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Dušan Kollár, Robert Šulka

ON SOME CLOSURE OPERATORS ON SEMIGROUPS

1. Introduction

In papers [1] and [2] some closure operators on semigroups were studied. In this article some other closure operators on semigroups are introduced, using nilpotency in semigroups. We study topologies, induced on a semigroup by these closure operators and characterize some classes of semigroups by means of these notions.

Let S be a semigroup and $U: 2^S \to 2^S$. The mapping U is called a closure operation on S in sense of Kuratowski if the following conditions hold for $M, M_1, M_2 \subseteq S$:

$M \subseteq U(M)$,

$U(\emptyset) = \emptyset$,

$U(U(M)) = U(M)$,

$U(M_1 \cup M_2) = U(M_1) \cup U(M_2)$.

The mapping U is called a closure operation on S in sense of Čech if the following conditions hold:

$M \subseteq U(M)$,

$U(\emptyset) = \emptyset$,

$U(U(M)) = U(M)$,

if $M_1 \subseteq M_2$, then $U(M_1) \subseteq U(M_2)$.

We mention that if U is a closure operation in sense of Kuratowski, then it is closure operation in sense of Čech. In fact if $M_1 \subseteq M_2$, then $M_1 \cup M_2 = M_2$ and this implies that $U(M_2) = U(M_1 \cup M_2) \supseteq U(M_1)$.

Let S be a semigroup, $M \subseteq S$. We denote

$N_1(M) = \{x \in S \mid x^n \in M$ for almost all $n \in \{1,2,\dots\}\}$,
$N_2(M) = \{x \in S \mid x^n \in M$ for infinitely many $n \in \{1,2,\dots\}\}$,

$N_3(M) = \{x\in S \mid x^n \in M$ for at least one $n\in\{1,2,\dots\}\}$.

In paper [2] it was shown, that the mapping

$$N_3\colon 2^S \longrightarrow 2^S,\ M \longmapsto N_3(M)$$

is closure operation in sense of Kuratowski. The open sets in the topology induced by N_3 on S are exactly those subsets of S, that are unions of some systems of subsemigroups of S, and the empty set. The system $\sum_3(S) = \{\langle a\rangle \mid a\in S\}$, where $\langle a\rangle$ is a cyclic semigroup generated by a, is the complete system of neighborhoods for this topology.

In paper [3] was proved, that the following relations hold for $M, M_1, M_2 \subseteq S$:

(i) $N_1(M) \subseteq N_2(M) \subseteq N_3(M)$,

(ii) if $M_1 \subseteq M_2$, then $N_i(M_1) \subseteq N_i(M_2)$ for $i=1,2,3$,

(iii) $N_2(M_1 \cup M_2) = N_2(M_1) \cup N_2(M_2)$.

2. The closure operation U_2

Let $U_2\colon 2^S \longrightarrow 2^S$, $U_2(M) = N_2(M) \cup M$.

Theorem 1. U_2 is a closure operation in sense of Kuratowski.

Proof. It is clear, that

a) $M \subseteq U_2(M)$ and b) $U_2(\emptyset)=\emptyset$.

c) $U_2(U_2(M)) = U_2(N_2(M)\cup M) = N_2(N_2(M)\cup M) \cup N_2(M) \cup M =$
$= N_2(N_2(M)) \cup N_2(M) \cup M \subseteq N_2(M) \cup M = U_2(M)$, hence $U_2(U_2(M)) \subseteq U_2(M)$.

From a) we get $U_2(M) \subseteq U_2(U_2(M))$. Therefore $U_2(U_2(M)) =$
$= U_2(M)$.

d) $U_2(M_1\cup M_2) = N_1(M_1\cup M_2) \cup M_1 \cup M_2 = N_1(M_1) \cup N_2(M_2) \cup M_1 \cup M_2 =$
$= U_2(M_1) \cup U_2(M_2)$. Hence $U_2(M_1\cup M_2) = U_2(M_1) \cup U_2(M_2)$.

Theorem 2. M is closed subset of S iff $N_2(M) \subseteq M$.

Proof. $U_2(M)=M \iff N_2(M)\cup M=M \iff N_2(M) \subseteq M$.

Theorem 3. M is an open subset of S iff $M \subseteq N_1(M)$.

Proof. M is open iff CM is closed and this holds iff $N_2(CM) \subseteq CM$.

