DEMONSTRATIO MATHEMATICA

Vol. XXIII No 2 1990

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A GOURSTAT PROBLEM FOR MANGERON POLYVIBRATING EQUATION IN R^n, $n \geq 3$

Introduction

Goursat problems for partial differential equations in R^n-space, where $n \geq 3$, have been extensively examined in the special case when the boundary conditions are given on the planes of coordinates (see [7], [8], [10], [13], [14], [15] and the references in [19]). The papers devoted to the other cases were less numerous and in almost all of them the boundary conditions were given either on one surface or on a set of characteristics (see [1], [9], [11] and [17]). The much harder case of a greater number of non-characteristic surfaces was examined in papers [3]-[6].

In the present paper we extend the results of [3] to the case of R^n, $n \geq 3$.

1. Let $\Omega = \{x(x_1, \dots, x_n) \in R^n: 0 \leq x_i \leq A_i < \infty;\ 1 \leq i \leq n\}$ be a parallelepiped in R^n-space ($n \in N^{*)}$, $n \geq 3$), m an element of the set N, $(Y, \|\cdot\|)$ a Banach space and $c : \Omega \to Y$ a given function.

We consider the following partial differential equation

$$L^m u(x) = c(x), \qquad x \in \Omega, \tag{1}$$

*) The symbol N denotes the set of all positive integers.

where $L = D_1 \dots D_n$, with $D_i = \frac{\partial}{\partial x_i}$, $i = 1,2,\dots,n$; $L^k = L(L^{k-1})$ for $1 \leq k \leq m$; $L^0 u = u$. Equation (1) is called Mangeron's polyvibrating equation of order nm with n independent variables.

By a solution of equation (1) in Ω we mean a function $u : \Omega \to Y$ such that $D_{i_1} \dots D_{i_l} L^k u \in C^{m-k-1}$ for $k = 0,1,\dots,m-1$; $1 \leq i_1 < \dots < i_l \leq n$; $l = 1,2,\dots,n$, satisfying equation (1) at each point $x \in \Omega$.

Admit the following notation

$$l_i(j) = \begin{cases} j & \text{for } 1 \leq j \leq i-1,\ 2 \leq i \leq n, \\ j+1 & \text{for } 1 \leq j \leq n-1,\ 1 \leq i \leq n-1, \end{cases}$$

$\Omega_i = \bigtimes_{j=1}^{n-1} \langle 0, A_{l_i(j)} \rangle$ for $i = 1,2,\dots,n$, and introduce the system of surfaces $S_1, S_2, \dots, S_n$ given by the equations

$$x_i = f_i(x^{(i)})$$

respectively, where $x^{(i)} = (x_1^i, x_2^i, \dots, x_{n-1}^i) \in \Omega_i$; $x_j^i = x_{l_i(j)}$, $i = 1,2,\dots,n$; $j = 1,2,\dots,n-1$.

In this paper we examine a linear Goursat-like problem that consists in finding a solution u of equation (1) in Ω, satisfying on S_i, $i = 1,2,\dots,n$, the boundary conditions of the form

$$L^r u(x) = N_{i,r}(x^{(i)}) \tag{2}$$

$(x \in S_i;\ x^{(i)} \in \Omega_i)$, where $N_{i,r} : \Omega_i \to Y$ $(i = 1,2,\dots,n;\ r = 0,1,\dots \dots m-1)$ are given functions.

We make the following assumptions

I. The functions $f_i : \Omega_i \to \langle 0, A_i \rangle$ are of class C^{m-n+2} and satisfy the conditions

(3) $$D^{l}_{I_i(j)} f_i(x^{(i)}) = 0$$

$(x^{(i)} \in \Omega_i$; $i = 1,2,\dots,n$; when $x^i_j = 0$ for some $j \in \mathbb{N}$; $j \leq n-1$; $l = 0,1,\dots,n-2)$, where $D^{l}_{I_i(j)} = D_{I_i(j)} D^{l-1}_{I_i(j)}$ for $1 \leq l \leq n-2$; $D^{0}_{I_i(j)} f_i = f_i$.

