetric groupoid. For mappings $f,g:S^m \to S^n$ we now define $(f\cdot g)(x):=f(x)\cdot g(x)$, $x\in S^m$. Since this binary operation is well-behaved with respect to homotopy, we get the structure of a symmetric groupoid on $[S^m;S^n]$. It turns out (cf. [1], (3.7), (3.9)), that the group structure and the structure coming from the reflecting product on $[S^m;S^n]$ are related in the manner described above. In the following we shall investigate group related symmetric groupoids from an algebraic point of view and find some criteria for isomorphism.

1. Proposition. Let $(R,\cdot)\leq(G,+)$. Then we have

(i) $$\sigma(\cdot)=\mathrm{id}-\tau(\cdot),$$

(ii) $$\tau(a-\tau(a)+\tau(b))=\tau(a)-a+b \quad \forall a,b\in R.$$

Proof. Simple calculation, using axioms (i) and (ii). ∎

Since in R the binary operation $\cdot$ can be expressed in terms of τ, we shall sometimes write (R,τ) instead of $(R,\cdot)$.

A kind of converse of Prop. 1 is the following

2. Remark. Given a subset R of G, and a mapping $\tau:R\to G$ satisfying $a-\tau(a)+\tau(b)\in R$ and $\tau(a-\tau(a)+\tau(b))=\tau(a)-a+b$ for $a,b\in R$, we can define a group related symmetric groupoid by $\sigma(a):=a-\tau(a)$, $a\cdot b:=\sigma(a)+\tau(b)$.

3. Definition. Isomorphisms between symmetric groupoids are defined in the usual way; for isomorphic symmetric groupoids $(R,\cdot)$ and $(R',\cdot')$ we write $(R,\cdot)\cong(R'\cdot')$.

In particular, let $(R,\cdot)\leq(G,+)$, $(R',\cdot')\leq(G,+)$ with $0\in R,R'$, where 0 denotes the neutral element of the group, and let λ be an isomorphism between $(R,\cdot)$ and $(R',\cdot')$. If, in addition, $\lambda(0)=0$, we call λ a distinguished isomorphism between symmetric groupoids.

Analogously we can define similar concepts coming from group theory, e.g. (distinguished) epimorphism.

Our first aim will be to simplify the conditions for the mapping τ. By the following proposition we get in a simple manner isomorphic copies of symmetric groupoids:

4. Proposition. Let $(R,\cdot)$ be a symmetric groupoid, $R\subseteq G$, with $(G,+)$ being an abelian group, $g\in G$. Let $R':=R+g$, and for $a',b'\in R'$ define $a'\cdot' b':=(a'-g)\cdot(b'-g)+g$. Then $(R',\cdot')\cong$ $\cong(R,\cdot)$.

Proof. The isomorphism $\lambda:R'\longrightarrow R$ is defined by $\lambda(a'):=$ $:=a'-g$, $a'\in R'$. ∎

5. Corollary. Let $(R,\cdot)\leq(G,+)$, $R':=R+g$ for $g\in G$ and a binary operation $\cdot'$ on the set R' be defined as in Prop. 4. Then there is a mapping $\tau':R'\longrightarrow G$ with

$$a'\cdot' b'=a'-\tau'(a')+\tau'(b'),\ a',b'\in R';$$

in other words, $(R',\cdot')$ is group related.

Proof. Let $\tau'(a'):=\tau(a'-g)$. ∎

6. Proposition. Let $(R,\tau)\leq(G,+)$, $0\in R$. Then there is a bijective map $\tau^*:R\longrightarrow R$ with $(R,\tau^*)=(R,\tau)$, $\tau^*(0)=0$ and $\tau^{*2}=\mathrm{id}$.

Proof. Let $a,b\in R$. Suppose $0-\tau(0)=:c\neq 0$.
If we let $\tau^*(a):=0\cdot a=c+\tau(a)$, then trivially follows $\tau^*(0)=0$, and we have

$$a\cdot b = a-\tau(a)+\tau(b) = a-(\tau(a)+c)+(\tau(b)+c) = a-\tau^*(a)+\tau^*(b).$$

It is justified to write (R,τ^*), since both τ^* and τ describe the binary operation $\cdot$ on R, i.e. the equation $\tau^*(a-\tau^*(a)+$ $+\tau^*(b))=\tau^*(a)-a+b$ holds. For $a=0$, $b\in R$ we get $\tau^*(\tau^*(b))=b$, and therefore $\tau^{*2}=\mathrm{id}$. Since $\tau^*(a)=0\cdot a$, the mapping τ^* has R as range.

