DEMONSTRATIO MATHEMATICA
Vol. XXIV No 1-2 1991

G.H. Bordalo

FRATTINI EXTENSIONS OF UNARY ALGEBRAS

Introduction

In the present work, we study the Frattini extensions of unary algebras, prove that every such algebra has a unique (up to isomorphism) minimal Frattini extension, and derive some consequences of that fact in the class of unary algebras and in the class of finite relative Stone lattices.

A non-empty finite poset $(T,\leq)$ is a (rooted) *tree* if it has a unique minimal element and every principal order ideal is a chain. Note that tree $(T,\leq)$ is a special kind of meet - semilattice.

As in [2] T can be endowed with a unary algebra structure by defining

$$f(x) = \begin{cases} x & \text{if } x \text{ is minimal in } (T,\leq) \\ \text{the immediate predecessor of } x & \text{otherwise.} \end{cases}$$

f^n is defined recursively by $f^n(x)=f(f^{n-1}(x))$, and $f^0(x)=x$, $\forall x\in T$.

It should be noted that with each unary operation f in a (non void) set is associated a quasi order $\leq_f$ given by

$$a\leq_f b \Leftrightarrow \exists n\in\mathbb{N}_0 = \mathbb{N}\cup\{0\} : a=f^n(b).$$

The results obtained in [2] enable us to state:

1. Theorem. For a finite tree $(T,\leq)$, the map $f : T \longrightarrow T$ defined above is the only possible map on T such that $\leq_f$ equals $\leq$, that is, the associated quasi order coincides with the initial one (and is therefore a partial order on T).

Note: An analogous statement holds for a finite disjoint union of trees (also called a forest).

So, each tree can be viewed both as a semilattice $S=(T,\wedge)$ and as a unary algebra $U=(T,f)$ where f induces the same order as $\wedge$ induces. Note that the subalgebras of U are exactly the non void order ideals of S; they form a distributive sublattice of the lattice of subalgebras of S.

From now on, unary algebra will stand for mono-unary algebra. Recall that every such finite algebra is a disjoint union of connected subalgebras.

2. Theorem [5]. Every mono-unary algebra is a disjoint union of connected mono-unary algebras, in a unique way. Every connected mono-unary algebra (A,f) has s subuniverse C with (C,f) isomorphic either to a finite cycle, or to ω with the successor function. The set $A-C$ can be given the order of a disjoint union of (rooted) trees so that f acts on $A-C$ by mapping the root of each tree into C and mapping each other element of $A-C$ into its unique predecessor.

So, each connected component is either I) a cycle, or II) a (not necessarily rooted) tree, or III) a union of rooted trees with roots in a cycle.

Of course I) and II) can be viewed as "degenerate" instances of III); a connected component of type I, consisting of a cycle only, will be called an *isolated cycle*. The connected component to which an element x belongs will be denoted Cx.

Consider now the quasi order $\leq_f$ associated with (A,f), and the equivalence relation: $aRb \Leftrightarrow a\leq_f b$ and $b\leq_f a$. The quotient set $\bar{A} = A/R$ is partially ordered by $\bar{x}\leq\bar{y}$ iff $x\leq_f y$ for $\bar{x},\bar{y}\in\bar{A}$; each class of R is a singleton or a cycle, and if $\bar{x} = \{x\}$, then $\bar{x}$ is maximal in $(\bar{A},\leq)$ iff x is maximal in $(A,\leq_f)$.

Note that R is a congruence relation on (A,f). So, to each unary algebra we associate a homomorphic image $\bar{A}$ that has no non-trivial cycles.

3. Theorem. Let (A,f) be a finite unary algebra, and $\bar{a}$ be a maximal element in $(\bar{A},\leq)$, i.e. $\bar{a} = \{a\}$ and $a \neq f(x)$

for all $x \neq a$, or $\bar{a}$ is an isolated cycle. Then $A-\bar{a}$ is a maximal subalgebra of (A,f), and every maximal subalgebra can be obtained in this way.