Moreover, the surfaces $S_1, S_2, \dots, S_n$ satisfy the relation $S_p \cap S_q = \{x \in \Omega;\ x_p = x_q = 0\}$, $p,q = 1,2,\dots,n$; $p \neq q$.

II. The functions $N_{i,r} : \Omega_i \longrightarrow Y$ of class $C^{m-r+n-2}$, satisfy the conditions*)

(4) $$N^{(k)}_{i,r}(x^{(i)}) = 0 \quad \text{for} \quad x^{(i)} \in \Omega_i;\ \prod_{j=1}^{n-1} x^i_j = 0,$$

$k = 0,1,\dots,m-r+n-3$, $i = 1,2,\dots,n$; $r = 0,1,\dots,m-1$,

(5) $$\|N^{(k)}_{i,r}(x^{(i)})\|_k \leq K_1 \Big[\min_{1 \leq j \leq n-1} x^i_j\Big]^{\theta} \quad \text{for} \quad x^{(i)} \in \Omega_i$$

$i = 1,2,\dots,n$; $k = m-r+n-2$; $r = 0,1,\dots,m-1$, where θ is a positive constant.

III. $c : \Omega \longrightarrow Y$ is a function of class C^{n-2}.

In what follows we will use the notation

(6) $$M_j = \max_{1 \leq i \leq n} \ \max_{|\alpha| = j} \ \sup_{\Omega_i} |D^{\alpha} f_i(x^{(i)})|,$$

$j = 1,2,\dots,m+n-2$, and α is the multi-index with $n-1$ components concerning the differentiation with respect to the coordinates of the point $x^{(i)}$ and let

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*) Here and in the sequel, $\|\cdot\|_k$, $k \geq 1$, denotes the norm in the space $L_k(B_1,B_2)$ of k-linear continuous functions, where B_1 and B_2 are Banach spaces. The symbol $\|\cdot\|_0$ is meant as the norm in B_2.

(7) $$M_k^* = \max\,(1, \max_{1\le j\le k} M_j), \qquad k = 1,2,\dots,m+n-2 .$$

2. Auxiliary theorems

Lemma 1. (cf. Lemma 1 in papers [2] and [3]). Let a function $u : \Omega \to Y$ be given by

(8) $$u(x) = R_m(x) + \sum_{j=1}^{m}\sum_{k=1}^{n} (x_k)^{m-j}\psi_{k,m-j}(x^{(k)}),$$

$x \in \Omega$; $x^{(k)} \in \Omega_k$; $k = 1,2,\dots,n$, where

(9) $$R_m(x) = [(m-1)!]^{-n}\int_0^{x_1}\dots\int_0^{x_n}\prod_{k=1}^{n}(x_k-\eta_k)^{m-1}\,c(\eta_1,\dots,\eta_n)\,d\eta_1\dots d\eta_n,$$

(10) $$\psi_{k,m-j}(x^{(k)}) = \left\{(m-j)!\,[(m-j-1)!]^{n-1}\right\}^{-1}\times$$

$$\times\int_0^{x_1^k}\dots\int_0^{x_{n-1}^k}\prod_{l=1}^{n-1}(x_l^k-\eta_l^k)^{m-j-1}\varphi_{k,m-j}(\eta_1^k,\dots,\eta_{n-1}^k)\,d\eta_1^k\dots d\eta_{n-1}^k$$

for $j = 1,2,\dots,m-1$; $\psi_{k,0} = \varphi_{k,0}$, where the functions $\varphi_{k,m-j} : \Omega \to Y$, $k = 1,2,\dots,n$; $j = 1,2,\dots,m$, possess continuous derivatives $D_{l_k(i_1)}\dots D_{l_k(i_l)}\varphi_{k,m-j}$, $1\le i_1<\dots<i_l\le n-1$; $l = 1,2,\dots,n-1$, of class C^{j-1}, respectively. Under these conditions u is a solution in Ω of equation (1) with $c \in C(\Omega)$.

Conversely, if u is a solution of (1) in Ω, then there are functions $\varphi_{k,m-j} : \Omega_k \to Y$, $k = 1,2,\dots,n$; $j = 1,2,\dots,m$, possessing the afore-said derivatives of class C^{j-1} such that (8)-(10) are satisfied at each point $x \in \Omega$.