Let $N_2(CM) \subseteq CM$. Then every element x having infinitely many powers x^n in CM belongs to CM. Hence for every element $x \in M$ almost all powers x^n belong to M. (Otherwise infinitely many powers x^n would belong to CM, and therefore $x \in CM$ would hold.) Hence $M \subseteq N_1(M)$.

Now let $M \subseteq N_1(M)$. Then all elements $x \in M$ have almost all powers x^n in M. Therefore if $x \in N_2(CM)$ i.e. infinitely many powers x^n belong to CM, then $x \notin M$. (If $x \in M$ would hold, then almost all powers x^n would belong to M and almost all powers x^n could not belong to CM.) Hence $x \in N_2(CM)$, implies $x \in CM$ i.e. $N_2(CM) \subseteq CM$.

Let $a \in S$. Let $G_{n_0}(a) = \{a^n | n \geq n_0,\ n \in \{1,2,\dots\}\}$ for all $n_0 \in \{1,2,\dots\}$. Then $O_{n_0}(a) = \{a\} \cup G_{n_0}(a)$ is a neighborhood of a for all $n_0 \in \{1,2,\dots\}$, because evidently

$$\{a\} \cup G_{n_0}(a) \subseteq N_1(\{a\} \cup G_{n_0}(a)).$$

Let $a \in S$, and a be of finite order. Then $\langle a \rangle = P(a) \cup G(a)$, where $G(a)$ is the maximal subgroup in $\langle a \rangle$ and $P(a) = \langle a \rangle \setminus G(a)$.

$O_0(a) = \{a\} \cup G(a)$ is a neighborhood of a, because $\{a\} \cup G(a) \subseteq N_1(\{a\} \cup G(a))$. $O_0(a)$ is clearly the smallest neighborhood of a.

Theorem 4. The system $\Sigma_2(S) = \{O_0(a) | a \in S$, a is of finite order$\} \cup \{O_{n_0}(a) | a \in S$, a is of infinite order, $n_0 \in \{1,2,\dots\}\}$ is the complete system of neighborhoods for the topology, generated on S by the closure operation U_2.

Proof. Let M be an open subset of S and $a \in M$. Since M is an open set, $a \in M \subseteq N_1(M)$ holds i.e. almost all powers a^n belong to M. Hence there exists an n_0 such that $a^n \in M$ holds for all $n \geq n_0$. This means, that $O_{n_0}(a) = \{a\} \cup G_{n_0}(a) \subseteq M$.

If a is of finite order, then $O_0(a) = \{a\} \cup G(a) \subseteq M$.

Every open set M is therefore a union of some sets of $\Sigma_2(S)$. This proves our Theorem.

Lemma 1. Let $a \in S$, $\langle a \rangle = G(a)$ and the cyclic group $\langle a \rangle$ generated by a be of higher order than 2. Then $a^{-1} \neq a$ and $O_0(a) = O_0(a^{-1})$.

Proof. Under these conditions $P(a) = \emptyset$. Let e be the identity of $G(a)$. Evidently $a\neq e$, $a^2\neq e$ (in other cases either $|\langle a\rangle|=1$ or $|\langle a\rangle|=2$ would hold).

Hence $a^n=e\,(n>2)$. Therefore $a^{-1}=a^{n-1}\neq a$. But both a and a^{-1} generate $\langle a\rangle$. This means, that
$O_0(a) = G(a) = \langle a\rangle = G(a^{-1}) = O_0(a^{-1})$, where $a\neq a^{-1}$.

Lemma 2. Let $a,b\in S$, $a\neq b$. Let a be of infinite order and b be of finite order. Then $O_1(a)\cap O_0(b)=\emptyset$.

Proof. Clearly $\langle a\rangle\cap\langle b\rangle=\emptyset$. But since
$O_1(a)\subseteq\langle a\rangle$ and $O_0(b)\subseteq\langle b\rangle$ we get $O_1(a)\cap O_0(b)=\emptyset$.

Lemma 3. Let $a,b\in S$, $a\neq b$ and a and b be of infinite order. Then the following relations hold:

a) there exist m_0 and n_0 such that $b\notin O_{m_0}(a)$ and $a\notin O_{n_0}(b)$,

b) If $b\in\langle a\rangle$, then $O_{m_0}(a)\cap O_{n_0}(b)\neq o$ for all m_0 and n_0.

Proof. a) If $b\notin\langle a\rangle$ and $a\notin\langle b\rangle$, then $b\notin O_1(a)=\langle a\rangle$ and $a\notin O_1(b)=\langle b\rangle$.