From now on, because of Cor. 5 and Prop. 6, we always assume $0\in R$, $\tau(0)=0$, and $\tau^2=\mathrm{id}$.

Group related symmetric groupoids satisfy some useful necessary conditions, which we shall gather in the next theorem.

7. Theorem. Let $(G,+)$ be an abelian group, $(R,\cdot)\leq(G,+)$, $a,b\in R$. Then $(R,\cdot)$ is entropic, right distributive and balanced, i.e. $a\cdot b=b$ implies $b\cdot a=a$. In addition, we have the identity

$$a\cdot b+b\cdot a=a+b\cdot$$

Proof. Let $a,b,x,y\in R$. As an example, we show the validity of the entropic law

$$(a\cdot x)\cdot(y\cdot b) = (a\cdot y)\cdot(x\cdot b).$$

Using τ to express the binary operation, we get $(a\cdot x)\cdot(y\cdot b)=$ $=2(a-\tau(a))-(x-\tau(x))-(y-\tau(y))+b$, and the assertion follows, since $(G,+)$ is abelian. The rest is shown by similar computations. ∎

For balanced symmetric groupoids, see e.g. [2]. Group related symmetric groupoids come under the concept of so called SIE-groupoids [3] (which means, that the binary operation satisfies axioms (i) and (ii) from symmetric groupoids, as well as the entropic law).

Symmetric groupoids $(R,\cdot)\leq(G,+)$ with $0\in R$ and $(G,+)$ abelian may be described by means of a certain subset U of R, defined as follows.

8. Definition. Let (R,τ) be a group related symmetric groupoid. We define $U:=\{u\in R,\ u=r-\tau(r),\ r\in R\}$. It can easily be seen, that $U=R\cdot 0=\sigma(R)$.

We list the relevant properties of U, and of the subgroup $\langle U\rangle$ of G generated by U, in the following

9. Proposition. Let $(R,\cdot)\leq(G,+)$, $0\in R$, $(G,+)$ be abelian, and U as in Def. 8. Then the following holds

(i) $$\tau_{|U}=-\mathrm{id}.$$

(ii) $$r\in R \Longrightarrow r+\langle U\rangle\subseteq R,$$

$$r\in R,\ u\in\langle U\rangle \Longrightarrow \tau(r+u)=\tau(r)+\tau(u).$$

In particular, $\langle U\rangle\subseteq R$.

(iii) $$\tau_{|\langle U\rangle}=-\mathrm{id},$$

$$u,v\in\langle U\rangle \Longrightarrow u\cdot v=2u-v.$$

(iv) $$u\in U \Longrightarrow u+2\langle U\rangle\subseteq U.$$

In particular, $2\langle U\rangle\subseteq U$.

Proof. (i) Let $u\in U$. Then $\exists r\in R$ with

$$\begin{aligned}\tau(u)&=\tau(r-\tau(r))\\&=\tau(r-\tau(r)+\tau(0))\\&=\tau(r)-r=-u.\end{aligned}$$

(ii) Let $r\in R$, $u_1\in U$. We first show $r+u_1\in R$. By definition of U resp. by bijectivity of τ there are $r', r_1 \in R$ with $u_1=r_1-\tau(r_1)$, $\tau(r')=r$, and we calculate

$$r+u_1=\tau(r')+r_1-\tau(r_1)=r_1\cdot r' \in R.$$

But $r+u_1 \in R$ implies $r+u \in R$ for $u \in \langle U\rangle$, and in particular $\langle U\rangle \subseteq R$. In order to prove

$$r\in R,\ u\in\langle U\rangle \implies \tau(r+u)=\tau(r)+\tau(u),$$

it is again sufficient to consider $u\in U$. As above, there exist $r_1, r' \in R$ with $u=r_1-\tau(r_1)$, $\tau(r')=r$, and consequently,

$$\begin{aligned}\tau(r+u)&=\tau(r_1-\tau(r_1)+\tau(r'))\\ &=\tau(r_1)-r_1+r'\\ &=\tau(r)-u\\ &=\tau(r)+\tau(u).\end{aligned}$$

(iii) The validity of $\tau_{|\langle U\rangle}=-\mathrm{id}$ is ensured by the last equation in the proof of (ii). For $u,v\in\langle U\rangle$ we have $u\cdot v=u-\tau(u)+\tau(v)=2u-v$.

