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COMPLETIONS OF UNIFORM UNIVERSAL ALGEBRAS AND STRONGLY UNIFORM BCK-ALGEBRAS

Let $\mathcal{U}$ be a uniformity on a universal algebra (A; F). $\mathcal{A}$ = (A; F; $\mathcal{U}$) is called a uniform algebra if each operation $f \in F$ is uniformly continuous on A (with respect to $\mathcal{U}$). It is shown that $\mathcal{A}$ is embeddable, as a dense subalgebra, in a complete uniform algebra of the same type as $\mathcal{A}$. If (A; $\mathcal{U}$) is Hausdorff, then $\mathcal{A}$ has a Hausdorff completion $\bar{\mathcal{A}}$, which is unique up to an isomorphism which fixes A pointwise, and the embedding $\mathcal{A} \to \bar{\mathcal{A}}$ may be described as a category reflection.

Special consideration is given to the case of (Hausdorff) uniform BCK-algebras and to conditions under which their (Hausdorff) completions are also BCK-algebras.

0. Introduction and preliminaries

We assume a familiarity with the theory of BCK-algebras (as general references, we recommend [2] and [3]). The binary operation on a BCK-algebra will be denoted by juxtaposition.

Let (A; ., 0) be a BCK-algebra. We denote by Id(A) (resp. Con(A)) the complete lattice of all ideals (resp. all $\{.,0\}$-congruen-

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ces) of A. If $I \in Id(A)$ and $\theta_I = \{(x,y) \in A\times A: x,y\ yx \in I\}$ then $\theta_I \in$ $\in Con(A)$ and A/θ_I is a BCK-algebra (see [3, p.12]). Conversely if $\theta \in Con(A)$ and $\theta(0) = \{x \in A: (x,0) \in \theta\}$ then $\theta(0) \in Id(A)$. If A is a member of a $\{.,0\}$-variety of BCK-algebras then the mappings $I \mapsto \theta_I$, $\theta \mapsto \theta(0)$ are mutually inverse lattice isomorphisms between Id(A) and Con(A) [2,p.108]. It should be noted that the class of all BCK-algebras is not a $\{.,0\}$-variety [8] and that there exist BCK-algebras A for which Id(A) and Con(A) are not isomorphic [5, Remark 9].

Throughout this paper, R shall denote the classical linearly ordered group of real numbers and ω the set of all non-negative integers. 0 shall denote both the real zero and the zero element of a BCK-algebra. If S and T are sets then for each $i \in T$, π_i shall denote the i-th projection map $S^T \to S$.

We assume a familiarity with the theory of uniform spaces. In this regard, our terminology accords with that of [4]. If $(S,\mathcal{U})$ is a uniform space and $U, V \in \mathcal{U}$, we define

$$U \circ V = \{(x,y) \in S\times S: \text{there exists } z \in S \text{ such that } (x,z) \in U \text{ and } (z,y) \in V\};$$

$$1U = U;\ (n+1)U = (nU) \circ U \quad (0 < n \in \omega).$$

Recall that every uniformity has a "symmetric base", i.e., a base consisting of sets U where $U = U^{-1} = \{(x,y) \in S\times S: (y,x) \in U\}$ [4, Theorem 6.6]. A uniform space $(S,\mathcal{U})$ is Hausdorff (i.e. the uniform topology on S induced by $\mathcal{U}$ is Hausdorff) if and only if $\bigcap \mathcal{U} = id_S = \{(x,x): x \in S\}$, if and only if $\bigcap \mathcal{B} = id_S$ for any base $\mathcal{B}$ for $\mathcal{U}$.

