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VECTOR SPACES OF REGULAR FUNCTIONAL CONFIGURATIONS*)

Introduction

This paper takes as one's target an extension to use the methods of algebra in investigation of regular functional configurations.

The notion of regular configuration is principal in the theory of conflicts initiated by Z. Pawlak [1]. This notion is extended as regular functional configuration in [2].

A particular kind of transitivity of connections between the objects is basic in the definition of regular configuration. For apex configurations this transitivity is defined through two-elements group $\{1,-1\}$ with ordinary multiplication. A problem posed in this paper consists in investigation of existence a group defining a transitivity of connections between the objects for any regular functional configuration. A positive proof gives possibilities to define the notions of finite field of the degrees of engagement and vector space of regular functional configurations. The notion of generator of a matrix of regular functional configuration (cf. [3] and [4]) is extended.

1. Classes of determined regular functional configurations

Let X be any finite set composed of n elements, μ a measure of Stieltjes on the set of all real numbers R such that $\mu(R) = 1$ and let $N_n = \{1,2,\dots,n\}$.

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Let $C = (X, \mu, \Phi)$ be a determined regular functional configuration in the sense of [2] and [5]. We have then

(1) $$f_{ij}(p) = f_{ji}(p),$$

(2) $$f_{ij}(p)\cdot f_{jk}(p) = f_{ik}(p)$$

for every $i,j,k \in N_n$, $p \in R$ and $(x_i, x_j) \in X^2$, where $\Phi(x_i, x_j) = f_{ij}$, $f_{ij} : R \to \{1; -1\}$ and f_{ij} is a measurable function with respect to μ.

Let F_μ be the set of all measurable functions $f : R \to \{1, -1\}$. For any $f \in F_\mu$ we denote by $\tilde{f}$ a class of the measurable functions equal to f on a set of measure 1 and by $\tilde{F}_\mu$ the set of all classes $\tilde{f}$ such that $f \in F_\mu$. Denote by $\tilde{\Phi}$ a mapping of X^2 into $\tilde{F}_\mu$ satisfied the analogous conditions to (1) and (2). By $\tilde{f}_o$ and $\tilde{f}_1$ we denote the classes of the constant functions $f_o : R \to \{1\}$ and $f_1 : R \to \{-1\}$.

It's evident that if ' is ordinary multiplication of functions, then $(\tilde{F}_\mu, \cdot)$ is an abelian group, $\tilde{f}_o$ is its identity element and the order of any element $\tilde{f}$ of $\tilde{F}_\mu$ is 2.

We define the notion of the degrees of engagement α_{ij} of a pair $(x_i, x_j) \in X^2$ in the class $(X, \mu, \tilde{\Phi})$ analogously to [2]:

$$\alpha_{ij} = \int_{-\infty}^{+\infty} \tilde{f}_{ij} \, d\mu,$$

where the integral is calculated for any function of the class $\tilde{f}_{ij}$.

2. Finite fields of degrees of engagement

Consider now any finite subset $\tilde{H}$ of $\tilde{F}_\mu$. By $(\tilde{H})$ denote a subgroup of $(\tilde{F}_\mu, \cdot)$ generated by $\tilde{H}$.

The group $(\tilde{H})$ is finite because the order of any its element is 2. The order of $(\tilde{H})$ is 2^r, where $r \leq \text{card}\, \tilde{H}$. It's known (cf. [6]), that then $(\widetilde{H})$ is an additive group of a finite field $((\widetilde{H}),\cdot,*)$ of characteristic 2 and this field is only one with respect to isomorphismes.

For any $\widetilde{g}_1,\dots,\widetilde{g}_n$ of $\widetilde{F}_\mu$ it's possible to define a class of determined regular functional configuration $(X,\mu,\widetilde{\Phi})$ as follows

$$\widetilde{\Phi}(x_i,x_j) = \widetilde{g}_i\cdot\widetilde{g}_j$$

for every $i,j\in N_n$.

Denote by $C_n(\widetilde{H})$ the set

$$\left\{(X,\mu,\widetilde{\Phi}) : \widetilde{\Phi}(x_i,x_j) = \widetilde{g}_i\cdot\widetilde{g}_j,\quad \widetilde{g}_i,\widetilde{g}_j\in(\widetilde{H}) \quad \text{for every} \quad i,j\in N_n\right\}.$$

Definition 1. A subgroup $(\widetilde{H})$ of the group $\widetilde{F}_\mu$ is called a standard subgroup of $\widetilde{F}_\mu$ iff

$$\widetilde{f}_1\in(\widetilde{H}), \tag{3}$$

$$\text{if } \widetilde{f}\neq\widetilde{g}, \text{ then } \int_{-\infty}^{+\infty}\widetilde{f}\,d\mu \neq \int_{-\infty}^{+\infty}\widetilde{g}\,d\mu \text{ for every } \widetilde{f},\widetilde{g}\in(\widetilde{H}). \tag{4}$$

Lemma 1. For every standard subgroup $(\widetilde{H})$ of $\widetilde{F}_\mu$ there exists a finite field isomorphic to $((\widetilde{H}),\bullet,*)$ such that every its element is a degree of engagement in a class of the set $C_n(\widetilde{H})$.