(iv) Let $u_1, u\in U$. Then $\exists r_1, r\in R$ with $u_1=r_1\cdot 0$, $u=r\cdot 0$, and $-u=0\cdot r\cdot 0$. We calculate

$$\begin{aligned}u+2u_1&=2u_1-(-u)\\ &=u_1\cdot(-u)\\ &=(r_1\cdot 0)\cdot((0\cdot r)\cdot 0)\\ &\overset{\text{Th. 7}}{=}(r_1\cdot(0\cdot r))\cdot 0\in U.\end{aligned}$$

But this implies $u+2\langle U\rangle\subseteq U$. ∎

To clarify the structure of $(R,\cdot)\leq(G,+)$, we introduce the following equivalence relations:

10. Definition. Let $(R,\cdot)\leq(G,+)$.

(a) For $r,r'\in R$ we define $r\sim r' :\iff r-r'\in\langle U\rangle$.

(b) For $u,u'\in\langle U\rangle$ we define $u\approx u' :\iff u-u'\in 2\langle U\rangle$.

11. Proposition. The relations $\sim$, $\approx$ are congruence relations, i.e. for elements $a,a',b,b'\in R$ ($\in\langle U\rangle$) with $a\sim b$ and $a'\sim b'$ we have $a\cdot a'\sim b\cdot b'$, (and analogously for $\approx$).

Proof. We show the assertion only for the relation $\sim$; in the case of $\approx$ the proof is done in the same way.

Prop. 9, (ii) implies for $a\sim b$:

$$\begin{aligned}&\tau(a)=\tau(b+a-b)=\tau(b)+\tau(a-b)\\ &\implies \tau(a-b)=\tau(a)-\tau(b).\end{aligned}$$

Let $a,a',b,b'\in R$ with $a\sim b$, $a'\sim b'$.

$$\begin{aligned}a\cdot a'-b\cdot b' &= a-\tau(a)+\tau(a')-b+\tau(b)-\tau(b')\\ &= a-b-(\tau(a)-\tau(b))+\tau(a')-\tau(b')\\ &= (a-b)-\tau(a-b)+\tau(a'-b')\\ &= 2(a-b)-(a'-b')\in\langle U\rangle.\end{aligned}$$

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Using Prop. 9 and Prop. 11, we can give the following description of group related symmetric groupoids. Here σ denotes the mapping known from Prop. 1.

12. Theorem. Let $(R,\cdot)\subseteq(G,+)$ be a symmetric groupoid, $0\in R$, $(G,+)$ be an abelian group. Then we have:

(1) R is a union of equivalence classes by $\langle U\rangle$ in G.

(2) U is a union of equivalence classes by $2\langle U\rangle$ in $\langle U\rangle$.

(3) $\sigma:R\longrightarrow U$ is a distinguished epimorphism between symmetric groupoids.

(4) Choose a set $\{r_i\}_{i\in I}$ of representatives of equivalence classes by $\langle U\rangle$ in G, such that $R=\bigcup_{i\in I} r_i+\langle U\rangle$.

The multiplicative structure on the symmetric groupoid $(R,\cdot)$ is determined by $\{\sigma(r_i)\}_{i\in I}$; furthermore, we have $\sigma(r_i+u)=\sigma(r_i)+2u$, $u\in\langle U\rangle$.

Let $r\sim r_i$, $s\sim r_j$, i.e. $\exists u,v\in\langle U\rangle$ with $r=r_i+u$, $s=r_j+v$, then $r\cdot s=r_i\cdot r_j+u\cdot v$.

(5) The symmetric factor groupoids $R/\langle U\rangle$, $U/2\langle U\rangle$ resp. $\langle U\rangle/2\langle U\rangle$ are trivial, i.e. let $\bar r_i,\bar r_j\in R/\langle U\rangle$, then for the canonically induced binary operation on $R/\langle U\rangle$, which for simplicity also will be written by $\cdot$, we have $\bar r_i\cdot\bar r_j=\bar r_j$, (and analogously for $U/2\langle U\rangle$ resp. $\langle U\rangle/2\langle U\rangle$).

Proof. Prop. 9, (ii), (iv) imply (1) and (2).

