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A CLOSED EPIGRAPH THEOREM

1. Various generalizations of the notion of convexity are known in the literature. One of them is the notion of d-convexity or convexity in Menger's sense (M-convexity) in a metric space (X,d) (cf. H. Busemann [1], В.П. Солтан [2], Menger [8], [9]). We shall confine ourselves to the notion of convexity in the so called G-space. In papers of В.П. Солтан [2]-[5] and J. Ger [6] the notion of convex function defined on a metric space was introduced. Below we recall those properties of G-spaces which will be useful in the sequel referring to [1] for further details.

Let (X,ρ) be a metric space and let $x,y,z \in X$ be three pairwise distinct points. We shall say that y lies between x and z and write $(x\ y\ z)$ if $\rho(x,z) = \rho(x,y) + \rho(y,z)$.

Definition 1 (cf. Busemann [1], Menger [9]). A metric space (X,ρ) is called M-convex (convex in Menger's sense) iff for every two distinct points $x,z \in X$ there exists a point $y \in X \setminus \{x,z\}$ such that $(x\ y\ z)$.

Definition 2 (cf. Busemann [1], Menger [9]). A metric space (X,ρ) is called finitely compact iff every bounded and infinite subset of X has at least one cluster point.

Alternatively, we say that (X,ρ) is finitely compact iff every bounded and closed subset of X is compact.

H. Busemann [1] introduced and investigated the notion of a G-space defined as follows:

Definition 3 (cf. Busemann [1]). A finitely compact M-convex metric space (X,ρ) is called a G-space provided that:

1^o for every point $p \in X$ there exists a positive number r_p such that for any two points x,y from the ball $K(p,r_p)$ centered at p and with radius r_p, there exists a point $z \in X$ such that $(x\ y\ z)$;

2^o for any two distinct points $x,y \in X$ and any points $z_1,z_2 \in X$ such that $(x\ y\ z_1)$, $(x\ y\ z_2)$ and $\rho(y,z_1) = \rho(y,z_2)$ one has $z_1 = z_2$. Condition 1^o is called the axiom of local prolongability, (ALP); condition 2^o expresses the uniqueness of prolongation.

In this paper the symbol X always denotes a G-space; the symbols $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{N}$ will stand for sets of reals, rationals and positive integers, respectively.

Fix any two distinct points $x,y \in X$. Let $I:[0,\rho(x,y)] \longrightarrow X$ be an isometry such that $I(0) = x$, $I(\rho(x,y)) = y$ or $I(\rho(x,y)) = x$, $I(0) = y$. Then the set $T(x,y) := I([0,\rho(x,y)])$ is called a segment joining the points x and y. Any two distinct points in a G-space X may be joined by a segment contained in X (cf. [1], [9]). Such a segment need not be unique. If this segment is unique then there exists exactly one isometry $I:[0,\rho(x,y)] \xrightarrow{\text{onto}} T(x,y)$ and such that $I(0) = x$ and $I(\rho(x,y)) = y$ (see [6] Remark 3).

Definition 4 (cf. Busemann [1]). A set $D \subset X$ is called convex iff for every two distinct points $x,y \in \mathrm{cl}D$ the segment $T(x,y)$ is unique and $T(x,y) \subset D$ if $x,y \in D$.

Let us note that if D is convex then the sets clD and intD are convex, too. (cf. [1]).

Definition 5 (cf. J. Ger [6]). Let $x,y \in \mathrm{cl}D$, $x \neq y$. Assume $I:[0,\rho(x,y)] \longrightarrow T(x,y)$ to be an isometry such that $I(0) = x$ and $I(\rho(x,y)) = y$. For every $\lambda \in [0,1]$ we define

$$\lambda x \oplus (1-\lambda) y := l((1-\lambda)\rho(x,y)).$$

Remark (cf. [6] Lemma 1). If $x,y \in \mathrm{cl}D$, $x \neq y$, $\lambda \in [0,1]$ and $z = \lambda x \oplus (1-\lambda) y$ then $\rho(z,x) = (1-\lambda)\rho(x,y)$ and $\rho(z,y) = \lambda\rho(x,y)$.

In the whole paper the symbol D denotes a non-empty, open and convex subset of X.

2. The following theorem holds true in any linear topological Baire space E: (cf. R. Ger [7]); if f is a J-convex function defined on an open and convex set $D_o \subset E$ and if the set

$$\text{epi } f := \left\{ (x,y) \in D_o \times \mathbf{R} : f(x) \leq y \right\}$$

is closed in $D_o \times \mathbf{R}$ then f is continuous. The goal of the present paper is to show that this result carries over the case of G-spaces. We start with the following

Definition 6 (cf. [6]). A function $f: D \to \mathbf{R}$ is called M-convex iff

$$\text{(I)} \qquad f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for $x,y \in D$ and every $\lambda \in [0,1]$. A function $f: D \to \mathbf{R}$ is JM-convex (Jensen M-convex) if (I) holds for all $x,y \in D$ and $\lambda = \frac{1}{2}$.