If $b\in\langle a\rangle$, then $b=a^k$ $(k>1)$. For $m_0>k$ we get $b=a^k\notin O_{m_0}(a)= =\{a^m\,|\,m\geq m_0\}\cup\{a\}$. Moreover $a\notin O_1(b)=O_1(a^k)=\{a^k\}\cup\{a^{kn}\,|\,n\geq 1\}=\langle a\rangle$.

b) We have again $b=a^k\,(k>1)$. Let $O_{m_0}(a)=\{a\}\cup\{a^m\,|\,m\geq m_0\}$, $O_{n_0}(b)=O_{n_0}(a^k)=\{a^k\}\cup\{a^{kn}\,|\,n\geq n_0\}$. If we choose $n\geq n_0$ such that $kn\geq m_0$, then $a^{kn}\in O_{n_0}(b)$ and $a^{kn}\in O_{m_0}(a)$. Hence $O_{m_0}(a)\cap$ $\cap O_{n_0}(b)\neq\emptyset$.

Lemma 4. Let $a,b\in S$, $a\neq b$ and a and b be of finite order. Let $\langle a\rangle$ and $\langle b\rangle$ be not groups of order higher than 2. Then either $a\notin O_0(b)$ or $b\notin O_0(a)$.

Proof. Let $b\notin\langle a\rangle$ and $a\notin\langle b\rangle$. Then $a\notin O_0(b)\subseteq\langle b\rangle$ and $b\notin O_0(a)\subseteq\langle a\rangle$.

Let $b\in\langle a\rangle$ and $a\notin G(a)$. Then $b=a^k$ $(k>1)$, $O_0(b)=O_0(a^k)= =\{a^k\}\cup G(a^k)\subseteq\{a^k\}\cup G(a)$. Therefore $a\notin O_0(b)$.

Now let $b\in\langle a\rangle$ and $a\in G(a)$. Then $|\langle a\rangle|=2$, $b=e$, where e is the identity of $G(a)$. Therefore $a\notin\{e\}=O_0(e)=O_0(b)$.

Lemma 5. Let $a\in S$, a be of finite order and let $|\langle a\rangle|>1$. Let $b\in G(a)$ and $b\neq a$. Then $b\in O_o(a)$.

Proof. If $b\in G(a)$, then $b\in\{a\}\cup G(a)=O_o(a)$.

Lemma 6. Let $a,b\in S$, $a\neq b$ and a and b be idempotents. Then $O_o(a)\cap O_o(b)=\emptyset$.

Proof. It is sufficient to observe, that $O_o(a)=\{a\}$ and $O_o(b)=\{b\}$.

From these lemmas we get the following Theorems.

Theorem 5. The topology induced on S by U_2 is a T_0-topology iff all finite cyclic subgroups of S are at most of order 2.

Proof. If the topology induced by U_2 on S is a T_0-topology, then by Lemma 1, all finite cyclic subgroups of S are at most of order 2. The second part of the proof follows from Lemma 2., 3. and 4.

Theorem 6. The topology induced on S by U_2 is a T_1-topology if all elements of finite order of S are idempotents.

Proof. If the topology induced by U_2 on S is a T_1-topology, then by Lemma 5, for every element of finite order $|\langle a\rangle|=1$. This means, that every element of finite order is an idempotent. The second part of the proof follows from Lemma 2.,3. and 6.

Theorem 7. The topology induced by U_2 on S is a T_2-topology iff all elements of S are idempotents.

Proof. If the topology induced by U_2 on S is a T_2-topology, then by Lemma 3, the semigroup S contains only elements of finite order and by Lemma 5, it contains only idempotents. The second part of the proof follows from Lemma 6.

Theorem 8. $U_2=N_3$ iff every cyclic subsemigroup of S is either a cyclic group or a cyclic semigroup $\langle a\rangle$ such that $|P(a)|=1$.

Proof. Comparing the complete systems of neighborhoods, we get that S does not contain elements of infinite order and that $<a>=\{a\}\cup G(a)$.

3. The closure operation U_1

Let $U_1: 2^S \longrightarrow 2^S$, $U_1(M)=M\cup N_1(M)$.

Lemma 7. For all $M, M_1, M_2 \subseteq S$, the following hold
$M\subseteq U_1(M)$,
$U_1(\emptyset)=\emptyset$,
if $M_1\subseteq M_2$, then $U_1(M_1)\subseteq U_1(M_2)$.