(3) follows from the remark after Def. 8, and since for $a,b\in R$ we get

$$\begin{aligned}\sigma(a\cdot b) &= a\cdot b-\tau(a\cdot b)\\ &= 2(a-\tau(a))-(b-\tau(b))\\ &= \sigma(a)\cdot\sigma(b).\end{aligned}$$

(4) By (1), any $r\in R$ can be uniquely represented as a pair (r_i,u), where r_i is the representative of the corresponding equivalence class, $u\in\langle U\rangle$, and $r=r_i+u$. By Prop. 1, the

multiplicative structure $\cdot$ on the symmetric groupoid R is known, if $\tau(r)$ is known for all $r\in R$.

Prop. 9, (ii) implies $\tau(r)=\tau(r_i+u)=\tau(r_i)-u=\tau(r_i)+\tau(u)$, therefore $\sigma(r)=\sigma(r_i)+2u$.

Now let $r,s\in R$; $r=r_i+u$, $s=r_j+v$. Then we calculate

$$\begin{aligned} r\cdot s &= r-\tau(r)+\tau(s) \\ &= r_i+u-\tau(r_i)+u+\tau(r_j)-v \\ &= r_i-\tau(r_i)+\tau(r_j)+2u-v \\ &= r_i\cdot r_j+u\cdot v. \end{aligned}$$

(5) The existence of all symmetric factor groupoids mentioned above is ensured by Prop. 11. We give the proof only for $R/\langle U\rangle$, for which we simply note, starting with an equation from (4)

$$\begin{aligned} r\cdot s &= r_i-\tau(r_i)+\tau(r_j)+2u-v \\ &= r_j+(r_i-\tau(r_i))-(r_j-\tau(r_j))+2u-v\in r_j+\langle U\rangle. \end{aligned}$$

■

13. Remark: Let G be an abelian group, $\mathcal{U}\leq G$ a subgroup, let $\{r_j, j\in J\}$ be a set of representatives of all classes of $G/\mathcal{U}$ with $r_0:=0$ as a representative of $\mathcal{U}$, and let $I\leq J$. Then on any union R of classes by $\mathcal{U}$ in G (without loss of generality suppose $\mathcal{U}\leq R$, i.e. $0\in I$), $R:=\bigcup_{i\in I} r_i+\mathcal{U}$, we can induce the structure of a group related symmetric groupoid by means of the following principle:

Let $u_i\in\mathcal{U}$, $i\in I$; let $u_0:=0$. We define

$$\begin{aligned} \sigma(r_i) &:= u_i, \quad i\in I, \\ \sigma(r_i+u) &:= \sigma(r_i)+2u, \quad i\in I,\ u\in\mathcal{U}, \\ \tau(r) &:= r-\sigma(r), \quad r\in R, \\ r\cdot s &:= \sigma(r)+\tau(s), \quad r,s\in R. \end{aligned}$$

Then $\tau(r)\in R$, as can easily be seen; furthermore one computes $\tau(r_i+u)=\tau(r_i)-u$ for $u\in\mathcal{U}$ and $r_i, i\in I$; and for $r=r_i+u$, $s=r_j+v$, $i,j\in I$, $u,v\in\mathcal{U}$:

$$\begin{aligned} \tau(r-\tau(r)+\tau(s)) &= \tau(r_i+u-\tau(r_i+u)+\tau(r_j+v)) \\ &= \tau(r_j+\underbrace{\tau(r_j)-r_j+r_i-\tau(r_i)+2u-v}_{\in\mathcal{U}}) \\ &= \tau(r_j)-\tau(r_j)+r_j-r_i+\tau(r_i)-2u+v \\ &= \tau(r_i+u)-(r_i+u)+(r_j+v) \end{aligned}$$

$$= \tau(r)-r+s.$$

Next we go in for the question of isomorphisms between group related symmetric groupoids. The problem becomes simpler by

14. Proposition. Let $(G,+)$ be an abelian group, $(R,\cdot)$, $(R',\cdot')\le(G,+)$, and $0\in R,R'$. Further, let $\lambda:(R',\cdot')\longrightarrow(R,\cdot)$ be a distinguished isomorphism between these symmetric groupoids, i.e. $\lambda(0)=0$. Then the following holds:

(1) $\lambda(r'+u')=\lambda(r')+\lambda(u')$ for $r'\in R'$, $u'\in\langle U'\rangle$.

In particular, λ respects classes, i.e. for $r',r''\in R'$ with $r'-r''\in\langle U'\rangle$ we have $\lambda(r')-\lambda(r'')\in\langle U\rangle$.