Lemma 2. The following inequality

$$(11)\qquad |D^{\alpha} f_i(x^{(i)})| \leq M^*_{n-1}\left(\min_{1\leq k\leq n-1} x^i_k\right)^{n-1-|\alpha|}, \quad x^{(i)} \in \Omega_i,$$

$i = 1,2,\ldots,n$; $|\alpha| \leq n-2$, is valid.

Lemma 3. For each number λ_f satisfying the condition

$$(12)\qquad 0 < \lambda_f < \left[(m-1)((2n-3)M^*_1)^{m+n-3+\theta}\right]^{-1/\theta}$$

there is a number $\delta = \delta(\lambda_f)$, $0 < \delta < \min(1, \min_{1\leq i\leq n} A_i)$, such that

$$(13)\qquad |D^{\alpha} f_i(x^{(i)})| \leq \lambda_f(\min_{1\leq j\leq n-1} x^i_j)^{n-2-|\alpha|}, \qquad |\alpha| \leq n-3$$

holds good for $x^{(i)} \in \Omega_i$; $\min_{1\leq j\leq n-1} x^i_j < \delta$; $i = 1,2,\ldots,n$.

Lemma 3 is a consequence of Lemma 2.

Lemma 4. The following inequality

$$(14)\qquad \|N^{(k)}_{i,r}(x^{(i)})\|_k \leq K_1[(m+n-2-r-k)!]^{-1}(\min_{1\leq j\leq n-1} x^i_j)^{m+n-2-r-k+\theta},$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $k = 0,1,\ldots,m+n-2-r$; $r = 0,1,\ldots,m-1$, holds good.

We prove Lemma 4 by using relations (4) and (5) and on applying the Taylor formula.

Now let us introduce the functions $a^i_{r,j} : \Omega_i \to R$ given by

$$(15)\qquad a^i_{r,j}(x^{(i)}) = \begin{cases} x^j_r & \text{for } r \neq i, \\ f_i(x^{(i)}) & \text{for } r = i, \end{cases}$$

when $i < j$, $r = 1,2,\ldots,n-1$; $x^{(i)} \in \Omega_i$; $1 \leq i,j \leq n$,

$$(16)\qquad a^{i}_{r,j}(x^{(i)}) = \begin{cases} x^{j}_{r} & \text{for } r \neq i-1, \\ f_{i}(x^{(i)}) & \text{for } r = i-1, \end{cases}$$

when $j<i$, $r = 1,2,\ldots,n-1$; $x^{(i)} \in \Omega_i$; $1 \leq i,j \leq n$.

We consider the function sequences $\left\{z^{\nu}_{\vec{k}(\alpha)}\right\}$ defined by

$$(17)\qquad z^{\nu}_{\vec{k}(\alpha)}(x^{(\nu)}) = (z^{\nu}_{1,\vec{k}(\alpha)}(x^{(\nu)}),\ldots,z^{\nu}_{n-1,\vec{k}(\alpha)}(x^{(\nu)}))$$

where

$$(18)\qquad z^{\nu}_{r,\vec{k}(\alpha)}(x^{(\nu)}) = a^{k_{\alpha-1}}_{r,k_\alpha}(z^{\nu}_{\vec{k}(\alpha-1)}(x^{(\nu)}))$$

for $\alpha = 2,3,\ldots$; $r = 1,2,\ldots,n-1$, and

$$(19)\qquad z^{\nu}_{r,\vec{k}(1)}(x^{(\nu)}) = a^{\nu}_{r,k_1}(x^{(\nu)})$$

for $r = 1,2,\ldots,n-1$ ($\vec{k}(\alpha) = (k_1,\ldots,k_\alpha)$; $1 \leq k_l \leq n$; $k_l \neq k_{l-1}$; $l = 1,2,\ldots,\alpha$; $k_0 = \nu$; $x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\ldots,n$.

Lemma 5. There is a number $\alpha_0 \in \mathbb{N}$ such that if $\mathbb{N} \ni \alpha > \alpha_0$ or $\min_{1\leq r\leq n-1} x^{\nu}_{r} \leq \delta$, then [1]

$$(20)\qquad \max_{\vec{k}(\alpha)} \min_{1\leq r\leq n-1} z^{\nu}_{r,\vec{k}(\alpha)}(x^{(\nu)}) \leq \mathrm{const}(\lambda_f)^{\alpha}\left(\min_{1\leq r\leq n-1} x^{\nu}_{r}\right)^{n-2}.$$

The validity of Lemma 5 results from Assumption 1, Corollary 1 in [18] and Lemma 3.

