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OVER AN ORTHOGONAL SYSTEM OF QUASIGROUPS

This paper deals with orthogonal systems of quasigroups and identities over them. V.J.Dementjeva in [3] dealt with closure conditions in k-nets, which she reduced to an investigation of universal identities. We start from 3-basic quasigroups and show a connection between universal identities and identities over an orthogonal system of 3-basic quasigroups.

1. 3-basic quasigroups

The quadruple $(Q_1,Q_2,Q_3; A^3_{12})$, where card Q_1=card Q_2=card $Q_3\neq 0$ and A^3_{12} is a map of $Q_1\times Q_2$ onto Q_3, is called a *3-basic quasigroup* if, in the equation $A^3_{12}(a_1,a_2)=a_3$, any two of the elements $a_1\in Q_1$, $a_2\in Q_2$, $a_3\in Q_3$ uniquely determine the remaining one. Similarly as in the classical case we define *parastrophic operations*

(1) $$A^m_{ij}(a_i,a_j) = a_m \Leftrightarrow A^3_{12}(a_1,a_2) = a_3$$

for all permutations (i,j,m) of the set $\{1,2,3\}$.

If $Q_1=Q_2=Q_3=Q$ we get usual *quasigroup*. For example we have two equivalent denotations for a quasigroup $(Q;.)$ or $((Q;A^3_{12})$.

Key words: 3-basic quasigroup, k-net, orthogonal system of 3-basic quasigroups, isotopy. universal identity, position of a term.

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In detail:

$x.y=z \Leftrightarrow A^3_{12}(x,y)=z,$ $\quad y^s(.)x=z \Leftrightarrow A^3_{21}(y,x)=z,$

$x\backslash z=y \Leftrightarrow A^2_{13}(x,z)=y,$ $\quad z^s(\backslash)x=y \Leftrightarrow A^2_{31}(z,x)=y,$

$z/y=x \Leftrightarrow A^1_{32}(z,y)=x,$ $\quad y^s(/)z=x \Leftrightarrow A^1_{23}(y,z)=x.$

It is clear that $A^m_{ij}(x,y)=A^m_{ji}(y,x)$ for all (i,j,m).

2. k-nets and orthogonal systems of quasigroups

A system $\mathcal{N}$ of objects of two sorts, called points and lines, with an incidence relation between them is a *k-net* iff

(i) the set of all lines is partitioned into k disjoint classes $\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_k$, $k\geq 3$,

(ii) two lines from different classes are incident to exactly one point,

(iii) in every line class there is exactly one line incident to a given point,

(iv) there are at least two points incident to every line.

A k-net is of *order n* if every line class $\mathcal{L}_i$ has exactly n lines. In a k-net of order n there are exactly n^2 points and kn lines. If Q_i, $1\leq i\leq k$, are sets such that card $Q_i=n$, then there exist bijections between Q_i and $\mathcal{L}_i$ and maps $A^m_{ij}:Q_i\times Q_j \longrightarrow Q_m$ such that every quadruple $(Q_i,Q_j,Q_m;A^m_{ij})$ forms a 3-basic quasigroup for all $i \neq j \neq m \neq i$ and $i,j, m \in \{1,2,\dots,k\}$. The geometric interpretation of the operations A^m_{ij} is the following: $A^m_{ij}(x,y)$ is the unique line of $\mathcal{L}_m$ incident to the unique point incident to the line x of $\mathcal{L}_i$ and the line y of $\mathcal{L}_j$. The set

$$\mathcal{A} = \{(Q_i,Q_j,Q_m;\ A^m_{ij});\ i \neq j \neq m \neq i;\ i,j,m\in\{1,\dots,k\}\}$$

forms a so-called *orthogonal system of 3-basic quasigroups* considering that every system

$$A^p_{ij}(x,y)=a,\quad A^q_{ij}(x,y)=b,\quad p \neq q, \tag{2}$$

has a unique solution for all $a \in Q_p$, $b \in Q_q$. This solution is

$x=A^i_{pq}(a,b)$, $y=A^j_{pq}(a,b)$. The system $\mathcal{A}$ is also called a *coordinate system of the k-net*.

If $Q_1=Q_2=\ldots=Q_k=Q$ and if we take the operations A^m_{12}, $m\geq 3$, only then we get an orthogonal system of quasigroups Σ in the meaning of V.D.Belousov ([1]). If we take all A^m_{ij} on Q, we obtain a covering system $\bar{\Sigma}$ ([1]).