Now, we may prove the following

Lemma 1. If $f: D \to \mathbf{R}$ is JM-convex and if its epigraph

$$\text{epi } f := \left\{ (x,y) \in D \times \mathbf{R} : f(x) \leq y \right\}$$

is closed in $D \times \mathbf{R}$, then f is M-convex.

Proof. From Theorem in [6] we get the inequality

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

valid for every $\lambda \in [0,1] \cap \mathbf{Q}$ and all $x,y \in D$. It means that $(\lambda x \oplus (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) \in \text{epi } f$. Let us take an arbitrary $\lambda_o \in (0,1)$ and let $(\lambda_n)_{n \in N}$ be a rational sequence such that $\lambda_o = \lim_{n \to \infty} \lambda_n$. Suppose that two distinct points x,y are fixed. From the fact that epi f is closed we obtain

$$\lim_{n \to \infty} (\lambda_n x \oplus (1-\lambda_n)y, \lambda_n f(x) + (1-\lambda_n) f(y)) \in \text{epi } f;$$

or, equivalently, if I is an isometry from $[0,\varrho(x,y)]$ onto $T(x,y)$ we get (see Definition 5):

$$\lim_{n \to \infty} (I(1-\lambda_n)\varrho(x,y)), \lambda_n f(x) + (1-\lambda_n) f(y)) \in \text{epi } f.$$

Therefore

$$(I((1-\lambda_o)\varrho(x,y)), \lambda_o f(x) + (1-\lambda_o) f(y)) \in \text{epi } f,$$

or, in other words,

$$(\lambda_o x \oplus (1-\lambda_o) y, \lambda_o f(x) + (1-\lambda_o) f(y)) \in \text{epi } f.$$

This means that

$$f(\lambda_o x \oplus (1-\lambda_o)y) \leq \lambda_o f(x) + (1-\lambda_o) f(y)$$

and ends the proof.

Lemma 2. Let $\lambda_o \in (0,1)$ and $x_o \in D$ be arbitrarily fixed points and let $r_o = r(x_o)$ be the number occurring in Definition 3. If the function $\varphi : D \to X$ is given by the formula

(1) $$\varphi(x) := \lambda_o x \oplus (1-\lambda_o)x_o, \quad x \in D,$$

then, for every $r \leq r_o$, we have

$$\varphi(K(x_o,r)) = K(x_o,\lambda_o r).$$

Proof. The mapping φ given by the formula (1) is a homeomorphism of D onto $\varphi(D)$ (see [6] Lemma 6). We shall show that for every $r \leqslant r_o$ the following inclusion

$$K(x_o,\lambda_o r) \subset \varphi(K(x_o,r))$$

holds. The opposite inclusion is fulfilled in view of Remark and formula (1). Let $z \in K(x_o,\lambda_o r)\setminus\{x_o\}$ be an arbitrarily fixed point and let $T(z,x_o)$ be the segment joining z and x_o. From Definition 3 we infer that there exists exactly one point y such that $(x_o\ z\ y)$, $z \in T(x_o,y)$ and $\rho(x_o,y) = \frac{1}{\lambda_o}\rho(z,x_o)$ since $\rho(x_o,y) = \frac{1}{\lambda_o}\rho(z,x_o) \leqslant \frac{1}{\lambda_o}\lambda_o r = r$. It means that $y \in K(x_o,r)$ and $\rho(z,x_o) = \lambda_o\rho(x_o,y)$. Therefore (see Remark) $z = \lambda_o y \oplus (1-\lambda_o)x_o$ and $z = \varphi(y) \in \varphi(K(x_o,r))$.

The following main result yields an analogue of the closed epigraph theorem proved by R. Ger in [7]. The point is that in our case no algebraic structure in the space considered is assumed. On the other hand one cannot treat our Theorem as a direct generalization of R. Ger's result from [7] because he had not assumed the metrizability of the underlying linear space and his functions were vector-valued. Both results however yield "convex analogues" of the classical Banach closed graph theorem.

Theorem. Let $f: D \to \mathbf{R}$ be a JM-convex function. If the epigraph of f is closed in $D \times \mathbf{R}$ then f is continuous in D.

Proof. Fix an $x_o \in D$ and put

$$A := \{x \in D:\ f(x) - f(x_o) \leqslant 1\}.$$

For an arbitrary $x \in D$ there exists an $n \in \mathbf{N}$ such that $f(x)-f(x_o) \leqslant 2^n$. From the fact that f is M-convex (see Lemma 1) we get

$$f\left(\frac{1}{2^n}x \oplus \left(1-\frac{1}{2^n}\right)x_o\right) - f(x_o) \leq \frac{1}{2^n}f(x) + \left(1-\frac{1}{2^n}\right)f(x_o) - f(x_o) \leq 1^{*)}.$$

Therefore, we have

(2) for every $x \in D$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^n}x \oplus \left(1-\frac{1}{2^n}\right)x_o \in A$.

