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REMARKS ON MULTIPLIERS FOR RIGHT INVERSES IN D-ALGEBRAS

Detailed studies of multipliers with integral operators in convolutional algebras were given recently by Dimovski [2] and Bozhinov [1]. The purpose of the present note is to indicate that some of these results can be obtained without any topological assumptions in a rather general form.

Denote by $L(X)$ the set of all linear operators with domains and ranges in a linear space X over a field $\mathcal{F}$ of scalars and by $L_o(X)$ the set of operators $A \in L(X)$ with $\operatorname{dom} A = X$.

Through the paper we shall admit the following assumptions, denoted by (A) for the sake of brevity:

(A) Let X be a D-algebra, i.e. a commutative algebra (over $\mathcal{F}$) with a right invertible operator $D \in L(X)$ such that $\operatorname{dom} D$ is subalgebra of X:

$$x, y \in \operatorname{dom} D \quad \text{implies} \quad xy \in \operatorname{dom} D.$$

We assume that $\ker D \neq \{0\}$. Let F be an initial operator for D corresponding to a right inverse R of D which has property

$$(1) \qquad \underset{0 \neq r \in \ker D}{\exists} \quad \underset{x \in X}{\forall} \quad Rx = rx \quad (\operatorname{dom} R = X).$$

(for all notations connected with right invertible operators and their properties, cf. the author [3]).

Formula (1) implies that

$$(2) \qquad R(xy) = xRy = yRx.$$

Indeed, $R(xy) = r(xy) = xry = xRy$. Similarly, $R(xy) = R(yx) = yRx$.

Theorem 1. Suppose that (A) holds and that $M \in L_o(X)$ satisfies the condition

(3) $$M(rx) = mx \quad \text{for all} \quad x \in X, \quad \text{where} \quad m = Mr.$$

(i) If $MR = RM$ then M is of the form

(4) $$Mx = D(mx) \quad \text{for all} \quad x \in X;$$

(ii) If M is of the form (4) then $MR = RM$ if and only if

(5) $$F(mx) = 0 \quad \text{for all} \quad x \in X.$$

Proof. (i) Suppose that $MR = RM$. Condition (3) implies that for all $x \in X$ we have

$$0 = (MR - RM)x = MRx - RMx = M(rx) - RMx = mx - RMx,$$

i.e. $mx = RMx$. Hence $mx \in \text{dom } D$ for all $x \in X$ and $Mx = DRMx = D(mx)$. (ii) Suppose that M is of the form (4). Then for all $x \in X$ we have

$$(MR - RM)x = M(rx) - RD(mx) = M(rx) - (I-F)(mx) =$$
$$= mx - mx + F(mx) = F(mx).$$

Thus $MR = RM$ if and only if $F(mx) = 0$ for all $x \in X$.

Corollary 1. Suppose that (A) holds and that $M \in L_o(X)$ is a <u>multiplier</u>, i.e.

(6) $$M(xy) = xMy = yMx \quad \text{for all} \quad x,y \in X.$$

Then M satisfies (i) and (ii) of Theorem 1. Indeed, by (6), we have $M(rx) = xMr = xm = mx$ for all $x \in X$, i.e. (3) is satisfied.

Theorem 2. Suppose that (A) holds, $M \in L_o(X)$ satisfies (3) and $MR = RM$. Then $\ker M = \{0\}$ if and only if $m = Mr$ is not a zero divisor.

Proof. By our assumptions, if $Mx = 0$ then $D(mx) = 0$. This implies that $mx = z \in \ker D$. By (5), we get $z = Fz = F(mx) = 0$. Hence $mx = 0$. If m is not a zero divisor, we find $x = 0$ and $\ker M = \{0\}$. If m is a zero divisor then there exists an $x \neq 0$ such that $RMx = MRx = M(rx) = mx = 0$. This implies that $Mx = DRMx = 0$, i.e. $0 \neq x \in \ker M$. Hence $\ker M \neq \{0\}$.

