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MULTIPLICATION GROUPS OF QUASIGROUPS: ELEMENTARY COMBINATORICS

1. Preliminaries

Assume that G is a set of permutations acting on a set Q. It might be difficult (or even impossible) to construct a quasigroup $(Q,\cdot)$ such that all rows and, simultaneously, all columns of the multiplication table belong to G. If such a quasigroup exists, it will have certain properties, following from the structure of G. It is a main purpose of this paper to exhibit such properties, thus continuing earlier work done in [9] and [10]. In particular, the results of [10] are extended, and elementary proofs are presented which have not been included there. Throughout this paper, it will always be assumed that G is a group of permutations. By this assumption, the techniques of (permutation) group theory will become available. For the needed group theoretic notations the reader is referred to Huppert [8] and Wielandt [18]. The basic notations of universal algebra can be found in Burris, Sankappanavar [5], or in [11]. However, the rest of this section will give some definitions and facts which are fundamental for this paper. Let $(Q,\cdot)$ be a *groupoid*, i.e. Q is a set and $(x,y) \longmapsto x\cdot y$ is a binary operation on Q. For each element $a\in Q$, the *left multiplication mapping* $L(a) : Q \longrightarrow Q$ is defined by $xL(a) := a\cdot x$ and, analogously, the *right multiplication mapping* $R(a) : Q \longrightarrow Q$ by $xR(a) := x\cdot a$ (in general, the image of x under a mapping M will be written as xM). If *all* left and right multiplication mappings are permutations of Q, then the groupoid $(Q,\cdot)$ will be called a *quasigroup*. In other words, a quasigroup is a groupoid with the property that in every row and in every column of the multiplication table each element of Q occurs exactly once. The *multiplication group* $Mlt(Q,\cdot)$ of a quasigroup $(Q,\cdot)$ is defined as the permutation group $\langle L(a),R(a) \mid a\in Q\rangle$, generated by all left and all right multiplication mappings. Hence, if G is a *group* of permutations, then the motivating question for this paper can be formulated as follows:

Question. Given a permutation group G acting on a set Q, which are quasigroups $(Q,\cdot)$ with $Mlt(Q,\cdot)\leq G$?

On every quasigroup $(Q,\cdot)$ two additional binary operations / and \ (*right division and left division*) can be defined by $x/y := xR(y)^{-1}$ and $x\backslash y := yL(x)^{-1}$. In some situations it is advantageous to consider the (universal) algebra $(Q,\cdot,/,\backslash)$ instead of $(Q,\cdot)$ (cf. Birkhoff [3], p. 160). This becomes obvious in the following proposition which is easy to prove:

Proposition 1. Let $(Q,\cdot)$ be a quasigroup. Then the algebras $(Q,Mlt(Q,\cdot))$ and $(Q,\cdot,/,\backslash)$ have the same congruence relations. If the base set Q is finite, then also $(Q,\cdot)$ has the same congruence relations.

For more detailed information on the close structural relationship between quasigroups and their multiplication groups one should consult e.g. Albert [1] and [2] (who introduced the concept of multiplication groups) or the book of Bruck [4].

A *loop* is a quasigroup $(Q,+)$ with neutral element, i.e. with an element $e\in Q$ satisfying $e+x = x+e = x$ for all $x\in Q$. There is a standard method how to convert a quasigroup $(Q,\cdot)$ into a loop: Choose elements $a,b\in Q$, and set $x+y := (x/a)\cdot(b\backslash y)$. Then + is a loop operation on Q with neutral element $e := b\cdot a$. The quasigroup operation can be regained from the loop operation as $x\cdot y = xR+yL$, where R and L stand for the right and the left multiplication mappings $R(a)$ and $L(b)$ of $(Q,\cdot)$. Hence part a) of the following proposition implies immediately part b). The proof of part a) is an easy exercise:

Proposition 2. a) Let $(Q,+)$ be a loop, R and L permutations on Q, and let $(Q,\cdot)$ be the quasigroup defined by $x\cdot y := xR+yL$. Then $Mlt(Q,\cdot)$ is the permutation group $\langle Mlt(Q,+),R,L\rangle$ on Q generated by $Mlt(Q,+)$, R and L.

b) For every quasigroup $(Q,\cdot)$ and each $e\in Q$ there exists a loop $(Q,+)$ with neutral element e and $Mlt(Q,+) \le$ $\le Mlt(q,\cdot)$.