Immediately we see that

$$(3)\qquad A^m_{ij}(x,A^j_{ip}(x,y)) = A^m_{ji}(A^j_{ip}(x,y),x) = A^m_{ip}(x,y)$$

for all $x \in Q_i$, $y \in Q_p$.

3. Isotopies

Let a line class $\mathcal{L}_i$ of the k-net $\mathcal{N}$ be labelled by elements of the set Q_i. If we take any set Q'_i, with card Q'_i=card Q_i and a bijective map $\alpha_i\colon Q_i \longrightarrow Q'_i$, we obtain a new labelling of $\mathcal{L}_i$ by elements of Q'_i, $i=1,2,\ldots,k$, and new quasigroup operations B^m_{ij} defined on $Q'_i \times Q'_j$ such that from the incidence it follows that

$$A^m_{ij}(x,y)=z \Leftrightarrow B^m_{ij}(\alpha_i x,\alpha_j y)=\alpha_m z \quad \text{and}$$

$$(4)\qquad A^m_{ij}(x,y)=\alpha_m^{-1}\, B^m_{ij}(\alpha_i x,\alpha_j y) \text{ for all } x \in Q_i,\ y \in Q_j.$$

The system

$$\mathcal{B} = \{(Q'_i,Q'_j,Q'_m;\ B^m_{ij});\ i \neq j \neq m \neq i;\ i,j,m \in \{1,\ldots,k\}\}$$

is a new coordinate system of $\mathcal{N}$. The k-tuple $(\alpha_1,\ldots,\alpha_k):=\bar{\alpha}$ of bijective maps is called an *isotopy* of the coordinate systems $\mathcal{A}$ and $\mathcal{B}$. Thus we have

Lemma 1. Isotopic orthogonal systems of 3-basic quasigroups are associated with the same k-net $\mathcal{N}$ and correspond to changes in the labelling of lines in the line classes of $\mathcal{N}$.

Every change of numbering of the line classes of $\mathcal{N}$ corresponds to a permutation ρ of the numbers $(1,2,\ldots,k)$.

An isotopy $\bar{\alpha} = (1,1,\alpha_3,\dots,\alpha_k)$ is a torsion by [1] -it changes the labellings of the lines in the line classes $\mathcal{L}_3, \mathcal{L}_4,\dots\mathcal{L}_k$, whereas an isotopy $\bar{\alpha}=(\alpha_1,\alpha_2,1,\dots,1)$ is a conjugation changing the labellings of the lines in $\mathcal{L}_1, \mathcal{L}_2$, i.e. changing the coordinates of the points.

4. Identities over $\mathcal{A}$

It is clear that an arbitrary term over a system $\mathcal{A}$ of orthogonal 3-basic quasigroups need not make sense. We must from terms carefully, respecting the places belonging to the various Q_i's.

In what follows let $x_i,y_i,z_i,\dots$ be variables over Q_i, $i=1,2,\dots,k$ and let t^i,t^i_n be terms over $\mathcal{A}$, $i=1,\dots,k$; $n=1,2,\dots$ .It is easy to show that only terms of the following forms are meaningful:

$$A^p_{ij}(x_i,x_j),\ A^m_{pq}(x_p,A^q_{sr}(y_s,y_r,)),\ A^m_{pq}(A^p_{ij}(x_i,x_j),y_q),$$

$$A^m_{pq}(A^p_{ij}(x_i,x_j),A^q_{sr}(y_s,y_r,)),\ \text{and so on.}$$

If $A^p_{ij}(x_i,x_j)=x_p$ for all triples (i,j,p), then

$$A^m_{pq}(A^p_{ij}(x_i,x_j),\ A^q_{sr}(x_s,x_r))= A^m_{pq}(x_p,x_q)=x_m.$$

Now an identity w over $\mathcal{A}$ consists of the following subterms:

$$(5)\qquad \begin{cases} A^m_{pq}(t^p_1,t^q_1),\ A^m_{pq}(t^p_1,A^q_{sr}(t^s_2,t^r_2)),\ A^m_{pq}(A^p_{ij}(t^i_1,t^j_1),t^q_2), \\ A^m_{pq}(A^p_{ij}(t^i_1,t^j_1),\ A^q_{sr}(t^s_2,t^r_2)),\dots, \end{cases}$$