Proof. The first two relations are evident. The third statement follows from the fact, that if $M_1\subseteq M_2$, then $N_1(M)\subseteq \subseteq N_2(M)$.

Lemma 8. If U_1 is a closure operation on S in sense of Čech, then S is a periodic semigroup.

Proof. Let S contain an element a of infinite order. Let

$$M=<a>\setminus(\{a\}\cup\{a^p \mid p \text{ is a prime}\}).$$

Then $a\notin M$, $a\notin N_1(M)$, hence $a\notin U_1(M)$.

On the other hand $<a>\setminus\{a\}\subseteq U_1(M)$. Therefore $a\in N_1(U_1(M))$, hence $a\in U_1(M)\cup N_1(U_1(M))=U_1(U_1(M))$.

We have obtained that $a\notin U_1(M)$ but $a\in U_1(U_1(M))$. The equality $U_1(U_1(M))=U_1(M)$ does not hold, i.e. U_1 is not a closure operation on S in sense of Čech.

Lemma 9. Let S be a periodic semigroup. Let $M\subseteq S$. Then $x\in U_1(M)=M\cup N_1(M)$ iff either $x\in M$ or $G(x)\subseteq M$.

Proof. Evidently $x\in N_1(M)$ iff $G(x)\subseteq M$. To end the proof it is sufficient to use the equality $U_1(M)=M\cup N_1(M)$.

Theorem 9. U_1 is a closure operation S in sense of Čech, iff S is a periodic semigroup.

Proof. By Lemma 8. if U_1 is a closure operation in sense of Čech, then S is a periodic semigroup.

Now we shall prove the converse statement.

Let S be an arbitrary semigroup and $M, M_1, M_2\subseteq S$. Then $M\subseteq M\cup N_1(M)=U_1(M)$ and $M_1\subseteq M_2 \Longrightarrow N_1(M_1)\subseteq N_1(M_2)$. Therefore $U_1(M)\subseteq \subseteq U_1(U_1(M))$.

We shall prove, that in a periodic semigroup S also the converse inclusion holds.

Let S be a periodic semigroup. If $x\in S$, then x is of finite order.

If $x\in U_1(U_1(M))$, then either $x\in U_1(M)$ or $G(x)\subseteq U_1(M)$.

Let $G(x)\subseteq U_1(M)$. Then for every $y\in G(x)$ either $y\in M$ or $G(y)\subseteq M$. But G(x) is a finite cyclic group and $y\in G(x)$. Therefore $y\in\langle y\rangle=G(y)\subseteq M$. Hence $y\in G(x)$ implies $y\in M$. This means that $G(x)\subseteq M$. But then $x\in U_1(M)$.

We have proved, that if $x\in U_1(U_1(M))$, then $x\in U_1(M)$ i.e. $U_1(U_1(M))\subseteq U_1(M)$ and this, together whith the inclusion $U_1(M)\subseteq U_1(U_1(M))$ gives the equality $U_1(U_1(M))=U_1(M)$.

This result together with Lemma 7 means that if S is a periodic semigroup, then U_1 in a closure operation on S in sense of Čech.

Theorem 10. Let U_1 be a closure operation on S in sense of Čech. Then a set $M\subseteq S$ is closed iff $N_1(M)\subseteq M$.

Proof. M is a closed set iff $M=U_1(M)$ i.e. iff $M=M\cup N_1(M)$ and this holds iff $N_1(M)\subseteq M$.

Theorem 11. Let U_1 be a closure operation on S in sense of Čech. Then a set $M\subseteq S$ is an open set iff $M\subseteq N_2(M)$.

Proof. M is open iff CM is closed and this holds iff $N_1(CM)\subseteq CM$.

Let $N_1(CM)\subseteq CM$. Then every element x having almost all powers x^n in CM belongs to CM. Therefore every element $x\in M$ has infinitely many powers x^n in M (if not, then x would be a member of CM). This means, that $M\subseteq N_2(M)$.

Now let $M\subseteq N_2(M)$. Then every element $x\in M$ has infinitely many powers x^n in M. Therefore every element x having almost all powers x^n in CM belongs to CM. This means, that $N_1(CM)\subseteq CM$.

Lemma 10. Let U_1 be a closure operation on S in sense of Kuratowski. Then S is a periodic semigroup.

The proof follows from Lemma 8. and from the fact, that U_1 is a closure operation in sense of Čech.

Lemma 11. If U_1 is a closure operation on S in sense of Kuratowski, then for every $a \in S$ such that a is of finite order and $a \notin G(a)$ we have $|G(a)|=1$.