Corollary 2. Suppose that all assumptions of Theorem 2 are satisfied. Then the equation

$$Mx = y, \tag{7}$$

where $y \in X$ is arbitrary, has at most a unique solution.

Theorem 1 can be generalized for higher powers of R in the following manner.

Theorem 3. Suppose that (A) holds, $M \in L_0(X)$ satisfies (3). Let $n \in \mathbb{N}$ be arbitrarily fixed. Then

(i) If $MR^n = R^nM$ then $Mx = D(mx)$ for all $x \in X$, where $m = Mr$ (i.e. M is of the form (4));

(ii) If M is of the form (4) then $MR^n = R^nM$ if and only if $F(mx) = 0$ for all $x \in X$ (i.e. (5) is satisfied).

Proof. (i) By an easy induction we prove that

$$R^n x = r^n x \quad \text{for all} \quad x \in X,\ n \in \mathbb{N}. \tag{8}$$

This implies

$$0 = (MR^n - R^nM)x = MR^n x - R^nMx = M(r^n x) - R^nMx =$$
$$= r^{n-1}xMr - R^nMx = r^{n-1}mx - R^nMx = R^{n-1}(mx) - R^nMx.$$

As before, we conclude that $mx \in \operatorname{dom} D$ for x X and

$$Mx = D^nR^nMx = D^nR^{n-1}(mx) = D(mx) \quad \text{for} \quad x \in X.$$

(ii) Since M is of the form (4), we find for $x \in X$

$$(MR^n - R^nM)x = M(r^n x) - R^n D(mx) = r^{n-1}xMr - r^{n-1}RD(mx) =$$
$$= r^{n-1}mx - r^{n-1}(I-F)(mx) = r^{n-1}mx - r^{n-1}mx + F(mx) = F(mx).$$

Therefore $MR^n = R^nM$ if and only if $F(mx) = 0$ for all $x \in X$.

Theorem 4. Suppose that condition (A) holds and there exists a $d \in \mathbf{R}$ such that D satisfies a Duhamel-like condition

(9) $$D(xy) = xDy + dxFy \quad \text{for all} \quad x \in X,\ y \in \mathrm{dom}\, D.$$

Then

(10) $$F(xy) = F(xFy) \quad \text{for all} \quad x \in X, \quad y \in \mathrm{dom}\, D.$$

Proof. Let $x \in X$, $y \in \mathrm{dom}\, D$ be arbitrary. By definition of F, (2) and (9), we find

$$F(xy) = (I-RD)xy = xy - R[xDy + dxFy] = xy - R(xDy) -$$
$$- dR(xFy) = xy - xRdy - drxFy = xFy - drxFy = (1-dr)xFy.$$

i.e.

(11) $$F(xy) = (1-dr)xFy \quad \text{for all} \quad x \in X, \quad y \in \mathrm{dom}\, D.$$

If we put in (11) Fy instead of y, we obtain

$$F(xFy) = (1-dr)xF^2y = (1-dr)xFy = F(xy).$$

Formula (11) immediately implies

Corollary 3. Suppose that all assumptions of Theorem 4 are satisfied. Then

(12) $$F(zx) = (1-dr)zFx \quad \text{for all} \quad z \in \ker D, \quad x \in \mathrm{dom}\, D.$$

Corollary 4. Suppose that all assumptions of Theorem 4 are satisfied and

(13) $(1-dr)x = 0$ for all $x \in X$.

Then

(14) $F(zx) = 0$ for all $z \in \ker D$, $x \in X$.

Example 1. Suppose that $X = C(\mathbb{R}_+)$, $D = \frac{d}{dt}$, $R = \int_0^t$, $(Fx)(t) = x(0)$. Define the multiplication in X as the convolution

(15) $$(x * y)(t) = \int_0^t x(t-s)y(s)ds \quad \text{for} \quad x,y \in X.$$

Clearly, X is a D-algebra, $\operatorname{dom} D = C^1(\mathbb{R}_+)$ is a subalgebra of X, $\ker D \neq \{0\}$. Moreover

$$(Rx)(t) = \int_0^t x(s)ds = x * r, \quad \text{where} \quad r(t) \equiv 1.$$

It is also well-known that condition (9) is satisfied with $d = 1$. Thus in this case we have $1 - dr = 0$, i.e. condition (13) is satisfied.