By part b) of this proposition, in the rest of this paper the interest is focused on loops: if there exists a quasigroup with multiplication group contained in a permutation group G, then there must exist a loop with the same property. Part a) was used in [9] in order to show that all finite symmetric, alternating, dihedral, general linear and projective general linear groups, and the Mathieu groups M_{11} and M_{23} occur as multiplication groups of quasigroups (i.e. the multiplication groups of these quasigroups are not only contained in the listed permutation groups, they are even *equal* to them): in all cases, the loop $(Q,+)$ can be chosen as an abelian group, and in most cases as a cyclic group (the exceptions are the alternating groups of even degree).

2. Main results

The following assumptions and denotations will be used throughout the rest of this paper: Let G always be a permutation group acting on a finite set Q, and let e be some (fixed) element of Q. The different orbits of the stabilizer G_e of e in G are denoted by $Q_1=\{e\}$, $Q_2,\ldots,Q_s$. All loops $(Q,+)$ are assumed to have e as neutral element.

For loops $(Q,+)$ with $Mlt(Q,+)\le G$, the stabilizer G_e will play an important role. This is, basically, due to the following simple observation on permutation groups:

Lemma 1. Let G be a permutation group acting on Q. For $A,B\in G$, let $eA=eB$. Then $Q_iA=Q_iB$ for all orbits Q_i of G_e.

The first of the next two lemmas shows, roughly speaking, that "modulo G_e" there is at most one loop with multiplication group contained in G. By the second of these lemmas, all such loops are commutative and associative "modulo G_e":

Lemma 2. Let $(Q,+)$ and $(Q,\circ)$ be loops with $\mathrm{Mlt}(Q,+)$ and $\mathrm{Mlt}(Q,\circ)$ contained in G, Then the following holds for all orbits Q_i of G_e and all $a \in Q$:

$$\{a+x \mid x \in Q_i\} = \{a \circ y \mid y \in Q_i\},$$

$$\{x+a \mid x \in Q_i\} = \{y \circ a \mid y \in Q_i\}.$$

Proof. The left multiplication mappings of $(Q,+)$ and of $(Q,\circ)$ are denoted by $L_+(a)$ and $L_\circ(a)$, respectively. Obviously, then $eL_+(a) = eL_\circ(a)$. Lemma 1 therefore implies $Q_iL_+(a) = Q_iL_\circ(a)$. This can be written in the form $\{a+x \mid x \in Q_i\} = \{a \circ y \mid y \in Q_i\}$. The second assertion can be proved analogously, by using right instead of left multiplication mappings.

Lemma 3. Let $(Q,+)$ be a loop with $\mathrm{Mlt}(Q,+) \subseteq G$. Then the following holds for all orbits Q_i of G_e and all $a,b \in Q$:

a) $\{a+x \mid x \in Q_i\} = \{y+a \mid y \in Q_i\}$,

b1) $\{(a+b)+x \mid x \in Q_i\} = \{a+(b+y) \mid y \in Q_i\}$,

b2) $\{(a+x)+b \mid x \in Q_i\} = \{a+(y+b) \mid y \in Q_i\}$,

b3) $\{(x+a)+b \mid x \in Q_i\} = \{y+(a+b) \mid y \in Q_i\}$.

Proof. a) Since $eL(a) = eR(a) = a$, Lemma 1 yields $Q_iL(a) = Q_iR(a)$ which can be written as $\{a+x \mid x \in Q_i\} = \{y+a \mid y \in Q_i\}$.

b1) From $eL(b) = eL(b)L(a)$ one obtains $Q_iL(a+b) = Q_iL(b)L(a)$, again by Lemma 1. This can be written in the claimed form.

b2) and b3) can be proved in the same way as b1).