Proof. Let $\varphi(\widetilde{f}) = \int_{-\infty}^{+\infty}\widetilde{f}\,d\mu$ for every $\widetilde{f}\in(\widetilde{H})$. From the condition (4) φ is an injective mapping of $(\widetilde{H})$ into the interval $\langle -1;1\rangle$.

Let K_r be the image $\varphi[(\widetilde{H})]$ and let card $K_r = 2^r$. From the definition of the set $C_n(\widetilde{H})$ every number α of K_r is a degree of engagement in a class $(X,\mu,\widetilde{\Phi})$ of $C_n(\widetilde{H})$.

We define the composition laws $\circ$ and $*$ in the set K_r as follows

$$\alpha\circ\beta = \varphi[\varphi^{-1}(\alpha)\cdot\varphi^{-1}(\beta)],$$

$$\alpha*\beta = \varphi[\varphi^{-1}(\alpha)*\varphi^{-1}(\beta)],$$

where $\cdot$ is ordinary multiplication of functions and $*$ is the multiplication in the field $((\widetilde{H}),\cdot,*)$. The mapping φ is an isomorphism of $((\widetilde{H}),\cdot,*)$ onto $(K_r,\circ,*)$, hence the lemma is proved.

Definition 2. The field K_r defined in the proof of Lemma 1 is called a finite field of degrees of engagement.

We denote $K_r \sim (\widetilde{H})$ iff there exists an isomorphism φ of the field K_r onto the field $(\widetilde{H})$.

Lemma 2. For every $r \in N$ any finite field of degrees of engagement K_r has the following properties

(5) $$K_1 = \{1,-1\},$$

(6) $$K_1 \subset K_r,$$

(7) $$\text{if} \quad \alpha \in K_r, \quad \text{then} \quad -\alpha \in K_r,$$

(8) $$0 \notin K_r.$$

Proof. The conditions (5) and (6) are satisfied because 1 is the zero and -1 is the identity element of the field K_r.

If $\alpha \in K_r$, then there exists $\widetilde{f} \in (\widetilde{H})$ such that $\alpha = \int_{-\infty}^{+\infty} \widetilde{f}\, d\mu$ and then $\widetilde{f}_1 \cdot \widetilde{f} \in (\widetilde{H})$ from the condition (3). We have $\int_{-\infty}^{+\infty} \widetilde{f}_1 \cdot \widetilde{f}\, d\mu = 0$ and $f \neq f_1 \cdot f$, thus the subgroup (H) is not standard.

It's possible to show, that for $r > 1$ a field K_r can be defined on the different sets of degrees of engagement. For $r = 2$, K_2 can be any set $\{1,-1,\alpha,-\alpha\}$ where $0 < \alpha < 1$.

3. Generators of matrices of regular functional configurations

Let $(K_r, \circ, *)$ be any fixed finite field of degrees of engagement. Let $G \in K_r^n$ and $\alpha \in K_r$. Denote $G = [g_i]_{n \times 1}$.

We define the following composition laws:

$$G \circ G = [g_{ij}]_{n\times n} = [g_i \circ g_j]_{n\times n} \quad \text{for every } G \text{ of } K_r^n,$$

$$\alpha \circ G = [\alpha \circ g_i]_{n\times 1} \quad \text{for every } \alpha \in K_r \text{ and } G \in K_r^n.$$

A matrix $M = [\alpha_{ij}]_{n\times n}$ is a matrix of class of regular functional configuration $(X,\mu,\tilde{\Phi})$ iff α_{ij} are the degrees of engagement in this class (cf. [4]).

Lemma 3. For every matrix $G \in K_r^n$ the matrix $G \circ G$ is a matrix of class of a regular functional configuration.

Proof. Let $g_1,\dots,g_n \in K_r$ and let $K_r \sim (\tilde{H})$. Then there exist the classes $\tilde{f}_1,\dots,\tilde{f}_n \in (\tilde{H})$ such that $\int_{-\infty}^{+\infty} \tilde{f}_i \, d\mu = g_i$ for every $i \in N_n$. Hence $g_i \circ g_j$ is a degree of engagement of the elements x_i and x_j in the class $(X,\mu,\tilde{\Phi})$ such that $\tilde{\Phi}(x_i,x_j) = \tilde{f}_i \cdot \tilde{f}_j$ for every $i,j \in N_n$. Therefore, the matrix $G \circ G$ is a matrix of this class.

Definition 3. A matrix $G \in K_r^n$ is called a generator over the field K_r of a matrix M of class of regular functional configurations iff $M = G \circ G$.

Denote by $M_n(K_r)$ the set $\{M : M = G \circ G \text{ and } G \in K_r^n\}$.

Lemma 4. Let $M \in M_n(K_r)$ and $M = G' \circ G'$. The set

$$\{G \in K_r^n : G = \alpha \circ G' \quad \text{and} \quad \alpha \in K_r\}$$

is the set of all generators of the matrix M.