Using formula (1.43) in [17] one can prove the following lemma (cf. Corollary 1.3 in [17])

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[1] Here and in the sequel const denotes a positive constant.

Lemma 6. The inequality

$$(21)\qquad \| z^{\nu(1)}_{k(\alpha)}(x^{(\nu)})\|_1 \le l!\,[M_1^*(2n-3)]^{\alpha l}\,[(n-1)^l M_l^*]^{l-1},$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $l = 1,2,\dots,m+n-2$; $\alpha = 1,2,\dots,$ holds good.

Let $R_{\nu,k}(x^{(\nu)}) = [R_k(x)]_{x=f_\nu(x^{(\nu)})}$, $x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $k = 1,2,\dots,m$, with R_k defined by (9) with the replacement of m by k.

Lemma 7. The inequality

$$(22)\qquad \| R^{(r)}_{\nu,k}(x^{(\nu)})\|_r \le \mathrm{const}\left(\min_{1\le j\le n-1} x_j^\nu\right)^{k+n-1-r},$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $r = 0,1,\dots,k+n-2$; $k = 1,2,\dots,m$, is valid.

We omit an inductive proof of this Lemma.

Let

$$(23)\qquad \mathcal{K}_{k,m-j} = \left\{ \varphi : \Omega_k \to Y \,;\, \| \varphi^{(r)}(x^{(k)})\|_r \le \le C(\varphi) \min_{1\le l\le n-1} x_l^{k\; j+n-2-r+\theta} \right\}$$

for $x^{(k)} \in \Omega_k$; $k = 1,2,\dots,n$; $r = 0,1,\dots,j+n-2$; $j = 1,2,\dots,m$, where $C(\varphi)$ is a positive constant depending in general on the function φ.

For a fixed positive integer s such that $2 \le s \le m$ and fixed function $\varphi_{k,m-j} \in \mathcal{K}_{k,m-j}$, $k = 1,2,\dots,n$; $j = 1,2,\dots,s-1$, we consider the functions

$$(24)\qquad Q_{s,j}(x) = \sum_{k=1}^{n} (x_k)^{s-j} \int_0^{x_1^k} \dots \int_0^{x_{n-1}^k} \prod_{l=1}^{n-1} (x_r^k - \varrho_r^k)^{s-j-1} \times$$

$$\times \varphi_{k,m-j}(\varrho_1^k, \dots, \varrho_{n-1}^k) d\varrho_1^k \dots d\varrho_{n-1}^k, \qquad x \in \Omega;$$

$$(25)\qquad Q_{\nu,s,j}(x^{(\nu)}) = \left[Q_{s,j}(x)\right]_{x_\nu = f_\nu(x^{(\nu)})},$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $j = 1,2,\dots,s-1$; $s = 2,3,\dots,m$.

We have the following lemma whose inductive proof will be also omitted.

Lemma 8. If $\varphi_{k,m-j} \in \mathcal{X}_{k,m-j}$, then

$$(26)\qquad \| Q^{(r)}_{\nu,s,j}(x^{(\nu)}) \|_r \leq$$

$$\leq \text{const} \max_{1\leq k\leq n} C(\varphi_{k,m-j}) \left(\min_{1\leq l\leq n-1} x_l^{\nu} \right)^{s+n-2-r+\theta},$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $j = 1,2,\dots,s-1$; $r = 0,1,\dots,s+n-2$; $s = 2,3,\dots,m$.

3. Solution of the problem (1), (2)

Let us observe that if u is a solution of (1) in Ω, then by Lemma 1, we have

$$(27)\qquad L^{m-s}u(x) = R_s^*(x) + \sum_{k=1}^{n} \varphi_{k,m-s}(x^{(k)}), \qquad x \in \Omega,$$

$x^{(k)} \in \Omega_k$; $k = 1,2,\dots,n$; $s = 1,2,\dots,m$, where[2)]

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2) As usual, we set $\sum_{k=\alpha}^{\beta} a_k = 0$ for $\beta < \alpha$.