Proof. We want to show that B is a maximal subalgebra of (A,f) iff i) $B = A-\{a\}$, with a maximal, or ii) $B = A-C$, with C an isolated cycle. Certainly if we have i) then B is maximal subalgebra of A, since $x \neq a \Rightarrow f(x) \neq a$; if we have ii) then again B is a maximal subalgebra of A, since C is an isolated cycle, and $\langle B \cup \{x_i\} \rangle = A$, $\forall x_i \in C$. Now let $B = A-X$ be a maximal subalgebra. If $X = \{x_i\}$ then certainly $a \neq x_i \Rightarrow f(a) \neq x_i$; if $|X|>1$, then $\forall x_i \in X$, $\langle A-(X-\{x_i\}) \rangle \supset B$, so $\langle A-(X-\{x_j\}) \rangle = A$; therefore $\langle x_i \rangle = X$, and X is cycle. The fact that $A-X$ is a subalgebra ensures that X is an isolated cycle.

Note that if A is infinite, the situation remains essentially the same, as long as there are maximal elements, or isolated cycles. If this is not a case, then A contains no proper maximal subalgebras.

Recall that the intersection of all proper maximal subalgebras of A is called the *Frattini subalgebra* of A. It is the set of all non-generators of A and it coincides with A exactly when there are no maximal subalgebras. So the results above enable us to state the following:

4. Theorem. The Frattini subalgebra of a unary algebra is the set of elements x such that

a) C_x is not an isolated cycle, and

b) $\exists y \neq x : f(y)=x$.

Suppose A is the Frattini subalgebra of a unary algebra B. We will then write $A=\phi(B)$ and call B a *Frattini extension* of A. The existence of a Frattini extension for every unary algebra is stated as follows:

5. Theorem. Let A be a unary algebra. There exists a unary algebra B such that $A=\phi(B)$. Moreover, the algebras of minimal cardinality satisfying this condition are all isomorphic.

Proof. We want to construct a Frattini extension B of A, with minimal cardinality. Theorem 4 shows that what we need to do is to adjoin an element to each maximal element in the connected components of type II and III, and an element to each isolated cycle.

Note that, if there are no maximal elements and no isolated cycles, then $A=\phi(A)$.

The minimal Frattini extension (unique up to isomorphism) of a unary algebra A will be denoted A^*. The procedure described above to obtain A^* from A ensures the following:

6. Corollary. The cardinality of the minimal Frattini extension A^* of a unary algebra A is given by :

$|A^*| = |A| + |\{\text{isolated cycles in } A\}| + |\text{maximal elements in all connected components of type II) and III)}|$

or equivalently:

$|A^*| = |A| + |\text{maximal elements of } (\bar{A},\le)|$.

If we iterate the application of ϕ to a finite algebra A, the situation is as follows: for $x\in A$, let n_x denote the minimum integer that takes x into a cycle, that is,

$$n_x = \min\{n : f^n(x) \text{ is in a cycle}\} = \text{height of } x \text{ in } (\bar{A},\le).$$

Then the following result is a straightforward consequence of Theorem 3.

7. Theorem. Let A be a finite unary algebra. Then $\phi^n(A)$ is void iff $n \ge \max_{x\in A} n_x+1$.

The map * defined in the class of unary algebras by $A \longrightarrow A^*$ has the following properties:

8. Theorem. Let (A,f), (B,g) be unary algebras. Then:

1) $A^*\cong B^* \Rightarrow A\cong B$.
2) If A is a subalgebra of B, then A^* is isomorphic to a subalgebra of B^*.
3) If B is any Frattini extension of A, then A^* is isomorphic to a subalgebra of B.