The following numerical values will be used in the rest of this section: Let f be the least positive integer such that each non-identity permutation of G fixes at most f elements. Define $n := |Q|$, $n_i := |Q_i|$ for every orbit Q_i of G_e, and $m := \sum \{n_i \mid n/n_i > f\}$.

The last two lemmas can be stated in a stronger form if the orbit Q_i is "small":

Lemma 4. Let $(Q,+)$ and $(Q,\circ)$ be loops with $Mlt(Q,+)$ and $Mlt(Q,\circ)$ contained in G. Let Q_i be an orbit of G_e with $n/n_i>f$. Then

$$a+x = a\circ x, \quad x+a = x\circ a$$

for all $a\in Q$, $x\in Q_i$.

Proof. Assume that $x\in Q_i$. By the first assertion of Lemma 2, the n-element set Q can be written as the (disjoint) union of n_i sets: $Q = \bigcup_{y\in Q_i} \{a\in Q \mid a+x=a\circ y\}$. Hence there must be an element $y\in Q_i$ such that $\{a\in Q \mid a+x=a\circ y\}$ contains at least n/n_i elements and thus, by assumptions, more than f elements. In other words, the right multiplication mappings $R_+(x)$ and $R_\circ(y)$ satisfy $aR_+(x) = aR_\circ(y)$ for more than f elements $a\in Q$, i.e. $R_+(x)R_\circ(y)^{-1}$ fixes more than f elements. Therefore $R_+(x)R_\circ(y)^{-1} = id$ (the identity mapping on Q) or, equivalently, $R_+(x) = R_\circ(y)$. Applying these two mappings to e yields $x = e+x = eR_+(x) = eR_\circ(y) = e\circ y = y$. This implies $R_+(x) = R_\circ(x)$, i.e. $a+x = a\circ x$ for all $a\in Q$. Analogously, one can show $x+a=x\circ a$.

Lemma 5. Let $(Q,+)$ be a loop with $Mlt(Q,+)\leq G$, and let Q_i be an orbit of G_e with $n/n_i>f$. Then

a) $a+x=x+a$ for all $a\in Q$, $x\in Q_i$,

b) $(a+b)+x = a+(b+x)$, $(a+x)+b = a+(x+b)$ and $(x+a)+b = x+(a+b)$ for all $a,b\in Q$, $x\in Q_i$.

Proof. a) Assume that $x\in Q_i$. Lemma 3a) implies $Q = \bigcup_{y\in Q_i} \{a\in Q \mid a+x=y+a\}$. The proof can now be continued like the proof of Lemma 2: There must be an element $y\in Q_i$ such that $\{a\in Q \mid a+x=y+a\}$ contains (strictly) more than f elements. This can be used in order to show $R(x)=L(y)$ and $x=y$, which then implies $a+x=x+a$ for all $a\in Q$.

b) Let $x\in Q_i$ and $b\in Q$. By Lemma 3b1), $Q = \bigcup_{y\in Q_i} \{a\in Q \mid (a+b)+x=a+(b+y)\}$. Again, there exists an element $y\in Q_i$ such that $\{a\in Q \mid (a+b)+x=a+(b+y)\}$ contains more than f elements.

Therefore $R(b)R(x) = R(b+y)$, and applying both mappings to e yields $b+x=b+y$. This implies $x=y$ and thus $R(b)R(x) = R(b+x)$, i.e. $(a+b)+x = a+(b+x)$ for all $a \in Q$. The other two equations can be proved in the same way.

The *center* Z of a loop $(Q,+)$ is defined as the set of all $z \in Q$ satisfying $a+z=z+a$, $(a+b)+z = a+(b+z)$, $(a+z)+b = a+(z+b)$ and $(z+a)+b = z+(a+b)$ for all $a,b \in Q$. It is well known that $(Z,+)$ is an abelian group which is a normal subloop of $(Q,+)$ (cf. Albert [1] or Bruck [4], pp. 57-60). Lemma 5 states that the set $Q' := \bigcup \{Q_i \mid n/n_i > f\}$ is contained in the center Z. Recall that m was defined as $m = \sum\{n_i \mid n/n_i > f\}$, i.e. as the cardinality of Q'. Hence the following theorem holds:

Theorem 1. If $(Q,+)$ is a loop with $\mathrm{Mlt}(Q,+) \leq G$, then the center of $(Q,+)$ contains at least m elements.