(28) $$R_s^*(x) = R_s(x) + \sum_{r=1}^{s-1} \left\{(s-r)!\left[(s-r-1)!\right]^{n-1}\right\}^{-1} \sum_{k=1}^{n} (x_k)^{s-r} \times$$

$$\times \int_0^{x_1^k} \cdots \int_0^{x_{n-1}^k} \prod_{j=1}^{n-1} (x_j^k - \varrho_j^k)^{m-j-1} \varphi_{k,m-r}(\varrho_1^k, \ldots, \varrho_{n-1}^k) d\varrho_1^k \ldots d\varrho_{n-1}^k .$$

Imposing on the function u (see (8)) the boundary conditions (2), we obtain a system of linear functional equations, with the unknown functions $\varphi_{k,m-s}$, $k = 1,2,\ldots,n$; $s = 1,\ldots,m$, of the form

(29) $$\varphi_{i,m-s}(x^{(i)}) = W_{i,m-s}\, x^{(i)} - \sum_{\substack{k_1=1 \\ k_1 \neq i}}^{n} \varphi_{k_1,m-s}\, z^i_{\vec{k}(1)}(x^{(i)})),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,m$, where

(30) $$W_{i,m-s}(x^{(i)}) = N_{i,m-s}(x^{(i)}) - R^*_{i,s}(x^{(i)}),$$

(31) $$R^*_{i,s}(x^{(i)}) = \left[R^*_s(x)\right]_{x_i = f_i(x^{(i)})}$$

and $z^i_{\vec{k}(1)}(x^{(i)}) = (z^i_{1,\vec{k}(1)}(x^{(i)}),\ldots,z^i_{n-1,\vec{k}(1)}(x^{(i)}))$ are given by formula (19). Let us denote

(32) $$V^{\alpha}_{i,m-s}(x^{(i)}) = (-1)^{\alpha} \sum_{\vec{k}(\alpha)} W_{k_\alpha,m-s}(z^i_{\vec{k}(\alpha)}(x^{(i)})),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,m$; $\vec{k}(\alpha) = (k_1,\ldots,k_\alpha)$; $1 \leq k_l \leq n$; $k_l \neq k_{l-1}$; $l = 1,2,\ldots,\alpha$; $k_0 = i$; $\alpha = 1,2,\ldots$.

We will solve (29) by using the iteration method (see [12]).

Theorem 1 (cf. Theorem 1 in [3]). If Assumptions I-III are satisfied then for each $1 \leq l \leq m$ the system of functions $\{\varphi_{i,m-s}\}$, $i = 1,2,\ldots,n$; $s = 1,2,\ldots,l$, given by

$$(33) \qquad \varphi_{i,m-s}(x^{(i)}) = W_{i,m-s}(x^{(i)}) + \sum_{\alpha=1}^{\infty} V^{\alpha}_{i,m-s}(x^{(i)})$$

is a solution of the system of functional equations

$$(34) \quad \varphi_{i,m-s}(x^{(i)}) = W_{i,m-s}(x^{(i)}) - \sum_{\substack{k_1=1 \\ k_1\neq i}}^{n} \varphi_{k_1,m-s}\, z^{i}_{\bar{k}(1)}(x^{(i)}),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,l$, in the set Ω. This is the unique solution of system (34) in the class $\mathcal{K}_l$ of all systems of functions $\{\varphi_{i,m-s}\}$, such that $\varphi_{i,m-s} \in \mathcal{K}_{i,m-s}$ for $i = 1,2,\ldots,n$; $s = 1,2,\ldots,l$ (see (23)).

Proof. Apply mathematical induction with respect to l. Let $l = 1$. We will prove the uniform convergence of the series $\sum_{\alpha=1}^{\infty} V^{\alpha(r)}_{i,m-1}(x^{(i)})$, (where $i = 1,2,\ldots,n$; $r = 0,1,\ldots,n-1$) in the set Ω_i. In virtue of formulas (30), (31) and Lemmas 4 and 7, we have

$$(35) \qquad \|W^{(r)}_{i,m-1}(x^{(i)})\|_r \leq \mathrm{const}\left(\min_{1\leq i\leq n-1} x^{i}_{j}\right)^{n-r-1+\theta},$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $r = 0,1,\ldots,n-1$.