Note that, by the Titchmarsh theorem, X is an algebra without zero divisors.

Corollary 5. Suppose that all assumptions of Theorem 4 are satisfied. Then

(16) $F(xy) = F[(Fx)(Fy)]$ for $x,y \in \operatorname{dom} D$.

Proof. If we put in (10) Fx instead of x, we find $F(yFx) = F[(Fx)(Fy)]$. But, again by (10), we have $F[(Fx)(Fy)] = F(yFx) = F(xy)$.

Theorem 5. Suppose that condition (A) holds, D satisfies (9), $M \in L_0(X)$ and $MR = RM$. Then

(i) $m \in \operatorname{dom} D$ and $Dm = dm$. In particular, if $d = 0$ then $m \in \ker D$;

(ii) If $d \neq 0$ and equation (7) has a unique solution for every $y \in X$ then $Fm \neq 0$.

Proof. (i) By definition, $m = Mr = D(mr) = mDr + dmFr = dmr = drm = dRm$ (since $Dr = 0$ and $Fr = r$). Hence $m \in \operatorname{dom} D$ and $Dm = dm$. If $d = 0$ then $Dm = 0$.

(ii) By Corollary 2, m is not a zero divisor. If we put in condition (5) Dx instead of x, we obtain $F(mDx) = 0$ for all $x \in \operatorname{dom} D$. Let $y \in X$ be arbitrary and let $x \in \operatorname{dom} D$ be the unique solution of (7). Theorem 4 and Corollary 5 together imply that

$$Fy = FMx = FD(mx) = F(mDx + dmFx) = F(mDx) + dF(mFx) =$$
$$= dF(mFx) = dF[(Fm)(Fx)].$$

If $Fm = 0$ then $Fy = 0$. This is a contradiction with our assumption that $y \in X$ is arbitrary. Hence $Fm \neq 0$.

Corollary 6. Suppose that all assumptions of Theorem 5 are satisfied. If $d \neq 0$ then equation (7) can be written in an equivalent form

(17) $$\left(1 - \frac{d}{2} r\right) mx = \frac{1}{2d} y, \quad \text{where} \quad y \in X \quad \text{is arbitrary.}$$

Proof. By Theorem 5, $Dm = dm$, $Fm \neq 0$. By definition, $Fm = (I-RD)m = m - rDm = m - drm = (1 - dr)m$. Hence for $x \in X$ we have $Mx = D(mx) = xDm + dxFm = x(dm) + dx(1 - dr)m = d(2 - dr)mx = 2d\left(1 - \frac{d}{2} r\right)m$, which implies (17).

Corollary 7. Suppose that assumptions of Theorem 5 are satisfied and $d \neq 0$. If the operator $I - \frac{d}{2} R$ is invertible then equation (17) can be written in an equivalent form

(17') $$mx = \frac{1}{2d}\left(I - \frac{d}{2} R\right)^{-1} y \quad \text{for all} \quad y \in X.$$

Proof. By definition, we have $(1 - \frac{d}{2}r)u = (I - \frac{d}{2}R)u$ for all $u \in X$. This and the invertibility of the operator $I - \frac{d}{2}R$ together imply (17').

Proposition 1. Suppose that an operator $D \in L(X)$ satisfies condition (9), $A, B \in L_0(X)$ are multipliers and $Az = z$ for all $z \in \ker D$. Then the operator $D_1 = DA + B$ also satisfies condition (9), i.e.

$$D_1(xy) = yD_1x + dyF_1x \quad \text{for all} \quad x \in \operatorname{dom} D, \quad y \in X,$$

where $F_1 = FA$ is a projection onto ker D (in particular, $A = I$).