(2) $\lambda_{|\langle U'\rangle}:\langle U'\rangle\longrightarrow\langle U\rangle$ is an isomorphism of groups.

Proof. The operation $\cdot'$ on R' may be described by mappings σ' and τ', and $\cdot$ on R by σ and τ.

$\lambda(r'\cdot's')=\lambda(r')\cdot\lambda(s')$ for $r',s'\in R'$ implies $\lambda(\sigma'(r')+$ $+\tau'(s'))=\sigma(\lambda(r'))+\tau(\lambda(s'))$.

For r' resp. $s'=0$ we get $\lambda(\sigma'(r'))=\sigma(\lambda(r'))$, $\lambda(\tau'(s'))=$ $=\tau(\lambda(s'))$.

Inserting this into the above equation one obtains

(H) $\qquad \lambda(\sigma'(r')+\tau'(s'))=\lambda(\sigma'(r'))+\lambda(\tau'(s'))$.

Now let $r'\in R'$, $u'_0\in U'$, i.e. there is $s'_0\in R'$ with $u'_0=s'_0-$ $-\tau'(s'_0)=\sigma'(s'_0)$. Since τ' is bijective on R', there is $r''\in R'$ with $\tau'(r'')=r'$. Condition (H) implies

$$\begin{aligned}\lambda(r'+u'_0) &= \lambda(\tau'(r'')+\sigma'(s'_0))\\ &= \lambda(\tau'(r''))+\lambda(\sigma'(s'_0))\\ &= \lambda(r')+\lambda(u'_0).\end{aligned}$$

Especially for $u',v'\in\langle U'\rangle$ we get $\lambda(u'+v')=\lambda(u')+\lambda(v')$, and for $r'\in R'$, $u'\in\langle U'\rangle$ we obtain $\lambda(r'+u')=\lambda(r')+\lambda(u')$. It remains to show, that $\lambda_{|\langle U'\rangle}$ is an isomorphism between the groups $\langle U'\rangle$ and $\langle U\rangle$: λ is injective by assumption; and by the above calculations, the inclusions $\lambda(U')\subseteq U$, $\lambda(\langle U'\rangle)\le\langle U\rangle$ are valid. In order to show the equality of $\lambda(\langle U'\rangle)$ and $\langle U\rangle$, we first prove the equality of $\lambda(U')$ and U.

Let $v\in U=R\cdot 0$, i.e. there is $r\in R$ with $v=r\cdot 0$. Since λ is bijective, there is $r'\in R'$ with $\lambda(r')=r$, and for $v':=r'\cdot 0\in U'$ we get $\lambda(v')=\lambda(r'\cdot 0)=\lambda(r')\cdot 0=r\cdot 0=v$.

Now let $u\in\langle U\rangle$, i.e. there are $u_1,\dots,u_k\in U$ with $u=u_1+\dots$ $\dots+u_k$. Since $\lambda(U')=U$, there are elements $u'_i\in U'$ with $\lambda(u'_i)=$ $=u_i$ for $i=1,\dots,k$, and consequently, $\lambda(u'_1+\dots+u'_k)=u$. It remains to show, that λ respects classes. For this, let $r',r''\in R'$ and $u'\in\langle U'\rangle$ with $r'-r''=u'$. It follows immediately $\lambda(r')=\lambda(r''+u')=\lambda(r'')+\lambda(u')$, hence $\lambda(r')-\lambda(r'')\in\langle U\rangle$. ∎

15. Remark. In the following we can restrict ourselves on the consideration of special isomorphisms between group related symmetric groupoids. For this remark let $(R',\cdot')\le(G,+)$ be a group related symmetric groupoid with its corresponding subgroup $\langle U'\rangle$ and the underlying set $R'=\bigcup_{i\in I} r'_i+ +\langle U'\rangle$ (according to Theorem 12, (1)), where $\{r'_j, j\in J\}$ denotes a set of representatives for all classes of $G/\langle U'\rangle$, which contains $r'_0:=0\in G$, and I a suitable subset of J with $r'_0\in\{r'_i, i\in I\}$.

By Theorem 12, (4), the operation $\cdot'$ on R' is determined by the values of $\sigma'(r'_i)\in\langle U'\rangle$, $i\in I$; in particular, let $\sigma'(0)=0$.

