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DECOMPOSITIONS AND EXTENSIONS OF HOMOMORPHISMS OF n-GROUPS

In this paper we present necessary and sufficient conditions for decompositions of n-groups determined by p-ideals to be semiselective and selective. Next, we apply these results to the theorems concerning extensions of homomorphisms of n-groups.

We shall consider n-groups for $n \geq 3$. Theorems, definitions, and notation related to the theory of n-groups are based on papers [1], [2], [4], [5], [6]. Definitions and theorems about decompositions of sets are based on the paper [3].

We begin with decompositions of n-groups. Let (A, f) be an n-group. A skew element for an arbitrary element $a \in A$ will be denoted by $\bar{a}$.

Let us quote an important theorem (cf[4]).

Theorem 1. An n-groupoid (A,f) is an n-group if and only if

$$f(x_1^n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \ldots \cdot \alpha^{n-1}(x_n) \cdot a \text{ , where:}$$

(i) $(A,\cdot)$ is a group,

(ii) $a \in A$ is a fixed element,

(iii) $\alpha \in Aut(A,\cdot)$, $\alpha(a) = a$, $\alpha^{n-1}(x) = a \cdot x \cdot a^{-1}$ for $x \in A$.

The system $(A,\cdot,\alpha,a)$ will be called a binary retract of an n-group (A,f) (this terminology is partly different from the terminology used in paper [2]). In the sequel the binary retract $(A, \cdot, \alpha, a)$ will often be treated as a group $(A,\cdot)$ and called simply the retract.

Let us recall the useful method of constructing a retract $(A, \cdot, \alpha, a)$ for an n-group (A,f) (cf. [6]).

Namely,

$$x\cdot y = f(x, \overset{(n-2)}{p}, y) \ ,$$
$$\alpha(x) = f(\bar{p}, x, \overset{(n-2)}{p}) \ ,$$
$$a = (\overset{(n)}{p})$$

for an arbitrary fixed element $p \in A$ and for all $x, y \in A$.
The set A with the operation $\cdot$ forms a group with an identity element $\bar{p}$. In this case we shall say that $(A, \cdot, \alpha, a)$ is a binary retract of an n-group (A,f) with respect to an element p and write $(A, \cdot, \alpha, a)_p$. In the paper [2] the above $(A,\cdot)$ is called a binary retract of an n-group (A,f) with respect to an element p.

Notice that the following statement results immediately from the definition of the operation $\cdot$ in the set A.

Proposition 1. If σ is a congruence on an n-group (A,f), then σ is a group congruence on a retract $(A, \cdot, \alpha, a)_p$, $p \in A$.

The converse theorem is not true as may be seen from the followin example.

Example. We shall give an example of a group congruence on a retract of the form $(A, \cdot, \alpha, a)_p$ of a 3-group (A,f) which is not a congruence on this 3-group (A,f).

Consider the Klein group $A = \{e, a, b, c\}$ i.e.

·	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

It is easy to check that the mapping $\alpha\colon A \longrightarrow A$ defined by $\alpha(e) = e$, $\alpha(a) = b$, $\alpha(b) = a$, $\alpha(c) = c$ is an automorphism of the second order of the Klein group. Let us define the 3-ary operation on the set A as follows

$$f(x_1, x_2, x_3) = x_1 \cdot \alpha(x_2) \cdot x_3$$

for arbitrary $x_1, x_2, x_3 \in A$. By Theorem 1 (A,f) is a 3-group.

Notice that the Klein group $(A,\cdot)$ is a retract of the form $(A, \cdot, \alpha, e)_e$ of the 3-group (A,f). Let σ be a congruence on the Klein $(A,\cdot)$ determined by the normal subgroup $\{e,a\}$. Therefore, we have $A/\sigma = \{\{e,a\},\{b,c\}\}$. The relation σ is not a congruence on the above 3-group (A,f). Indeed, for $f(e, e, e) = e$ and $f(e, a, e) = b$ we have $(e, b) \notin \sigma$.