Proof. Since $F_1z = FAz = Fz = z$ for all $z \in \ker D$, we conclude that F_1 is a projection onto ker D. By our assumptions, for all $x \in \operatorname{dom} D$, $y \in X$ we have

$$D_1(xy) = DA(xy) + B(xy) = D(yAx) + yBx = yDAx + d(FAx)y +$$
$$+ yBx = y(DA + b)x + d(FAx)y = yD_1x + dyF_1x.$$

Condition (9) can be written in a symmetric form

(18) $$D(xy) = \frac{1}{2}(xDy + yDx) + \frac{d}{2}(xFy + yFx) \text{ for all } x, y \in \operatorname{dom} D.$$

Indeed, the commutativity of X and condition (9) together imply that for $x, y \in \operatorname{dom} D$ we have

$$D(xy) = \frac{1}{2}[D(xy) + D(yx)] = \frac{1}{2}[yDx + dyFx + xDy + dxFy] =$$
$$= \frac{1}{2}(xDy + yDx) + \frac{d}{2}(xFy + yFx).$$

Formula (18) shows that X is a D-algebra with $c_D = \frac{1}{2}$ and with the non-Leibniz component $f_D(x,y) = \frac{d}{2}(xFy + yFx)$. (cf. [3], Chapter 6).

Note. If dim ker D = 1 then ker D = lin{r} and $Fx = \varphi(x)r$ for $x \in X$, where φ is a linear functional defined on X and such that $\varphi(r) = 1$.

Theorem 6. Suppose that all assumptions of Theorem 5 are satisfied and m = Mr is not a zero divisor. Consider an initial value problem

(19) $$Dx - ax = y, \quad Fx = x_o,$$

where $a, y \in X$, $x_o \in \ker D$ are given. Then $x = rv + x_o$ is a solution of (19) if and only if v is a solution of the equation

(20) $$(1 - ra)v = y + ax_o.$$

Both equations have simultaneously a unique solution (or not).

Proof. Consider the auxiliary problem

(21) $$Du = y, \quad Fu = x_o,$$

where $y \in X$, $x_o \in \ker D$ are given. This problem has a unique solution $u = ry + x_o$. Indeed, write $w = u - x_o$. Then $Fw = Fu - Fx_o = x_o - x_o = 0$, $Dw = D(u - x_o) = Du - Dx_o = Du = y$. By (9), we get $Mw = D(mw) = mDw + dmFw = mDw = my$. By (5), we have $F(mw) = 0$. Thus, we obtain $mw = mw - F(mw) = (I - F)(mw) = RD(mw) = RMw = rMw = rmy$. This implies that $m(w - ry) = 0$. Since m is not a zero divisor, we conclude that $w - ry = 0$, i.e. $w = ry$ and $u = w + x_o = ry + x_o$.

If v satisfies the equation (20) and $x = rv + x_o$ then $Dx = D(rv + x_o) = DRv + Dx_o = v$, $Fx = F(rv + x_o) = FRv + Fx_o = x_o$ and $Dx - ax = v - a(rv + x_o) = (1 - ar)v - ax_o = y + ax_o - ax_o = y$. Hence, x is a solution of the problem (19). On the other hand, in order to solve (19), write $u = (1-ra)x$. Then we have

$$Du = Dx - D(rax) = Dx - DR(ax) = Dx - ax = y,$$

$$Fu = F(x - rax) = Fx - FR(ax) = x_o.$$

Hence, u is a solution of the problem (21), i.e. $u = ry + x_o$. By (9), we find

$$Mu = D(mu) = mDu + dmFu = my + dmx_o = m(y + dx_o).$$

But, by our definition, if we write $v = Dx$, we get

$$m(y+dx_o) = Mu = M(x-rax) = Mx - M(rax) = D(mx) - D(mrax) =$$
$$= mDx + dmFx - mraDx - dmraFx = (m-mra)(Dx+dFx) =$$
$$= (m-mra)(v+dx_o) = m(1-ra)(v+dx_o),$$

which implies

$$0 = m[(1-ra)(v+dx_o) - (y+dx_o)] = m[(1-ra)v - y - d(rax_o)] .$$

Since m is not a zero divisor, we conclude that

$$(1-ra)v = y+drax_o = y + ax_o Dr + dax_o Fr = y + D(rax_o) =$$
$$= y + DR(ax_o) = y + ax_o$$

since $Dr = 0$ and $Fr = r$, by our assumption that r ker D. This means that v satisfies (20). Since $v = Dx$, we have $x = Rv + Fx = rv + x_o$. This and (20) together imply that x is a solution of (19).