Suppose $\mathcal{U}$ is a subgroup of G, isomorphic to $\langle U'\rangle$, and $\rho:\langle U'\rangle\longrightarrow\mathcal{U}$ a group isomorphism, furthermore $\{r_k, k\in K\}$ a set of representatives of all classes of $G/\mathcal{U}$ with $r_0=0$; in addition we assume $\pi:I\longrightarrow K$ to be an injective mapping with $\pi(0)=0$. Then $R:=\bigcup_{i\in I} r_{\pi(i)}+\mathcal{U}$ becomes a symmetric groupoid isomorphic to R' by defining $\sigma(r_{\pi(i)}):=\rho(\sigma'(r'_i))$ and by the principle from Rem. 13, such that $\langle\rho(R)\rangle=\mathcal{U}$. Denote the binary operation on R by $\cdot$. An isomorphism $\lambda:(R',\cdot')\longrightarrow(R,\cdot)$ is given by $\lambda(r'_i+u'):=r_{\pi(i)}+\rho(u')$.

Consequently, without loss of generality, we can restrict ourselves to the consideration of distinguished isomorphisms $\lambda:(R,\cdot)\longrightarrow(R,\cdot')$ with $\langle\rho(R)\rangle=\langle\sigma'(R)\rangle$, such that -defining $\mathcal{U}:=\langle\rho(R)\rangle$- one has $r_i-\lambda(r_i)\in\mathcal{U}$ for representatives r_i of classes modulo $\mathcal{U}$.

To describe isomorphisms between group related symmetric groupoids, we make use of the representation for symmetric groupoids introduced above (Theorem 12 to Rem. 15). The following theorem sharpens Prop.14.

16. Theorem. Let $(R,\cdot),(R,\cdot')\le(G,+)$ be symmetric groupoids with $0\in R$, with $\cdot$ resp. $\cdot'$ described by σ resp. σ'. Let $\langle\sigma(R)\rangle=\langle\sigma'(R)\rangle=:\mathcal{U}$, and write $R=\bigcup_{i\in I} r_i+\mathcal{U}$. Furthermore, suppose $\lambda:R\longrightarrow R$ being a bijection with $\lambda(0)=0$ and $r_i-\lambda(r_i)\in\mathcal{U}$ $\forall i\in I$. Then the following are equivalent:

(1) $\lambda:(R,\cdot')\longrightarrow(R,\cdot)$ is a distinguished isomorphism between symmetric groupoids.

(2)(a) $\lambda(r+u)=\lambda(r)+\lambda(u)$, $r\in R$, $u\in\mathcal{U}$; in particular, $\lambda_{|\mathcal{U}}$ is an isomorphism of groups.

(b) $\sigma(r)=\lambda(\sigma'(r))+2(r-\lambda(r))$.

Proof. "(1) $\Longrightarrow$ (2)" (a) is implied by Prop. 14.

(b) In the proof of Prop. 14 we have shown $\sigma(\lambda(r))=$ $=\lambda(\sigma'(r))$. Since λ respects classes, i.e. $r-\lambda(r)\in\mathcal{U}$, we calculate

$$\begin{aligned}\sigma(r)&=\sigma(\lambda(r)+r-\lambda(r))\\ &\overset{\text{Th.12, (4)}}{=}\sigma(\lambda(r))+2(r-\lambda(r))\\ &=\lambda(\sigma'(r))+2(r-\lambda(r)).\end{aligned}$$

"(1) $\Longleftarrow$ (2)" Let $r\in R$. There are $i\in I$, $u\in\mathcal{U}$ with $r=r_i+u$. Since

$$r-\lambda(r)=r_i+u-\lambda(r_i+u)\overset{(a)}{=}r_i-\lambda(r_i)+u-\lambda(u)\in\mathcal{U}$$

we obtain from (b)

$$\lambda(\sigma'(r)) = \sigma(r)+2(\lambda(r)-r) = \sigma(r+\lambda(r)-r) = \sigma(\lambda(r)).$$

So we calculate

$$\begin{aligned}\lambda(r\cdot' s) &= \lambda(r_i\cdot' r_j+u\cdot v)\\ &= \lambda(r_j+\sigma'(r_i)-\sigma'(r_j)+u\cdot v)\\ &= \lambda(r_j)+\lambda(\sigma'(r_i))-\lambda(\sigma'(r_j))+2\lambda(u)-\lambda(v)\\ &= \lambda(r_j)+\sigma(\lambda(r_i))-\sigma(\lambda(r_j))+2\lambda(u)-\lambda(v)\\ &= \lambda(r_j)+\lambda(v)+\sigma(\lambda(r_i))+2\lambda(u)-\sigma(\lambda(r_j))-2\lambda(v)\\ &= \lambda(r_j+v)+\sigma(\lambda(r_i+u))-\sigma(\lambda(r_j+v))\\ &= \lambda(r_i+u)\cdot\lambda(r_j+v)\\ &= \lambda(r)\cdot\lambda(s).\end{aligned}$$

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The subsequent corollaries describe isomorphisms from a slightly different point of view.