where the length of the subterms t^i_n, $1\le i\le k$, $n=1,2$, can be ≥ 1 (written $l(t^i_n)\ge 1$). Note $l(t^i_n)=1 \Leftrightarrow t^i_n=x_i$ for some $x_i\in Q_i$ and

$$l(t^i_n)\ge 2 \Leftrightarrow t^i_n=A^i_{gh}(t^g_{n+1},t^h_{n+1}).$$

It is clear that for all triples (i,j,m), $i\ne j\ne m\ne i$, the following trivial identities are true:

$$(6)\quad \begin{cases} A^m_{ij}(x,A^j_{im}(x,y))=y, \\ A^p_{ij}(x,A^j_{im}(x,y))=A^p_{im}(x,y), \\ A^q_{pj}(A^p_{im}(x,y),A^j_{im}(x,y)) = A^q_{im}(x,y) \text{ for all } x\in Q_i,\ y\in Q_m. \end{cases}$$

Similarly for subterms $t^i\in Q_i$ and $t^m\in Q_m$:

$$(7)\quad \begin{cases} A^m_{ij}(t^i,A^j_{im}(t^i,t^m)) = t^m, \\ A^p_{ij}(t^i,A^j_{im}(t^i,t^m)) = A^p_{im}(t^i,t^m), \\ A^q_{pj}(A^p_{im}(t^i,t^m),A^j_{im}(t^i,t^m)) = A^q_{im}(t^i,t^m). \end{cases}$$

If an identity w contains at least one the left terms from (7), then it can be replaced by the corresponding right term from (7) and we obtain a new identity w′ equivalent to w, written $w \Leftrightarrow w'$, with $l(w')\le l(w)-1$. In this case the identity w is called *reducible.*

Lemma 2. Let w be a non-reducible and non-trivial identity over $\mathcal{A}$. Then w contains three different variables at least.

Proof. It is easy to verify that every identity w containing $x_i\ne x_j$ only necessarily has among its subterms some term from the left sides of (7) aus thus w is reducible. After reductions we obtain a trivial identity w′, $w'\Leftrightarrow w$.

Lemma 3. The minimal length of any non-reducible and non-trivial identity w over $\mathcal{A}$ is six.

Proof. By Lemma 2 every non-reducible and non-trivial identity w contains three different variables at least and every variable occurs two times at least, because any identity has no isolated variables.

Let Σ be an orthogonal system of usual quasigroups. An identity w over Σ is said to be *universal* if the validity of w over every Σ' isotopic to Σ follows from the validity of w over Σ. If we transfer this definition to an orthogonal system of 3-basic quasigroups in the natural way, then we can conclude:

Theorem 1. Any identity w over an orthogonal system of 3-basic quasigroups is universal.

Proof. Let t^m be some subterm of w of the type

$$t^m = A^m_{pq}(A^p_{ij}(t^i_1,t^j_1),\ A^q_{sr}(t^s_2,t^r_2)).$$

Then under an isotopy $\bar{\alpha}$ we have

$$\alpha_m t^m = A^m_{pq}(A^p_{ij}(\alpha_i t^i_1,\alpha_j t^j_1), A^q_{sr}(\alpha_s t^s_2,\alpha_r t^r_2)),\ \text{where if}$$

$t^i_1 = A^i_{gh}(t^g_3,t^h_3)$, then under $\bar{\alpha}$ we get $\alpha_i t^i_1 = A^i_{gh}(\alpha_g t^g_3,\alpha_h t^h_3)$ and so on. In the end we obtain for some n subterm $t^i_n = A^i_{df}(x_d,x_f)$ for which $\alpha_i t^i_n = A^i_{df}(\alpha_d x_d,\alpha_f x_f)$. Thus from the identity w we obtain under $\bar{\alpha}$ the identity $\bar{\alpha}(w)$ which is of the same form as w, where instead of individual variables $x_i, y_j, \ldots$ there occur variables $\alpha_i x_i, \alpha_j y_j, \ldots$ .Thus w is universal.

Now let $\mathcal{A}$ be an orthogonal system of 3-basic quasigroups $(Q_i,Q_j,Q_m;A^m_{ij})$, $i,j,m\in\{1,\ldots,k\}$ and let $\mathcal{B}$ be an orthogonal system of usual quasigroups (Q,B^m_{ij}) with card Q=card Q_i, $i\in\{1,\ldots,k\}$. We shall investigate the connection between corresponding identities over $\mathcal{A}$ and over β.