The next result was already shown in Albert [1]. The proof which is presented here uses the fact that the 1-element orbits Q_i belong automatically to the center of $(Q,+)$, due to Lemma 5:

Theorem 2. Let $(Q,+)$ be a loop with $\mathrm{Mlt}(Q,+) \leq G$. Then the center of $(Q,+)$ contains a subloop which is isomorphic to the center $Z(G)$ of G. Moreover, $Z(G) \leq \mathrm{Mlt}(Q,+)$.

Proof. Assume that $(Q,+)$ is a loop with $\mathrm{Mlt}(Q,+) \leq G$. Let $C \in Z(G)$. Then, for all $x \in Q$, $xR(eC) = x+eC = eCL(x) = eL(x)C = (x+e)C = xC$. Hence $R(eC) = C$ (and, analogously, $L(eC) = C$). This shows that $Z(G) \leq \mathrm{Mlt}(Q,+)$. If $C \in Z(G)$ and $A \in G_e$, then $eCA = eAC = eC$. Therefore $\{eC\}$ is a 1-element orbit of G_e. Thus, by Lemma 5, $Q'' := \{eC \mid C \in Z(G)\}$ is contained in the center of $(Q,+)$. It is an easy exercise to show that Q'' is a subloop of $(Q,+)$, and that $Z(G)$ and Q'' are isomorphic (as loops), via the mapping $C \longmapsto eC$.

By the preceding results, it is no surprise that a loop $(Q,+)$ with $\mathrm{Mlt}(Q,+) \leq G$ is

(i) uniquely determined, (ii) an abelian group,

provided that G_e has "sufficiently many sufficiently small" orbits:

Theorem 3. a) If $m>f$, then there is at most one loop $(Q,+)$ with $\mathrm{Mlt}(Q,+)\le G$.

b) If $m\ge f$, then each loop $(Q,+)$ with $\mathrm{Mlt}(Q,+)\le G$ is an abelian group.

Proof. a) Assume that $(Q,+)$ and $(Q,\circ)$ are loops with multiplication groups contained in G. As an immediate consequence of Lemma 4, for each $a\in Q$, the left multiplication mappings $L_+(a)$ and $L_\circ(a)$ agree on the m-element set $Q' = \bigcup\{Q_i \mid n/n_i>f\}$. Hence $L_+(a) = L_\circ(a)$, since $m>f$. This shows that $a+x = a\circ x$ holds for all $a,x\in Q$.

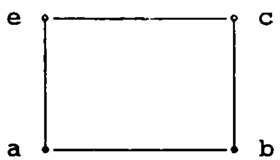
b) In order to show that $(Q,+)$ is commutative, it is sufficient to prove that $L(a) = R(a)$ for all $a\in Q$. If $a\in Q'$, this follows immediately from Lemma 5a). If $a\notin Q'$, then $L(a)$ and $R(a)$ agree on the $(m+1)$ - element set $Q'\cup\{a\}$: for $x\in Q'$, Lemma 5a) implies $xL(a) = xR(a)$, while $aL(a) = aR(a)$ trivially holds. Hence $m\ge f$ implies $L(a) = R(a)$. Associativity can be shown in the same way: The associative law holds in $(Q,+)$ if and only if $L(a+b) = L(b)L(a)$ for all $a,b\in Q$. If $a\in Q'$, then this follows directly from Lemma 5b). Now assume that $a\notin Q'$. Then $L(a+b)$ and $L(b)L(a)$ agree on the $m+1$ elements of $Q'\cup\{a\}$. Again, this implies $L(a+b) = L(b)L(a)$.