If $\alpha > \alpha_0$ or $\min\limits_{1\leq j\leq n-1} x^{i}_{j} < \delta$, then the relations (20), (21) and (35) imply the inequality

$$(36) \quad \| V^{\alpha(r)}_{i,m-1}(x^{(i)})\|_r \leq \mathrm{const}\left[(n-1)(\lambda_f)^{n-r-1+\theta}((2n-3)M^{*}_{1})^{r}\right]^{\alpha} \times \\ \times \left(\min_{1\leq j\leq n-1} x^{i}_{j}\right)^{n-r-1+\theta},$$

$i = 1,2,\dots,n;\ r = 0,1,\dots,n-1$. If $\alpha \leq \alpha_0$ and $\min\limits_{1\leq j\leq n-1} x^i_j > \delta$, then

$$(37)\qquad \| V^{\alpha(r)}_{i,m-1}(x^{(i)})\|_r \leq \mathrm{const}\left(\min_{1\leq j\leq n-1} x^i_j\right)^{n-r-1+\theta},$$

$i = 1,2,\dots,n;\ r = 0,1,\dots,n-1$.

As $0<(n-1)(\lambda_f)^{n-r-1+\theta}((2n-3)M^*_1)^r<1$ (cf. (12)) for $r = 0,\dots,n-1$, the series appearing in formula (33) with $s = 1$, as well as the series obtained by its differentiation of order $r=1,2,\dots,n-1$, are uniformly convergent in Ω_i, whence and by (33), (36) and (37) the functions $\varphi_{i,m-1}$, $i = 1,2,\dots,n$, given by (33) for $s = 1$ belong to the classes $\mathcal{K}_{i,m-1}$ respectively. We will show that $\{\varphi_{i,m-1}\}$, $i = 1,\dots,n$, satisfies the system of functional equations

$$(38)\qquad \varphi_{i,m-1}(x^{(i)}) = W_{i,m-1}(x^{(i)}) - \sum_{\substack{k_1=1\\k_1\neq i}}^{n} \varphi_{k_1,m-1}(z^i_{\vec{k}(1)}(x^{(i)})).$$

Setting $k_\beta = l_{\beta-1}$ and $\alpha-1 = \mu$ for $\alpha \geq 2$ in (32), we get (cf. (18), (19))

$$z^i_{\vec{k}(\alpha)}(x^{(i)}) = z^{k_1}_{\vec{l}(\mu)}\left(z^i_{\vec{k}(1)}(x^{(i)})\right),$$

where $\vec{l}(\mu) = (l_1,\dots,l_\mu)$; $l_j \neq l_{j-1}$; $j = 1,2,\dots,\mu$; $l_0 = k_1$.

Basing on this relation and using (32), (33), we obtain the following sequence of equalities

$$\varphi_{i,m-1}(x^{(i)}) = W_{i,m-1}(x^{(i)}) - \sum_{\substack{k_1=1\\k_1\neq i}}^{n} \left\{ W_{k_1,m-1}\left(z^i_{\vec{k}(1)}(x^{(i)})\right) + \right.$$

$$+\sum_{\mu=1}^{\infty}(-1)^{\mu}\sum_{\vec{l}(\mu)} W_{l_\mu,m-1}\left(z^{k_1}_{\vec{l}(\mu)}\left(z^{i}_{\vec{k}(1)}\left(x^{(i)}\right)\right)\right)\Bigg\} =$$

$$= W_{i,m-1}(x^{(i)}) - \sum_{\substack{k_1=1\\ k_1\neq i}}^{n} \varphi_{k_1,m-1}\left(z^{i}_{\vec{k}(1)}(x^{(i)})\right),$$

which imply that the system of functions $\{\varphi_{i,m-1}\}$, $i = 1,2,\dots,n$, given by (33) satisfies (38) in the set Ω, as required.