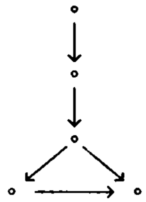
Proof. 1) Suppose $\psi^* : A^* \longrightarrow B^*$ is an isomorphism. Then $a\le_f b \Leftrightarrow \psi^*(a)\le_g\psi^*(b)$, since $a=f^n b \Leftrightarrow \psi^*(a)=\psi^*(f^n(b))=g^n\psi^*(b)$.

Consider the maximal classes in $\overline{A^*}$: $\bar{x}$ is maximal in $\overline{A^*}$ iff $\overline{\Psi^*(x)}$ is maximal in $\overline{B^*}$. It is easily verified that $\psi^*_{|\phi(A^*)} = \psi^*_{|A} : A \longrightarrow B$ is an isomorphism.

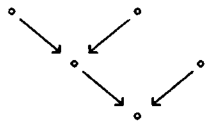
2) Take a subalgebra A of B. We know how to construct A^* and B^*. To prove that A^* is isomorphic to a subalgebra of B^*, we only need to consider again the elements in maximal classes of $\bar{A}$: if $\bar{x} = \{x\}$ is maximal in $\bar{A}$ and also in $\bar{B}$, then there must exist an element covering x both in B^* and in A^*, that is, an element $y \neq x$ s.t. its image, under f, is x, both in A^* and in B^*; if $\bar{x} = \{x\}$ and $\bar{x}$ is maximal in $\bar{A}$ but not maximal in $\bar{B}$, then there is $y \neq x$ in B s.t. $\bar{y}$ covers $\bar{x}$ in $\bar{B}$. But since $\bar{x}$ is maximal in $\bar{A}$, there must be an element covering x also in A^*. Now suppose $\bar{x}$ is a cycle. Then either it is also an isolated cycle in B, or there is an element $y \notin \bar{x}$ in B such that $g(y) \in \bar{x}$; the same reasoning as before applies, again bearing in mind the construction of A^* and B^*.

3) Let $\phi(B)=A$, and construct A^*. Since A satisfies conditions a) and b) of Theorem 3 relative to B, for each $\bar{x} \in Max(\bar{A},\leq)$, $\exists y \in B$, $y \neq x$: $g(y) \in \bar{x}$; for each $\bar{x} \in Max(\bar{A},\leq)$, pick such one y(x). Then $A \cup \left(\bigcup_{\bar{x} \in Max\bar{A}} y(x) \right) = A^* \leq B$. Since A is a subalgebra of B, we have only to consider the elements in A^*-A and to verify that, for each such x, its image in A^* is the same as in B. But this is immediately ensured by our choice of the elements $y(x), \bar{x} \in Max\bar{A}$.

Note: It is sufficient to consider any unary algebra consisting of isolated cycles to see that the map * is not surjective. We can in fact describe those unary algebras that are minimal Frattini extensions: they are those algebras (A,f) such that $|A| = |\phi(A)| + |\text{maximal elements of } \overline{(\phi(A),\leq)}|$, by Corollary 6 above. So, for example,



is a minimal Frattini extension (5=4+1), whereas



is not ($5 \neq 2+1$).

It was first noted in [4, p. 110] that a distributive lattice L is isomorphic to the lattice of subuniverses Sub A of a unary algebra (A,f) precisely when the set of filters containing a given prime filter in L, forms a chain under set inclusion. In the finite case, that condition is equivalent to the fact that each interval in L is a Stone lattice, that is, L is a (finite) relative Stone lattice ([1], [6]);

Following [2], [3], we will call an RS lattice a finite distributive lattice such that each closed interval is Stone.

An element x of a unary algebra (A,f) is *fixed* when f(x)=x. (i.e. {x} is a trivial cycle in (A,f)).

9. **Theorem** [2]. A finite poset P with more than one element is an RS lattice with n atoms if and only if it is isomorphic to Sub A, for some finite unary algebra (A,f) with n fixed points and no non-trivial cycles. Moreover, for each RS lattice L there is exactly one unary algebra A without non-trivial cycles such that $L \cong$ Sub A.