Examples. (1) Regular permutation groups yield the most trivial examples for Theorem 3: If G is a regular permutation group operating on Q, then $f=0$ and $m=n$. Hence, by Theorem 3, there exists at most one loop $(Q,+)$ with $\mathrm{Mlt}(Q,+)\le G$, and this loop must be an abelian group. But this is obvious: For each $a\ne e$, the right multiplication mapping $R(a)$ and the left multiplication mapping $L(a)$ must both be the unique fixed point free permutation which maps e to a. Therefore $(Q,+)$ is isomorphic to G. In particular, a loop $(Q,+)$ with $\mathrm{Mlt}(Q,+)\le G$ exists if and only if G is an abelian group.

The same holds in the more general situation when G is a Frobenius group operating on Q. Again, there is at most one loop $(Q,+)$ with $\mathrm{Mlt}(Q,+)\le G$. Such a loop exists if and only if the Frobenius kernel of G is abelian. In this case, $(Q,+)$

is isomorphic to the Frobenius kernel.

(2) For each integer $n \geq 3$, the dihedral group D_n is defined as the group of symmetries of the regular n-gon. In particular, D_n is a group of order 2n operating on an n-element set and containing a cyclic subgroup of order n. If $n>4$, then $m=n$ and $f=1$ (for n odd) or $f=2$ (for n even). Hence, by Theorem 3, the cyclic group of order n is the only loop with multiplication group contained in D_n. The situation is different for the group D_4: In this case $m=f=2$, i.e. part a) of Theorem 3 can be applied but not part b). The following multiplication tables represent two different loops with multiplication groups contained in D_4, namely the cyclic group of order 4, and Klein's four-group. The positions in the multiplication tables which are *not* uniquely determined by Lemma 4 are marked with circles:



regular 4-gon

+			
e	a	b	c
a	ⓑ	c	ⓔ
b	c	e	a
c	ⓔ	a	ⓑ

cyclic group of order 4

+			
e	a	b	c
a	ⓔ	c	ⓑ
b	c	e	a
c	ⓑ	a	ⓔ

Klein's four-group

Two loops with multiplication groups contained in D_4

(3) The general linear group $GL(d,q)$ is defined as the group of all linear automorphism of a d-dimensional vector space over $GF(q)$. Regard $GL(d,q)$ as a permutation group acting on the non-zero vectors of the underlying vector space. Let $(Q,\cdot)$ be a loop with $Mlt(Q,\cdot) \leq GL(d,q)$. By Theorem 2, $Mlt(Q,\cdot)$ contains the center of $GL(d,q)$, i.e. all scalar

multiplications $x \longmapsto \lambda x$, $\lambda \in GF(q)$. This shows that certain subgroups of general linear groups cannot contain multiplication groups of quasigroups. For instance, a quasigroup with multiplication group contained in the special linear group SL(d,q) cannot exist, unless q-1 divides d.

It can easily be seen that the loops with $Mlt(Q,\cdot) \le GL(d,q)$ are exactly the multiplicative loops of finite semifields (cf. Dembowski [6] and [10] for more information).

Remarks. (1) The reader may find out which of the results stated in this section also hold in the infinite case. For instance, Theorem 2 is valid for infinite loops as well.

(2) In order to show that the inequality of Theorem 3b) cannot be improved, one should find a permutation group G with m = f-1 and a loop (Q,+) which is *not* an abelian group such that $Mlt(Q,+) \le G$. The groups PSL(2,q) might be good candidates (in their natural permutation representation, these groups satisfy m = f-1).

(3) The numerical assumptions on the orbits of G_e which have been used in this section (e.g. in Theorems 1 and 3) are certainly a very rough tool. It might prove fruitful to exploit not only numerical but also structural properties of G_e. A good example of this kind was presented in a recent paper by Kepka and Niemenmaa [15]: They proved that, if G_e is a cyclic group, then each loop (Q,+) with $Mlt(Q,+) \le G$ must be an abelian group.

(4) If the permutation group G is doubly transitive, then the methods of this section do not yield any information on the loops with $Mlt(Q,+) \le G$. In this case one can apply results on doubly transitive permutation groups, due to the classification of finite simple groups. This works in particular for quasigroups of prime order, since the permutation groups of prime degree are well known (cf. Feit [7] and [10]).