Now we shall prove that (38) has exactly one solution such that $\varphi_{i,m-1} \in \mathcal{X}_{i,m-1}$ $(i = 1,2,\dots,n)$. To this end let us note that if $\{\varphi_{i,m-1}\}$, $i = 1,2,\dots,n$, is a solution of (38), then

$$\varphi_{i,m-1}(x^{(i)}) = W_{i,m-1}(x^{(i)}) + \sum_{\alpha=1}^{s-1} V^{\alpha}_{i,m-1}(x^{(i)}) +$$

$$+ (-1)^{s}\sum_{\vec{k}(s)} \varphi_{k_s,m-1}\left(z^{i}_{k(s)}(x^{(i)})\right), \qquad s \geq 1.$$

As, by assumption, $\varphi_{i,m-1} \in \mathcal{X}_{i,m-1}$, Lemma 5 implies that $\sum_{\vec{k}(\alpha)} \varphi_{k_s,m-1} \circ z^{i}_{\vec{k}(s)} \rightrightarrows 0$, when $s \to \infty$.

Thus, the functions $\varphi_{i,m-1}$, $i = 1,2,\dots,n$, are expressed by formula (33) for $s = 1$, which ends the proof of Theorem 1 for $l = 1$, $m = 1$.

Let us assume that $m \geq 2$ and that Theorem 1 is valid for $1 \leq l \leq l_0$, where $1 \leq l_0 < m$. We will prove that it is also valid for $l = l_0+1$, i.e. that

(39) $$\varphi_{i,m-(l_0+1)}(x^{(i)}) = W_{i,m-(l_0+1)}(x^{(i)}) + - \sum_{\substack{k_1=1 \\ k_1 \neq i}}^{n} \varphi_{k_1,m-(l_0+1)}\left(z^{i}_{\bar{k}(1)}(x^{(i)})\right),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\dots,n$, possesses exactly one solution $\left\{\varphi_{i,m-(l_0+1)}\right\}$, $i = 1,2,\dots,n$, such that $\varphi_{i,m-(l_0+1)} \in \mathcal{K}_{i,m-(l_0+1)}$, with the functions $\varphi_{i,m-(l_0+1)}$ given by formula (33) for $s = l_0+1$.

By the inductive assumption, definitions (30), (31) and Lemmas 4, 7, 8, the inequality

(40) $$\| W^{(r)}_{i,m-(l_0+1)}(x^{(i)}) \|_r \leqslant \mathrm{const} \left(\min_{1 \leqslant j \leqslant n-1} x^{i}_{j} \right)^{l_0+n-1-r+\theta},$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\dots,n$; $r = 0,1,\dots,l_0+n-1$, holds good.

If $\alpha > \alpha_0$ or $\min\limits_{1 \leqslant j \leqslant n-1} x^{i}_{j} < \delta$, then inequalities (20), (21), (40) imply

(41) $$\| V^{(r)}_{i,m-(l_0+1)}(x^{(i)}) \|_r \leqslant \mathrm{cons} \left[(n-1)(\lambda_f)^{l_0+n-1-r+\theta} ((2n-3)M^{*}_{1})^{r} \right]^{\alpha} \times \left(\min_{1 \leqslant j \leqslant n-1} x^{i}_{j} \right)^{l_0+n-1-r+\theta},$$

$i = 1,2,\dots,n$; $r = 0,1,\dots,l_0+n-1$. And if $\alpha \leqslant \alpha_0$ and $\min\limits_{1 \leqslant j \leqslant n-1} x^{i}_{j} \geqslant \delta$, $i = 1,2,\dots,n$; $r = 0,1,\dots,l_0+n-1$, holds true.

Since $0 < (n-1)(\lambda_f)^{l_0+n-1-r+\theta} ((2n-3)M^{*}_{1})^{r} < 1$ (cf. (12)) for $r = 0,\dots,l_0+n-1$, we can conclude, analogously as in the case $s = 1$, that the functions $\varphi_{i,m-(l_0+1)}$ $(i = 1,\dots,n)$ defined by formula (33) (for $s = l_0+1$) belong to the class $\mathcal{K}_{i,m-(l_0+1)}$, respectively.