Proof. Let $a \in S$, a be of finite order, $a \notin G(a)$, $a^i \neq a^k$, $a^i, a^k \in G(a)$ i.e. $|G(a)|>1$. Then $M_1=\{a,a^i\}$ and $M_2=\{a,a^k\}$ are open sets (Theorem 11.), but $M_1 \cap M_2=\{a\}$ is not an open set (by Theorem 11.). This is a contradiction to the fact, that U_1 is a closure operation.

The foregoing lemmas imply the following theorem.

Theorem 12. Let U_1 be a closure operation on S in sense of Kuratowski. Then S is a periodic semigroup and for every element a such that $a \notin G(a)$, $|G(a)|=1$ holds.

Theorem 13. Let U_1 be a closure operation on S in sense of Čech. Then $\sum_1(S)=\{\{a\}|a \in G(a),\ a \in S\} \cup \{\{a,a^k\}|a \notin G(a),\ a^k \in G(a),\ a \in S\}$ is a complete system of neighbourhoods of the topology induced on S by U_1.

Proof. If $a \in G(a)$, then $\{a\}$ is an open set, that contains a, hence $\{a\}$ is a neighborhood of a.

If $a \notin G(a)$, $a^k \in G(a)$, then $\{a,a^k\}$ is an open set containing a, hence $\{a,a^k\}$ is a neighborhood of a.

On the other hand, let M be an open set and $a \in M$. By Theorem 9, S is a periodic semigroup, therefore the element a is of finite order.

If $a \notin G(a)$, then since M is an open set and $a \in M$, there exist infinitely many powers a^n belonging to M, i.e. there exists a power a^k such that $a^k \in G(a)$ and $a^k \in M$. Therefore $\{a,a^k\} \subseteq M$, where $a^k \in G(a)$.

If $a \in G(a)$ and $a \in M$, then $\{a\} \subseteq M$, where $a \in G(a)$.

This implies, that every open subset of S is a union of some subsystem of the system $\sum_1(S)$. Therefore $\sum_1(S)$ is a complete system of neighborhoods of the topology, induced on S by U_1.

Theorem 14. Let S be a periodic semigroup and for every element $a \in S$ such that $a \notin G(a)$ let $|G(a)|=1$. Then U_1 is a closure operation on S in sense of Kuratowski.

Proof. In view of Theorem 9, it is sufficient to show, that $U_1(M_1 \cup M_2)=U_1(M_1)\cup U_1(M_2)$.

a) $M_1 \subseteq M_2 \cup M_2 \Rightarrow U_1(M_1)\subseteq U_1(M_1\cup M_2)$,
$M_2\subseteq M_1\cup M_2 \Rightarrow U_1(M_2)\subseteq U_1(M_1\cup M_2)$.
Hence
$U_1(M_1)\cup U_1(M_2)\subseteq U_1(M_1\cup M_2)$.

b) Let $x\in U_1(M_1\cup M_2)$. Then either $x\in M_1\cup M_2$ or $G(x)\subseteq M_1\cup M_2$.

If $x\in M_1\cup M_2$, then either $x\in M_1$ or $x\in M_2$. Hence either $x\in U_1(M_1)$ or $x\in U_1(M_2)$. In both cases $x\in U_1(M_1)\cup U_1(M_2)$.

If $x\in G(x)\subseteq M_1\cup M_2$, then $x\in M_1\cup M_2$ and we again have that $x\in U_1(M_1)\cup U_1(M_2)$.

If $x\notin G(x)$, then $G(x)=\{x^k\}\subseteq M_1\cup M_2$. Hence either $x^k\in M_1$ or $x^k\in M_2$ i.e. either $G(x)=\{x^k\}\subseteq M_1$ or $G(x)=\{x^k\}\subseteq M_2$. Therefore either $x\in U_1(M_1)$ or $x\in U_1(M_2)$. In both cases $x\in U_1(M_1)\cup U_1(M_2)$.

This means, that
$$U_1(M_1\cup M_2)\subseteq U_1(M_1)\cup U_1(M_2),$$
what together with
$$U_1(M_1)\cup U_1(M_2)\subseteq U_1(M_1\cup M_2)$$
gives the equality
$$U_1(M_1\cup M_2)=U_1(M_1)\cup U_1(M_2).$$
Now the proof is finished.