(5) The investigations of this paper were partly motivated by the results in Johnson [12] and Smith [16] on *centralizer rings* of multiplication groups of quasigroups. In particular, the fundamental Lemmas 2 and 3 reflect properties of the

centralizer ring: e.g., Lemma 3b1) states that $\bigcup\{R(x)\mid x\in Q_i\}$ *centralizes* L(a). In [13], Johnson and Smith used centralizer rings of multiplication groups in order to develop a character theory of finite quasigroups which generalizes the ordinary character theory of finite groups.

(6) The question which groups occur as multiplication groups of quasigroups seems to be even more difficult if one considers *abstract* groups instead of *permutation* groups. The following negative results are known: Hamiltonian groups and Heineken-Mohamed groups are never isomorphic to multiplication groups of quasigroups (cf. Kepka [14] or Smith [17]).

3. Further properties of multiplication groups

The following properties are known to hold for all permutation groups which contain the multiplication group of a quasigroup:

Theorem 4. Let G be a permutation group acting on a finite set Q, and let $(Q,\cdot)$ be a quasigroup with $\mathrm{Mlt}(Q,\cdot)\le G$. Then the following holds for an arbitrary element $e\in Q$:

a) G is transitive on Q,

b) the (universal) algebra (Q,G) is congruence permutable, i.e. $H_1H_2 = H_2H_1$ for all subgroups H_1,H_2 of G with $G_e\le H_1,H_2$,

c) $N(G_e) = G_e\cdot Z(G)$, where $N(G_e)$ denotes the normalizer of G_e in G,

d) the centralizer ring $V(Q,G)$ of G is commutative,

e) the number of orbits of G_e is less or equal the number of conjugacy classes of G.

Sketch of proof. a) follows from the fact that $\{L(a)\mid a\in Q\}$ is a transitive subset of G.

b) By Proposition 1, $\mathrm{Con}(Q,G)\le\mathrm{Con}(Q,\mathrm{Mlt}(Q,\cdot)) =$ $= \mathrm{Con}(Q,\cdot,/,\backslash)$. Hence the congruence of (Q,G) permute, since $\mathrm{Con}(Q,\cdot,/,\backslash)$ has this property if $(Q,\cdot)$ is a quasigroup (cf. [11]). By using the transitivity of G, it is straightforward to show that $\theta \longmapsto \{A\in G \mid (e,eA)\in\theta\}$ defines a bijective mapping from $Con(Q,G)$ onto the set of all subgroups of G which contain G_e, and that θ_1 and θ_2 permute if and only if the corresponding subgroups H_1 and H_2 satisfy $H_1H_2 = H_2H_1$.

c) Obviously, $N(G_e)\geq G_e\cdot Z(G)$. In order to show $N(G)\leq G_e\cdot Z(G)$, it may be assumed that there is a loop $(Q,+)$ with neutral element e and $Mlt(Q,+)\leq G$ (Proposition 2). Let $A\in N(G_e)$. For $x := eA$ then $G_x = A^{-1}G_eA = G_e$. Hence $\{x\}$ is a 1-element orbit of G_e. It is an interesting exercise to show that this implies $R_+(x)\in Z(G)$ (hint: $L_+(qA)A^{-1}L_+(q)^{-1}\in G_e$). Moreover, $AR_+(x)^{-1}\in G_e$, and thus $A = AR_+(x)^{-1}R_+(x)\in G_e\cdot Z(G)$. Note that c) was already proved by Smith in [17].

d) Was proved by Johnson [12] and by Smith [16].

e) Holds if $V(Q,G)$ is commutative (cf. Wielandt [18], p. 87).

It is unknown whether the sum of conditions of the above theorem is sufficient for a permutation group to contain the multiplication group of a quasigroup. The following problem might be interesting in connection with the results of Section 2:

Problem. Find a permutation group G acting on a finite set Q with $m\geq f$ such that G does *not* contain an abelian subgroup which is regular on Q.

Such a permutation group cannot contain the multiplication group of a quasigroup: Theorem 3b) would imply each loop $(Q,+)$ with $Mlt(Q,+)\leq G$ to be an abelian group, and $Mlt(Q,+)$ would thus be a regular abelian subgroup of Q.

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