Using an argument analogous to that for $l = 1$ one can prove that the system of the said functions is the only solution of (39) such that $\varphi_{i,m-(l_0+1)} \in \mathcal{K}_{i,m-(l_0+1)}$. Hence and from the inductive assumption we can conclude that Theorem 1 is valid for $l = l_0+1$. Using the induction principle we end the proof of Theorem 1.

Lemma 1 and Theorem 1 imply the following theorem.

Theorem 2. If Assumptions I-III are satisfied then the Goursat-like problem (1), (2) possesses a solution (8), where the functions $\psi_{k,m-j}$ are defined by (10) with $\varphi_{k,m-j}$ given by (33) and satisfying the system of functional equations (29). The solution (8) is unique in the set of all solutions u of equation (1) such that the system of functions $\{\varphi_{k,m-j}\}$, $k = 1,2,\dots,n$; $j = 1,2,\dots,m$, appearing in (10) belongs to the class $\mathcal{K}_m$.

REFERENCES

[1] S.A. Aldashev: On a nonlocal boundary value problem and the Goursat problem for a higher-order equation (in Russian), Izv. Akad. Nauk Kazakh. SSR Ser. Fiz-Mat. 3 (1982), 51-54.

[2] A. Borzymowski: A Goursat problem for some partial differential equation of order 2p, Bull. Acad. Pol. Sci.: Math. 32 (1984), 577-580.

[3] A. Borzymowski, A. Sadowski: A Goursat problem for Mangeron's polyvibrating equation in R^3, Applicable Anal. 23 (1987), 319-332.

[4] A. Borzymowski, A. Sadowski: A boundary value problem for the Mangeron polyvibrating equation, Rend. Mat. 6 (1986), 565-585.

[5] S.C. Chu: On a mixed boundary value problem for linear third order hyperbolic partial differential equations, Rend. Circ. Mat. Palermo 13 (1964), 199-208.

[6] S.C. Chu and J.B. Diaz: The Goursat problem for the partial differential equation $u_{xyz} = f(x,y,z,u,u_x,\dots,u_{yz})$, A Mirage, J. Math. Mech. (16) (1967), 709-713.

[7] J. Conlan: An existence theorem for the equation $u_{xyz} = f$, Arch. Rational Mech. Anal. 9 (1962), 64-76.

[8] J. Conlan, J.B. Diaz: Existence of solutions of n-th order hyperbolic partial differential equations, Contrib. Diff. Equations 2 (1963), 277-289.

[9] Z. Denkowski, A. Pelczar: On the existence and uniqueness of solutions of some partial differential functional equations, Ann. Polon. Math. 35 (1978), 261-304.

[10] I.I. Glick: On an analog of the Euler-Cauchy polygon method for the partial differential equation $u_{x_1 \dots x_n} = f(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_2 \dots x_n})$, Contrib. Diff. Equations 2 (1963), 1-59.

[11] M.G. Gonzales: Resolucion del problema de Cauchy para ecuaciones en derivades parciales totalmente hiperbolicas, Collect. Math. 21 (1970), 199-283.

[12] M. Kuczma: Functional equations in a single variable, PWN, Warszawa, 1968.

[13] M. Kwapisz, B. Palczewski, W. Pawelski: Sur l'existence et l'unicité des solutions du certaines équations differentielles du type $u_{xyz} = f(x,y,z,u,u_x,\dots,u_{yz})$, Ann. Polon. Math. 11 (1961), 75-106.

[14] L. Nappi: Un teorema di esistenza e di unicita per il problema di Darboux in n variabili, Matematiche 23 (1968), 224--241.

[15] J. Persson: Global linear Goursat problems for functions of Gevrey-Lednev type, Ann. Scuola Norm. Sup. Pisa, 23 (1969), 387-412.

[16] J. Persson: Non-characteristic Cauchy problems and generalized Goursat problems in R^n. J. Math. Mech. 18 (1969), 1087-1094.

[17] A. Sadowski: Goursat-like boundary value problems for some partial differential equations with the number of independent variables greater than two (in Polish), Doctoral Thesis, Warsaw University of Technology, Warszaw 1986.

[18] A. Sadowski: Concerning some sequences of iterates in R^n-space, Demonstratio Math. 23 (4) (1990).
[19] W. Walter: Differential and integral inequalities, Springer-Verlag, Berlin, 1970.

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Received December 19, 1988.