Let U_1 be a closure operation on S in sense of Kuratowski. If $a\notin G(a)$, then $G(a)=\{a^k\}$. Let us denote $O(a)=\{a,a^k\}$. If $a\in G(a)$, let $O(a)=\{a\}$. Clearly $O(a)$ is the smallest neighborhood of the element a in the topology induced on S by U_1.

Lemma 12. Let U_1 be a closure operation on S in sense of Kuratowski. Let $a,b\in S$, $a\neq b$. Then the following statements hold:

a) If $a\notin G(a)=\{a^m\}$, $O(a)=\{a,a^m\}$ and $b\notin G(b)=\{b^n\}$, $O(b)=\{b,b^n\}$, then $b\notin O(a)$ and $a\notin O(b)$.

b) If $a\notin G(a)=\{a^k\}$, $O(a)=\{a,a^k\}$ and $b\in G(b)$, $O(b)=\{b\}$, then $a\notin O(b)$.

c) If $a\in G(a)$, $O(a)=\{a\}$ and $b\in G(b)$, $O(b)=\{b\}$, then $O(a)\cap O(b)=\emptyset$.

d) If $a\notin G(a)=\{a^k\}$, $O(a)=\{a,a^k\}$, then $a^k\neq a$ and $a^k\in O(a)$.

Proof. a) Let $b\in O(a)$. Then $b=a^m$, i.e. $\{b\}=\{a^m\}=G(a)=G(b)$. Hence $b\in G(b)$ and therefore $O(b)=\{b\}$. But this is a contradiction to the fact, that $O(b)=\{b,b^n\}$, where $b\neq b^n$, because $b\notin G(b)=\{b^n\}$. Therefore $b\notin O(a)$.

Similarly one can prove, that $a\notin O(b)$.

The statements b), c), d) are evident.

Theorem 15. U_1 is closure operation on S in sense of Kuratowski iff S is a periodic semigroup and for every element a such that $a\notin G(a)$, $|G(a)|=1$.

The topology induced by U_1 on S is a T_o-topology.

The proof follows from Theorem 12 and 14, and from Lemma 12.

Theorem 16. Let U_1 be a closure operation on S in sense of Kuratowski. Then the topology induced by U_1 on S is a T_1-topology iff S is a periodic semigroup whose every cyclic subsemigroup is a group.

The topology, induced by U_1 on S is then the discrete topology.

The proof follows from Lemma 12.

4. Connections between U_1, U_2 and N_3

${}^1O(a)$ will denote the smallest neighborhood of the element a in the topology, induced on S by U_1, ${}^2O(a)$ the smallest neighborhood of the element a in the topology induced on S by U_2.

Theorem 17. $U_1=U_2$ on S iff S is a periodic semigroup and for every $a\in S$, $|G(a)|=1$ holds.

Proof. a) If $U_1=U_2$, Then U_1 is a closure operation in sense of Kuratowski and S must be clearly a periodic semigroup. Hence it is sufficient to consider elements of finite order.

If $a\notin G(a)=\{a^k\}$, then ${}^1O(a)={}^2O(a)$ means, that $\{a,a^k\}=$ $=\{a\}\cup G(a)$ i.e. $G(a)=\{a^k\}$. Hence $|G(a)|=1$.

If $a \in G(a)$, then $^1O(a)={}^2O(a)$ means, that $\{a\}=\{a\}\cup G(a)$ i.e. $G(a)=\{a\}$. Hence $|G(a)|=1$.

b) If for every $a \in S$, $|G(a)|=1$, then by Theorem 15, U_1 is a closure operation in sense of Kuratowski and we can easily see, that the complete systems of neighborhoods of the topologies, induced on S by closure operations U_1 and U_2 are equal. Hence $U_1=U_2$.

Theorem 18. $U_1=N_3$ on S iff a^2 is an idempotent for any $a \in S$.

Proof. We know, that $U_1(M) \subseteq U_2(M) \subseteq N_3(M)$ for every $M \in 2^S$. This means, that $U_1=N_3$ iff $U_1=U_2=N_3$. From Theorem 17 and 8, it follows, that $U_1=N_3$ iff S is a periodic semigroup and every element of S is either idempotent or
$\langle a \rangle = P(a) \cup G(a)$, where $|P(a)| = |G(a)|=1$.
Hence $U_1=N_3$ iff a^2 is an idempotent for any $a \in S$.

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INSTITUTE OF MATHEMATICS, SLOVAK TECHNICAL UNIVERSITY,
812-19 BRATISLAVA, CZECHOSLOVAKIA

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