Theorem 7. Suppose that all assumptions of Theorem 4 are satisfied and $d = 0$. Then

(22) $$r^n \in \ker D^n \text{ for every } n \in \mathbf{N} .$$

Proof. By our assumption that $d = 0$, we get from (9) that $D(zx) = zDx$ for all $z \in \ker D$, $x \in X$. This implies

(23) $$D^n(zx) = zD^n x \quad \text{for all} \quad z \in \ker D, \quad x \in X, \quad n \in \mathbf{N}.$$

By our assumption, $r \in \ker D$, i.e. $Dr = 0$. Suppose that $D^n r^n = 0$ for an arbitrarily fixed $n \geq 2$. Then, by (23), we obtain that

$$D^{n+1} r^{n+1} = D^{n+1}(r \cdot r^n) = rD^{n+1} r^n = rD(D^n r^n) = 0$$

which proves (22).

Theorem 7 permits us to apply all previous considerations to higher powers of R^n in another way. Namely, we have

Corollary 8. Suppose that D, R, F satisfy condition (A) and (9) with $d = 0$, i.e. we have

$$D(xy) = xDy \quad \text{for} \quad x \quad X, \ y \quad \operatorname{dom} D.$$

Let $n \in \mathbf{N}$ be arbitrarily fixed. Then the operators $D_o = D^n$, $R_o = R^n$ and

$$F_o = \sum_{k=0}^{n-1} R^k F D^k$$

satisfy also condition (A) with $r_o = r^n$.

Proof. Theorem 7 implies that $r_o = r^n \in \ker D_o = \ker D^n$. By the Taylor formula, an initial operator F_o for $D_o = D^n$ corresponding to $R_o = R^n$ is

$$F_o = I - R_o D_o = I - R^n D^n = \sum_{k=0}^{n-1} R^k F D^k.$$

By (8), we have $R_o x = R^n x = r^n x = r_o x$ for all $x \in X$.

To conclude our considerations, we should point out that the assumption (1) is, indeed, not very restrictive. This is shown by

Example 2. Suppose that X, D, R, F are defined as in Example 1. We have there: $r(t) \equiv 1$ and $d = 1$. Also, since $\dim \ker D = 1$, we may write $\ker D = \operatorname{lin}\{r\}$. Suppose that $R_1 \neq R$ is a right inverse of D. Then an initial operator F_1 corresponding to R_1 is $F_1 x = \varphi(x) r$

for $x \in X$, where φ is a linear functional defined on X such that $\varphi(r)=1$. This implies that $R_1x = Rx - \varphi(Rx)r = Rx - \varphi(rx)r$ for $x \in X$.

Consider now the so-called Berg-Dimovski convolution determined by the functional φ:

$$(24) \qquad ((x \underset{\varphi}{*} y)(t) = \varphi_\tau\left\{\int_\tau^t x(t+\tau+\sigma)y(\sigma)d\sigma\right\} \quad \text{for} \quad x,y \in X$$

(cf. [2], [1]), where the subscript "τ" means that the functional φ_τ acts on the variable τ. Again X is a D-algebra with respect to the multiplication defined by (24). And we also have

$$(r \underset{\varphi}{*} x)(t) = \varphi_\tau\left\{\int_\tau^t x(\sigma)d\sigma\right\} = \varphi_\tau\left\{\int_0^t x(\sigma)d\sigma - \int_0^\tau x(\sigma)d\sigma\right\} =$$

$$= \int_0^t x(\sigma)d\sigma - \varphi_\tau\left\{\int_0^\tau x(\sigma)d\sigma\right\} = [Rx - \varphi(rx)r](t) = (R_1x)(t).$$

This means that again condition (1) is satisfied.

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