Let $\varphi_n : \mathrm{cl}D \longrightarrow X$ be a mapping given by the formula

(3) $$\varphi_n(x) := \frac{1}{2^n}x \oplus \left(1-\frac{1}{2^n}\right)x_o, \quad x \in \mathrm{cl}D.$$

By virtue of (2) we obtain the inclusion

$$D \subset \bigcup_{n\in\mathbb{N}} \varphi_n^{-1}(A \cap D_n) \quad \text{where} \quad D_n := \varphi_n(\mathrm{cl}D), \quad n \in \mathbb{N}.$$

Note that the function φ_n given by (3) has the form (1); consequently Lemma 2 may be applied. Since D is open and nonempty subset of a complete metric space, by the classical theorem of Baire, D is of the second Baire category whence

(4) $$\mathrm{int}\,\mathrm{cl}\,\varphi_n^{-1}(A \cap D_n) \neq \emptyset$$

for some $n \in \mathbb{N}$. We are goint to show that

$$\mathrm{int}\,\mathrm{cl}(A \cap D_n) \neq \emptyset.$$

To this aim we shall first prove the following inclusion

(5) $$\mathrm{cl}\varphi_n^{-1}(A \cap D_n) \subset \varphi_n^{-1}(\mathrm{cl}(A \cap D_n)).$$

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*) This inequality may also be derived directly from the JM-convexity of f without using Lemma 1 (see Theorem 1 from [6]) because the coefficients occurring here are rational.

Indeed, take an $x \in \mathrm{cl}\varphi_n^{-1}(A \cap D_n)$. Then there exists a sequence $(a_k)_{k\in\mathbf{N}}$, $a_k \in A \cap D_n$, $k \in \mathbf{N}$, such that $x = \lim_{k\to\infty} \varphi_n^{-1}(a_k)$. Let $b_k := \varphi_n^{-1}(a_k)$. Then $x = \lim_{k\to\infty} b_k$, and $\varphi_n(b_k) = a_k$. From (3) we have $a_k = \frac{1}{2^n} b_k \oplus \left(1 - \frac{1}{2^n}\right) x_o$. We have also $a_k \in T(b_k, x_o)$ and $\rho(a_k, x_o) = \frac{1}{2^n}\rho(b_k, x_o)$. Since $\lim_{k\to\infty} b_k = x$ we have $\lim_{k\to\infty} a_k = a$ and $a = \frac{1}{2^n} x \oplus \left(1 - \frac{1}{2^n}\right) x_o$ (see [6] Corollary 1) and from (3) we get $a = \varphi_n(x)$ or, equivalently, $x = \varphi_n^{-1}\left(\lim_{k\to\infty} a_k\right) = \varphi_n^{-1}(a) \in \varphi_n^{-1}(\mathrm{cl}(A \cap D_n))$.
From Lemma 2 we obtain that φ_n is an open mapping, and so

$$\mathrm{int}\, \varphi_n^{-1}(\mathrm{cl}(A \cap D_n)) \subset \varphi_n^{-1}(\mathrm{int}\, \mathrm{cl}(A \cap D_n)).$$

This, (4) and inclusion (5) imply that

$$\emptyset \neq \mathrm{int}\, \mathrm{cl}(\varphi_n^{-1}(A \cap D_n)) \subset \mathrm{int}\, \varphi_n^{-1}(\mathrm{cl}(A \cap D_n)) \subset \varphi_n^{-1}(\mathrm{int}\, \mathrm{cl}(A \cap D_n)).$$

Therefore

$$U := \mathrm{int}\, \mathrm{cl}(A \cap D_n) \neq \emptyset.$$

Now, we shall prove that the set $A \cap D_n$ is closed in D. To show this let us fix a $z \in D \setminus (A \cap D_n)$; then $(z, f(x_o)+1) \in (D \times \mathbf{R}) \setminus \mathrm{epi}\, f$. From the fact that the set $(D \times \mathbf{R}) \setminus \mathrm{epi}\, f$ is open we get the existence of a neighbourhood U_z of the point z and a number $\delta > 0$ such that $(U_z \times (f(x_o)+1-\delta,\ f(x_o)+1+\delta) \subset (D \times \mathbf{R}) \setminus \mathrm{epi}\, f$. So, for every $x \in U_z$, we have $(x, f(x_o)+1) \in (D \times \mathbf{R}) \setminus \mathrm{epi}\, f$ whence $f(x) > f(x_o)+1$. Consequently, $x \in D \setminus A$ which means that $(A \cap D_n)$ is closed in D. Moreover

$$\emptyset \neq U \cap A \cap D_n = U \cap D \cap \mathrm{cl}(A \cap D_n) = U \cap D$$

and $U \cap D$ is an open, nonempty subset of A. We have shown that int $A \neq \emptyset$. The function f is M-convex and upper bounded on A. From Corollary 2 in [6] we obtain that f is continuous in A.

Corollary. If $f: D \to \mathbb{R}$ is JM-convex and lower semicontinuous in D then f is M-convex and continuous in D.

The proof follows from Lemma 1, Theorem and the fact that if f is lower semicontinuous function then epi f is closed (see R. Sikorski [10], exercise 5, p.131).

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