Proposition 2 (cf. [2]). Let (B,f) be an n-subgroup of an n-group (A,f). Let $p\in B$ be an arbitrary fixed element. Then the retract $(B, \cdot, \alpha, a)_p$ is a subgroup of the retract $(A, \cdot, \alpha, a)_p$.

Definition 1 (cf. [5]). Let (A,f) be an n-group and $p\in A$. A p-ideal of n-group (A,f) is a set of the form p/σ for some congruence relation σ in (A,f).

Let I be a p-ideal of an n-group (A,f) and let $\sigma \subset A\times A$ be a relation defined in the following way

$$x\sigma y \leftrightarrow f(x, \bar{y}, \overset{(n-3)}{y}, p) \in I$$

for arbitrary $x,y \in A$. Theorems 2.5 and 2.6 of [5] imply that the relation σ is the only congruence on the n-group (A,f) for which the p-ideal I is its equivalence class. Therefore, we shall say that the p-ideal I determines the congruence σ on the n-group (A,f). The decomposition of the set A designated by the congruence σ is actually determined by the p-ideal I.

In the sequel we shall need the following definitions (cf. [3]). Let A be an arbitrary set, B an arbitrary subset of the set A and let $\{A_t\}_{t\in T}$ and $\{B_s\}_{s\in S}$ be decompositions of the sets A and B, respectively.

Definition 2. We shall say that $\{B_s\}_{s\in S}$ is the subdecomposition of the decomposition $\{A_t\}_{t\in T}$ if the following condition holds

$$(1) \qquad \forall\, s\in S\ \exists\, t\in T :\ B_s \subset A_t.$$

Definition 3. We shall say that the decomposition $\{B_s\}_{s\in S}$ is semiselective for the decomposition $\{A_t\}_{t\in T}$ if the condition (1) holds and

(2) $\forall\, t\in T\ \forall\, s_1, s_2 \in S : [(B_{s_1} \subset A_t \wedge B_{s_2} \subset A_t) \Rightarrow B_{s_1} = B_{s_2}]$.

Definition 4. We shall say that the decomposition $\{B_s\}_{s\in S}$ is selective for the decomposition $\{A_t\}_{t\in T}$ if the conditions (1) and (2) hold and

$$\forall\, t\in T\ \exists\, s\in S :\ B_s \subset A_t.$$

Let (A,f) be an n-group and $p\in A$. Consider subsets X and Y of the set A. Farther on we shall use the following notation

$$f(X, \overset{(n-2)}{\bar{p}}, Y) = \{f(x, \overset{(n-2)}{\bar{p}}, y) \in A\colon x\in X,\ y\in Y\}.$$

Theorem 2. Let (B,f) be an n-subgroup of an n-group (A,f) and $p\in B$. If D_A (resp. D_B) stands for the decomposition of the set A (resp. B) determined by the $\bar{p}$-ideal A_0 (resp. $\bar{p}$-ideal B_0) of (A,f) (resp. (B,f)). Then

1^o D_B is a subdecomposition of D_A iff

(3) $$B_0 \subset A_0\ ;$$

2^o D_B is semiselective for D_A iff

(4) $$B_0 = B \cap A_0\ ;$$

3^o D_B is selective for D_A iff the condition (4) holds and

(5) $$A = f(B, \overset{(n-2)}{\bar{p}}, A_0).$$

Proof. Let us consider the retracts $(A, \cdot, \alpha, a)_p$ and $(B, \cdot, \alpha, a)_p$ of the n-groups (A,f) and (B,f), respectively. The skew element $\bar{p}$ is an identity element of the groups $(A, \cdot, \alpha, a)_p$ and $(B, \cdot, \alpha, a)_p$. Therefore, by Proposition 1 A_0 and B_0 are normal subgroups of the groups $(A, \cdot, \alpha, a)_p$ and $(B, \cdot, \alpha, a)_p$, respectively. The normal subgroups A_0 and B_0 dictate the decompositions D_A and D_B, respectively. Notice that under these assumptions the condition (5) is equivalent to the condition $A = B \cdot A_0$. Now, it is enough to apply Theorems 1-3 of [3].