17. Corollary. Let $(R,\cdot),(R,\cdot')\in(G,+)$ be symmetric groupoids with $0\in R$, and $\cdot$, $\cdot'$ on R be described by σ, σ', furthermore let $<\sigma(R)>=<\sigma'(R)>=:\mathcal{U}$, and $R=\bigcup_{i\in I} r_i+\mathcal{U}$. Then the following are equivalent:

(1) $(R,\cdot)\cong(R,\cdot')$.

(2) $\exists\ \lambda:\mathcal{U}\longrightarrow\mathcal{U}$ isomorphism of groups, s.th. $\sigma(r_i)-\lambda(\sigma'(r_i))\in 2\mathcal{U}\ \ \forall i\in I$.

Proof. "(1) $\Longrightarrow$ (2)" by Rem. 15 and Theorem 16.
"(1) $\Longleftarrow$ (2)" Let $\lambda:\mathcal{U}\longrightarrow\mathcal{U}$ be an isomorphism of groups, let $\sigma(r_i)-\lambda(\sigma'(r_i))\in 2\mathcal{U}\ \ \forall i\in I$. We construct an extension of λ on whole R, such that $\lambda:(R,\cdot')\longrightarrow(R,\cdot)$ becomes an isomorphism between symmetric groupoids: Choose for each $r_i, i\in I$, an element $\lambda(r_i)\in R$ satisfying $2\lambda(r_i)=2r_i-(\sigma(r_i)-\lambda(\sigma'(r_i)))$ and $r_i-\lambda(r_i)\in\mathcal{U}$ (in particular: $\lambda(0):=0$). For $i\in I$, $u\in\mathcal{U}$ we define $\lambda(r_i+u):=\lambda(r_i)+\lambda(u)$.

From above we obtain $\sigma(r_i)=\lambda(\sigma'(r_i))+2r_i-2\lambda(r_i)$. By an easy calculation we get $\sigma(r)=\lambda(\sigma'(r))+2(r-\lambda(r))$ for $r\in R$. Applying Theorem 16 completes the proof. ∎

18. Corollary. Let $(R,\cdot')\subseteq(G,+)$ be a symmetric groupoid with $0\in R$, where $\cdot'$ on R is described by σ', put $\mathcal{U}:=<\sigma'(R)>$, and write $R=\bigcup_{i\in I} r_i+\mathcal{U}$. Let $\lambda:R\longrightarrow R$ be a bijection with $r_i-\lambda(r_i)\in\mathcal{U}$, and suppose that $\lambda_{|\mathcal{U}}:\mathcal{U}\longrightarrow\mathcal{U}$ is an isomorphism of groups. λ describes an isomorphism between group related symmetric groupoids iff $\lambda(r_i+u)=\lambda(r_i)+\lambda(u)\ \ \forall i\in I,\ u\in\mathcal{U}$.

Proof. "$\Longrightarrow$" Let $\cdot$ be a further binary operation on R, described by σ, let $\lambda:(R,\cdot')\longrightarrow(R,\cdot)$ be an isomorphism between symmetric groupoids. Since $<\sigma(R)>=\mathcal{U}$, the assertion follows by Theorem 16.

"$\Longleftarrow$" Let $\lambda(r_i+u)=\lambda(r_i)+\lambda(u)\ \ \forall i\in I,\ u\in\mathcal{U}$. We define $\sigma(r):=\lambda(\sigma'(r))+2(r-\lambda(r))$; by this and an easy calculation, for $r\in R$ with $r=r_i+u$, $u\in\mathcal{U}$ we obtain $\sigma(r_i+u)=\sigma(r_i)+2u$.

By Rem. 13 we conclude, that $\sigma(r)$ induces on R the binary operation $\cdot$ of a symmetric groupoid, defined by $r\cdot s=s+\sigma(r)-\sigma(s)$, $r,s\in R$. Applying Theorem 16 completes the proof. ∎

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