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FRAME FUNCTIONS AND COMPLETENESS

Using the methods of quantum logics, we present a very simple criterion showing that an inner product space S is complete iff it possesses at least one nonzero frame function. As corollary we obtain the criterion that S is complete iff, for some unit vector from the completion of S, the Bessel equality holds for any maximal orthonormal system in S. It generalizes the criterion of S.P. Gudder and S. Holland [10] who required that any maximal orthonormal system to be a basis.

1. Introduction

Let S be an inner product space over the field C of real or complex numbers, and with an inner product $(.,.)$. There are many characterizations of the completeness of an inner product space via algebraic-topological properties ([8-10]), algebraic conditions ([1-3]) or measure-theoretic properties ([5,6,11]).

We introduce three families of closed subspaces of S which will be interesting for us:

(1) F(S) is the set of all $\perp$-closed subspaces of S, i.e., of all subspaces M of S for which we have $M = M^{\perp\perp}$, where $A^{\perp} = \{x \in S: (x,y) = 0 \text{ for any } y \in A\}$, which, in its turn, is an orthocomplemented, complete lattice.

(2) $V(S)$ is the set of all subspaces M of S such that $M = \{u_i\}^{\perp\perp}$ and $M^{\perp} = \{v_j\}^{\perp\perp}$ for every maximal orthonormal system (MONS, for short) $\{u_i\}$ and $\{v_j\}$ in M and $M^{\perp}$, respectively, which is an orthocomplemented poset.

(3) $E(S)$ is the set of all splitting subspaces of S, i.e., of all subspaces M of S for which the condition $M + M^{\perp} = S$ holds, which is an orthocomplemented orthomodular orthoposet. S is complete iff $E(S)$ is a quantum logic [3].

It is evident that

$$E(S) \subseteq V(S) \subseteq F(S),$$

and S is complete iff $E(S) = F(S)$. Moreover, it is an open problem whether for an incomplete S $E(S) = V(S)$.

A (completely additive) state on $E(S)$ is a mapping $m: E(S) \longrightarrow [0,1]$ such that (i) $m(S) = 1$; (ii) $m(\bigvee_{t \in T} M_t) = \sum_{t \in T} m(M_t)$ whenever $\{M_t : t \in T\}$ is a system of mutually orthogonal splitting subspaces for which the join $\bigvee_{t \in T} M_t$ exists in $E(S)$. Analogously we define a state on $V(S)$ or $F(S)$, respectively.

The following is true: S is complete iff $E(S)$ ($V(S)$ [4]) ($F(S)$ [5]) possesses at least one (completely additive) state [6].

In the present note we give a numerical characterization of the completeness using the notion of a frame function.

2. The main result

Denote by $\varphi(S)$ the unit sphere in S, that is, $\varphi(S) = \{x \in S : \|x\| = 1\}$. A frame function is a mapping $f: \varphi(S) \longrightarrow [0,\infty)$ such that there is a finite constant W called the weight of f such that $\sum_i f(x_i) = W$ for any MONS $\{x_i\}$ in S. The notion of frame function for Hilbert spaces has been introduced by Gleason [7].

Theorem. An inner product space is complete iff it possesses at least one nonzero frame function.

Proof. The necessary condition is evident.

For the sufficiency, we may assume without loss of generality that $W = 1$. Let M be a splitting subspaces of S. Let $\{x_i\}$ and $\{y_i\}$ be two MONS's in M and let $\{z_j\}$ be an MONS in $M^\perp$. The simple verification shows that $\{x_i\} \cup \{z_j\}$ and $\{y_i\} \cup \{z_j\}$ are two MONS's in S. Then

$$(1) \qquad \sum_i f(x_i) + \sum_j f(z_j) = 1 = \sum_i f(y_i) + \sum_j f(z_j)$$

which gives $\sum_i f(x_i) = \sum_i f(y_i)$.

Define a mapping $m\colon E(S) \longrightarrow [0,1]$ as follows

$$(2) \qquad m(M) = \begin{cases} \sum_i f(x_i) & \text{if } M \neq \{0\} \\ 0 & \text{if } M = \{0\}, \end{cases}$$

where $\{x_i\}$ is an MONS in M. We claim to show that m is a (totally additive state on $E(S)$. Indeed, the property (1) entails

$$m(M) + m(M^\perp) = 1$$

for any $M \in E(S)$. Let $M = \bigvee_{t \in T} M_t \in E(S)$, where $\{M_t \colon t \in T\}$ is a system of mutually orthogonal subspaces from $E(S)$, and let $\{x_t^i\}_i$ be an MONS in M_t for any $t \in T$. Then $\bigcup_{t \in T} \{x_t^i\}$ is an MONS in M. Hence

$$m(M) = \sum_{t,i} f(x_t^i) = \sum_t \sum_i f(x_t^i) = \sum_t m(M_t).$$

In view of the criterion from [6], the existence of a (totally additive) state on E(S) implies the completeness of S. Q.E.D.

Note 1. In the proof of Theorem, we used the simple observation: if $\{x_i\}$ is an MONS in a splitting subspace M, then $\{x_i\}^{\perp\perp} = M$. For any $\perp$-closed subspace this is true iff S is complete. Therefore (1) and (2) may define a (completely additive) state on V(S) but not on F(S).

Corollary. An inner product space S is complete iff there is a unit vector $y \in \bar{S}$ ($\bar{S}$ denotes the completion of S) such that

$$y = \sum_i (y, x_i)\, x_i \tag{3}$$

for any MONS $\{x_i\}$ in S.

Proof. Let us put $f(x) = |(y,x)|^2$, $x \in \varphi(S)$. The straightforward verification shows that f is a frame function with a unit weight. Q.E.D.

Note 2. If, in Corollary, we find a unit vector $y \in S$ such that (3) holds, then we obtain the generalization of the criterion of Gudder and Holland [8] that any MONS in S is a basis. On the other hand, in this case, we have easily that for every $x \in \varphi(S)$

$$x = \sum_i (x, x_i)\, x_i \tag{4}$$

for any MONS $\{x_i\}$ in S. Indeed, if we define a unitary linear operator $U: S \to S$ such that $Ux = y$ and $Uz = z$ for all $z \perp x, y$, then (3) implies (4) for each $x \in S$. Therefore, for this particular case, Corollary may be proved also using the criterion of Gudder and Holland.

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