The above results will be used to solve some problems related to extensions of homomorphisms of n-groups.

Theorem 3. Let (B,f) be an n-subgroup of an n-group (A,f) and $p \in B$. Let $h: B \longrightarrow C$ be a homomorphism from the n-group (B,f) into an n-groupoid (C,f). Let B_0 be a p-ideal determined by the congruence kerh on the n-group (B,f). Then the homomorphism $h: B \longrightarrow C$ from the n-group (B,f) into the n-grupoid (C,f) can be extended to a homomorphism $\tilde{h}: A \longrightarrow C$ from the n-group (A,f) into the n-groupoid (C,f) under the assumption that $\mathrm{Im}\,\tilde{h} = \mathrm{Im}\,h$ if and only if there exists a $\bar{p}$-ideal A_0 of the n-group (A,f) such that the conditions (4) and (5) are satisfied.

Proof. (i) It is easy to notice that $\ker h \subset \ker\tilde{h}$. Let A_0 be a $\bar{p}$-ideal determined by the congruence $\ker\tilde{h}$ on the n-group (A,f). Hence $B_0 \subset B \cap A_0$. Now take an arbitrary $x \in B \cap A_0$. Since $x \in A_0$, we have $\tilde{h}(x) = \tilde{h}(\bar{p}) = h(\bar{p})$. But $x \in B$, so $\tilde{h}(x) = h(x)$. Hence $h(x) = h(\bar{p})$, i.e. $x \in B_0$. Thus the condition (4) is fulfilled. Since $\mathrm{Im}\,\tilde{h} = \mathrm{Im}\,h$, for an arbitrary $c \in A$ there exists a $b \in B$ such that $\tilde{h}(c) = h(b)$. Hence $\tilde{h}(c) = \tilde{h}(b)$. Thus $(b,c) \in \ker\tilde{h}$. Consequently, the decomposition of the set B determined by the $\bar{p}$-ideal B_0 is selective for the decomposition of the set A which is determined by the $\bar{p}$-ideal A_0. Thus using Theorem 2(3^o) we get (5).

(ii) Let us assume that there esists a $\bar{p}$-ideal A_0 of the n-group (A,f) such that the conditions (4) and (5) hold. Thus, the decomposition $\{B_s\}_{s\in S}$ of the set B determined by the $\bar{p}$-ideal B_o is selective for the decomposition $\{A_t\}_{t\in T}$ of the set A which is determined by the $\bar{p}$-ideal A_0. Let σ be a congruence on the n-group (A,f) determined by the $\bar{p}$-ideal A_0. Define the mapping $\tilde{h}: A \longrightarrow C$ by

$$\tilde{h}(x) = h(b)$$

for an arbitrary $x \in A$, where $b \in B$ is an arbitrary fixed element such that $(x,b) \in \sigma$. Let us notice that $\mathrm{Im}\,\tilde{h} = \mathrm{Im}\,h$. Take arbitrary elements $x_1, x_2, \ldots, x_n \in A$. There exist elements $b_1, b_2, \ldots, b_n \in B$ such that $(x_1, b_1) \in \sigma$, $(x_2, b_2) \in \sigma$, ..., $(x_n, b_n) \in \sigma$. Since σ is a congruence on the n-group (A,f) we have $(f(x_1, x_2, \ldots, x_n), f(b_1, b_2, \ldots, b_n)) \in \sigma$. Hence

$\tilde{h}(f(x_1, x_2, \ldots, x_n)) = h(f(b_1, b_2, \ldots, b_n)) = f(h(b_1), h(b_2), \ldots, h(b_n)) = f(\tilde{h}(x_1), \tilde{h}(x_2), \ldots, \tilde{h}(x_n))$. Thus $\tilde{h}$ is a homomorphism from the n-group (A,f) into the n-groupoid (C,f).