In the class of all finite unary algebras, consider now the equivalence relation ~ defined by $U_1 \sim U_2 \Leftrightarrow \text{Sub } U_1 \cong \text{Sub } U_2$. Then each equivalence class of ~ contains exactly one algebra without non trivial cycles, and therefore there is an onto, one-one correspondance between the equivalence classes of unary algebras and the class of all RS lattices.

The representation of an RS lattice L mentioned in Theorem 9 above is obtained by endowing the poset of join irreducibles J(L) with a unary algebra structure. Under this representation, the unary subalgebras correspond exactly to the non void order ideals of $(J(L),\leq)$.

We can now use the preceeding results on unary algebras to prove the following:

10. Theorem. Given any RS lattice L, there exists a unique up to isomorphism RS lattice L_1 such that:

1) $L \cong (\wedge x : x$ is a coatom in $L_1]$.
2) L is a homomorphic image of L_1.
3) If L_2 is an RS lattice satisfying $L\cong(\wedge x$, x is a coatom in $L_2]$ then L_1 is a homomorphic image of L_2.

Proof. 1) Take the unary algebra $A=(J(L),f)$ that represents L, that is, $L\cong \mathrm{Sub}\ A$; in the case of unary algebras with no cycles, forming A^* is equivalent to adjoining new maximal elements on each top of each tree. We thus get another unary algebra A^*, also without cycles. Take $L_1\cong\mathrm{Sub}\ A^*$. Then, the subalgebras of A (that is, the elements of L) are certainly contained in A, and A corresponds exactly to the meet of the coatoms in L_1, that is, to the intersection of maximal subalgebras in A^*.

2) Since J(L) is embedded in the carrier of A^*, by Birkhoff's duality, the corresponding lattices L and L_1 must be such that L is a homomorphic image of L_1.

3) Suppose $L\cong(\wedge$ coatom of $L_2]$. Then $1_{\mathrm{Sub}\ A}=A$ corresponds to the meet of the coatoms of L_2, that is, to the intersection of the maximal subalgebras in $A_2=(J(L_2),f)$. So $A=\emptyset(A_2)$, and therefore A^* is a subalgebra of A_2, by Theorem 8.3. So, the carrier of A^* is order embedded in $J(L_2)$, and by duality there is an homomorphism from L_2 onto $\mathrm{Sub}\ A^* = L_1$.

Call a finite lattice L_1 an extension of a lattice L if $L\cong(\wedge$ coatoms in $L_1]$. Note that an extension of an RS lattice does not have to be an RS lattice. In fact the extension of minimal cardinality of any lattice L (obtained just by adjoining a new 1) is RS iff L is a chain. However:

11. Corollary. Every RS lattice L has up to isomorphism a unique RS lattice extension of minimal cardinality.

Proof. Consider the unary algebra $A = (J(L), f)$ and take its minimal Frattini extension A^*. By Theorem 10, Sub A^* is the minimal RS extension of L.

REFERENCES

[1] G. **Bordalo:** Strongly n-normal lattices, Indagationes Math. 46, 2 (1984) 113-125.

[2] G. **Bordalo:** Representing finite relative Stone algebras by monounary algebras, Contributions to General Algebra, 5 (1987) 45-60.

[3] G. **Bordalo:** A duality between unary algebras and their subuniverse lattices, Portugaliae Math. 46, 4 (1989) 431-439.

[4] **B. Jónsson:** Topics on Universal Algebra. Lecture Notes in Mathematics, Springer Verlag, 1972.

[5] **R. McKenzie, G. Nulty, W. Taylor:** Algebras, Lattices, Varieties. vol. I, Wadsworth, 1987.

[6] **J. Varlet:** Contribution à l'étude des lattis pseudocomplementés e des lattis de Stone, These de Doctorat, Université de Liège, 1963/4.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LISBON,
LISBON, PORTUGAL.

Received May 9, 1989.