Theorem 2. Let w_A be an identity over $\mathcal{A}$ and let w_B arise from w_A by substitutions between subterms

$$(8)\qquad \begin{cases} t_{1A} = A^m_{ij}(x,y) := B^m_{ij}(x,y) = t_{1B}, \\ t_{3A} = A^m_{ij}(t_{1A},t_{2A}) := B^m_{ij}(t_{1B},t_{2B}) = t_{3B}, \end{cases}$$

where at least one of t_{1A}, t_{2A} has length >1. Then w_B is a universal identity over $\mathcal{B}$.

Proof. The identity w_B as well as w_A consists of subterms of the types (5) only. Then similarly as in Theorem 1, from w_B under any isotopy $\bar{\beta} = (\beta_1,\ldots,\beta_k)$ we obtain the identity $\bar{\beta}(w_B)$ which is of the same form as w_B. Thus w_B is a universal identity.

Corollary. Conversely, let w_B be an identity over $\mathcal{B}$. Then w_B is universal if the corresponding expression w_A arising from w_B by (8), is an identity over $\mathcal{A}$, i.e. w_A has a meaning over $\mathcal{A}$.

5. Positions of terms

The question as to when an identity w over an orthogonal system $\mathcal{B}$ of usual quasigroups is universal can be investigated independently of multi-basic systems. For this we shall define positions of subterms.

Let t_1, t_2, t_3 be terms over $\mathcal{B}$ and let

$$B^m_{ij}(t_1,t_2) = t_3.$$

Then we say that terms t_1 and t_2 are in *inner positions* i and j with respest to B^m_{ij}, written:

$$\widehat{\mathrm{pos}}^m_{ij}(t_1) = i, \quad \widehat{\mathrm{pos}}^m_{ij}(t_2) = j,$$

and the term t_3 is in an *outer position* m with respect to B^m_{ij}, written: $\overset{\smile}{\mathrm{pos}}{}^m_{ij}(t_3) = \overset{\smile}{\mathrm{pos}}(B^m_{ij}(t_1,t_2)) = m$.

Illustration: Let

$$B^2_{13}(B^1_{24}(x,y),z) = B^2_{31}(y,B^4_{23}(x,z)).$$

Then we have

$$\widehat{\mathrm{pos}}^1_{24}(x) = 2 = \widehat{\mathrm{pos}}^4_{23}(x), \qquad \widehat{\mathrm{pos}}^1_{24}(y) = 4 \neq \widehat{\mathrm{pos}}^2_{31}(y) = 3,$$

$$\widehat{\mathrm{pos}}^2_{13}(z) = 3 = \widehat{\mathrm{pos}}^4_{23}(z),$$

$$\widehat{\mathrm{pos}}^2_{13}(B^1_{24}(x,y) = 1 = \overset{\smile}{\mathrm{pos}}(B^1_{24}(x,y)),$$

$$\widehat{\mathrm{pos}}^2_{31}(B^4_{23}(x,z) = 1 \neq \overset{\smile}{\mathrm{pos}}(B^4_{23}(x,z)) = 4,$$

$$\overset{\smile}{\mathrm{pos}}(B^2_{13}(B^1_{24}(x,y),z)) = 2 = \overset{\smile}{\mathrm{pos}}(B^2_{31}(y,B^4_{23}(x,z))).$$

Position condition (P): Every subterm of a given identity occurs always in the same position, independently of whether this position is inner or outer.

Theorem 3. If an identity w over $\mathcal{B}$ satisfies (P), the w is universal.

Proof. If w satisfies (P), then w consists of subterms of the types (5) and w makes sense over an orthogonal system of 3-basic quasigroups $\mathcal{A}$ with card Q_i = card Q for all $i\in\{1,\dots,k\}$. By Theorem 2 w is universal.

Theorem 4. Let w be an identity over $\mathcal{B}$ satisfying (P). Then $\delta(w)$ is universal for all permutations δ of $\{1,\dots,k\}$, where $\delta(w)$ denotes the identity arising from w by replacement of B^{m}_{ij} by $B^{\delta m}_{\delta i,\delta j}$ for all $i \neq j \neq m \neq i$; $i,j,m\in\{1,\dots,k\}$.

Proof. From the assumption that w satisfies (P) it follows that every subterm of w occurs in one position only. Then also every subterm of $\delta(w)$ occurs in one positon only, because δ is a permutation of $\{1,\dots,k\}$. So $\delta(w)$ satisfies (P), and therefore $\delta(w)$ is universal for all permutations δ.

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