Theorem 4. Let (B,f) be an n-subgroup of an n-group (A,f) and $p \in B$. Let $h: B \longrightarrow C$ be a homomorphism from the n-group (B,f) onto an n-group (C,f). Let B_0 be a $\bar{p}$-ideal determined by the congruence kerh on the n-group (B,f). The n-group (C,f) can be extended to an n-group $(\tilde{C},f)$ in such a way that there exists a homomorphism $\tilde{h}: A \longrightarrow \tilde{C}$ from the n-group (A,f) onto the n-group $(\tilde{C},f)$ which is the extension of the homomorphism h if and only if there exists a $\bar{p}$-ideal A_0 of the n-group (A,f) such that the condition (4) holds.

Proof. To prove (4) it is enough to take as A_0 the $\bar{p}$-ideal determined by the congruence ker$\tilde{h}$ on the n-group (A,f) and to apply the same argument as in the proof of the condition (4) in Theorem 3. Let A_0 be a $\bar{p}$-ideal of the n-group (A,f) such that the condition (4) holds. It follows from Theorem 2 (2^o) that the decomposition of the set B determined by the $\bar{p}$-ideal B_0 is semiselective for the decomposition of the set A determined by the $\bar{p}$-ideal A_0. Put $B_t = h^{-1}(\{t\})$ for any $t \in C$. In this manner the elements of the quotient n-group $(B/B_0,f)$ are indexed one-to-one by means of the elements of the n-group (C,f). Let $\tilde{C}$ be an arbitrary set such that $C \subset \tilde{C}$ and card$\tilde{C}$ = cardA/A_0. With the aid of the elements of the set $\tilde{C}$ we index one-to-one the elements of the A/A_0 in such a way that

$$\forall\, t_1 \in C \quad \forall\, t_2 \in \tilde{C} : \quad [B_{t_1} \subset A_{t_2} \Leftrightarrow t_1 = t_2] \; .$$

In the set $\tilde{C}$ we define an n-ary operation f as follows:

$$f(t_1,t_2,\ldots,t_n) = t_{n+1} \Leftrightarrow f(A_{t_1},A_{t_2},\ldots,A_{t_{n+1}}$$

for arbitrary $t_1, t_2, \ldots, t_{n+1} \in \tilde{C}$. It is easy to notice that $(\tilde{C},f)$ is an n-group isomorphic to the quotient n-group $(A/A_0,f)$. Moreover, (C,f) is an n-subgroup of the n-group $(\tilde{C},f)$. The function $\tilde{h}: A \longrightarrow \tilde{C}$ defined by

$$\tilde{h}(x) = t \Leftrightarrow x \in A_t$$

for an arbitrary $x \in A$ is a homomorphism from the n-group (A,f) onto the n-group $(\tilde{C},f)$. The homomorphism $\tilde{h}$ is the extension of the homomorphism h. The proof is complete.

REFERENCES

[1] V. D. Belousov: n-ary quasigroups, (in Russian), Kišinev, 1972.

[2] W. A. Dudek, J. Michalski: On a generalization of Hosszú theorem, Demonstratio Math., 15 (1982), 783-805.

[3] A. Grząślewicz: On extensions of homomorphisms, Aequationes Math., 17 (1978), 199-207.

[4] M. Hosszú: On the explicit form of n-group operations, Publ. Math., Debrecen, 10 (1963), 88-92.

[5] J. D. Monk, F.D. Sioson: On general theory of m-groups, Fund. Math., 72 (1971), 233-244.

[6] E. I. Sokolov: The Gluskin-Hosszú theorem for Dörnte n-groups, Nets and quasigroups (in Russian), Mat. Issled., 39 (1976), 187-189.

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