Proof. If $\alpha \in K_r$ and $G \in K_r^n$, then $(\alpha \circ G) \circ (\alpha \circ G) = G \circ G$, hence if $M = G' \circ G'$, then $\alpha \circ G'$ is a generator of the matrix M over the field K_r for every $\alpha \in K_r$.

If $G \circ G = G' \circ G'$, $G = [g_i]_{n\times 1}$ and $G' = [g'_i]_{n\times 1}$, then $g_1 \circ g_i = g'_1 \circ g'_i$ for every $i \in N_n$, thus $g_i = (g_1 \circ g'_1) \circ g'_i$ and we have $G = \alpha \circ G'$, where $\alpha = g_1 \circ g'_1$.

Lemma 5. For every matrix M of $M_n(K_r)$ there exists one generator $G_o = [g_i]_{n\times 1}$ and only one such that $g_1 = 1$.

Proof. Let $M \in M_n(K_r)$ and $M = G' \circ G'$, where $G' = [g_i']_{n\times 1}$. From Lemma 4, $g_1' \circ G'$ is only one generator $G_o = [g_i]_{n\times 1}$ of the matrix M such that $g_1 = 1$.

Definition 4. The matrix $\bar{G} = [\bar{g}_i]_{(n-1)\times 1}$ is called a reduced generator of a matrix M of $M_n(K_r)$, iff $M = G \circ G$, $G = [g_i]_{n\times 1}$, $g_1 = 1$ and $\bar{g}_i = \bar{g}_{i-1}$ for every $i = 2,3,\dots,n$.

Let $A = [a_{ij}]_{k\times m}$, $B = [b_{ij}]_{k\times m}$ and $a_{ij}, b_{ij}, \lambda \in K_r$.

We define the following composition laws

$$A \oplus B = [a_{ij} \circ b_{ij}]_{k\times m}, \tag{9}$$

$$\lambda * A = [\lambda * a_{ij}]_{k\times m}. \tag{10}$$

Theorem. The set $M_n(K_r)$ with the composition laws $\oplus$ and $*$ defined by (9) and (10) is a vector space isomorphic to the vector space K_r^{n-1} over the field K_r.

Proof. Let $\psi : M_n(K_r) \longrightarrow K_r^{n-1}$ be a mapping such that for every M of $M_n(K_r)$, $\psi(M)$ is a reduced generator of M. From Lemmas 3 and 5 the mapping ψ is bijective.

For every $M, M' \in M_n(K_r)$ we have $\psi(M \oplus M') = \psi([g_i \circ g_j \circ g_i' \circ g_j']_{n\times n})$, where $g_1 = g_1' = 1$ and then $\psi(M \oplus M') = \bar{G} \oplus \bar{G}'$, where $\bar{G}$ and $\bar{G}'$ are the reduced generators of M and M'.

If $\lambda \in K_r$, then $\psi(\lambda * M) = \psi([\lambda * (g_i \circ g_j)]_{n\times n})$, where $g_1 = 1$. We have $\lambda * (g_i \circ g_j) = (\lambda * g_i) \circ (\lambda * g_j)$, therefore $\psi(\lambda * M) = \lambda * \psi(M)$.

It's proved that ψ is an isomorphism of $(M_n(K_r), \oplus, *)$ onto the vector space K_r^{n-1} over the field K_r.

4. Vector spaces of determined regular functional configurations

Consider now any standard subgroup $(\tilde{H})$ of the group $\tilde{F}_\mu$ and a finite field of degrees of engagement $K_r \sim (\tilde{H})$.

Let $\varphi: C_n(\tilde{H}) \longrightarrow K_r^{n-1}$ be a mapping such that for every C of $C_n(\tilde{H})$, $\varphi(C)$ is a reduced generator of the matrix of C. The mapping φ is bijective. We define an addition of configurations of $C_n(\tilde{H})$ and a multiplication of configuration by an element of K_r as follows

(11) $$C_1 + C_2 = \varphi^{-1}(\bar{G}_1 \oplus \bar{G}_2),$$

(12) $$\lambda \cdot C = \varphi^{-1}(\lambda * \bar{G}),$$

where $\bar{G}_1$, $\bar{G}_2$ and $\bar{G}$ are the reduced generators of matrices of configurations C_1, C_2 and C. From (11) and (12), the bijective mapping φ is an isomorphism of $C_n(\tilde{H})$ onto the vector space K_r^{n-1} over the field K_r. Let $C_n(\tilde{H})$ and $C_m(\tilde{H})$ be two vector spaces of determined regular functional configurations and let φ and ψ be the isomorphismes of $C_n(\tilde{H})$ and $C_m(\tilde{H})$ onto K_r^{n-1} and K_r^{m-1}. For every linear mapping $h : K_r^{n-1} \longrightarrow K_r^{m-1}$, the mapping $\psi^{-1} \circ h \circ \varphi$ is a linear mapping of the vector space $C_n(\tilde{H})$ onto vector space $C_m(\tilde{H})$.

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