Where products of uniform spaces arise, it will be assumed that they are endowed with the product uniformity (see [4, pp.180-184]). In particular, if (S,d) is a pseudo-metric space and $0 < m \in \omega$, then S^m will be considered to have been endowed with the so-called "max-" pseudo-metric, also denoted by d, i.e.

$$d(x,y) = \max\{d(x(i),y(i)) : 1 \le i \le m\} \qquad (x,y) \in S^m).$$

The following notational convention will be adopted: Let $(S, \mathcal{U})$ be a uniform space with $D \in \mathcal{U}$. If $0 < m \in \omega$ and f is an m-ary operation on S, we define

$$f^{-1}(D) = \{(x,y) \in S^m \times S^m : (f(x), f(y)) \in D\}.$$

Let $t = (F; ar)$ be a "type" of algebras, i.e. F is a set of operation symbols and $ar: F \to \omega$ is the arity function. We define

$$F^+ = \{f \in F : ar(f) > 0\}$$

(i.e., F is the set of all non-constant operation symbols in F).
A structure $\mathcal{A} = (A; F; \mathcal{U})$ is called a uniform algebra (resp. a complete uniform algebra; a Hausdorff uniform algebra) of type t if:

(i) $(A; F)$ is a universal algebra of type t,

(ii) $(A; \mathcal{U})$ is a uniform space (resp. a complete uniform space; a Hausdorff uniform space) and

(iii) for each $f \in F$, the mapping $f_{\mathcal{A}} : A^{ar(f)} \to A$ is uniformly continuous in $\mathcal{U}$. (In the absence of any possible confusion we shall denote $f_{\mathcal{A}}$ by f).

In particular, if $\mathcal{U}$ is the (pseudo-) metric uniformity induced by a (pseudo-) metric d on A and $(A; F; \mathcal{U})$ is a uniform algebra of type t then $(A; F; \mathcal{U})$ will be called a (pseudo-) metric algebra of type t. Such algebras and their completions were studied in [6].□

1. In this paper, we consider uniform algebras and their completions. While the results are quite general, they arose out of a consideration of uniformities on BCK-algebras: a uniform BCK-algebra $(A; \cdot, 0, \mathcal{U})$ will be called a strongly uniform BCK-algebra if it satisfies the following condition: for any Cauchy nets $(x_\gamma; \gamma \in \Gamma)$ and

$(y_\gamma;\ \gamma\in\Gamma)$ in A such that the nets $(x_\gamma y_\gamma;\ \gamma\in\Gamma)$ and $(y_\gamma x_\gamma;\ \gamma\in\Gamma)$ each converge to 0, the net $((x_\gamma,y_\gamma);\ \gamma\in\Gamma)$ in $A\times A$ is eventually in every element of $\mathcal{U}$. Note that such an algebra also satisfies

$$(xy,0),\ (yx,0)\in\bigcap\mathcal{U} \implies (x,y)\in\bigcap\mathcal{U}$$

for all $x,y\in A$. We cite some examples.

1.1. A structure $(A;\ .,\ 0;\ n)$ is called a pseudo-normed BCK-algebra if $(A;\ .,\ 0)$ is a BCK-algebra and n is a pseudo-norm on A, that is, a real valued function on A satisfying $n(0)=0$ and $n(x)\leq \leq n(xy)+n(y)$ for all $x,y\in A$. We call n a norm (and $(A;\ .,\ 0;\ n)$ a normed BCK-algebra) if in addition, $x=0$ whenever $n(x)=0$. (Pseudo-) normed BCK-algebras were introduced and studied in [6]: in particular if A is interpreted as an algebra of sets and $\cdot$ as the operation of set difference then a pseudo-norm on A has some of the properties of a measure; on the other hand if A is an implicational calculus and xy denotes the propositional formula $y\to x$ then pseudo-norms on A may be interpreted as "falsity-valuations". A (pseudo-) norm n on a BCK-algebra A defines a (pseudo-) metric d_n on A: we set $d_n(x,y)=n(xy)+n(yx)$, $x,y\in A$. In [6, Lemma 2] it was proved that the BCK-operation $\cdot : A^2\to A$ is a uniformly continuous mapping, so $(A;\ .,\ 0;\ d_n)$ is a (pseudo-) metric BCK-algebra. The set

$$\mathcal{B}=\Big\{\{(x,y)\in A\times A : d_n(x,y)<\varepsilon\} : 0<\varepsilon\in R\Big\}$$

is a symmetric base for the (pseudo-) metric uniformity $\mathcal{U}$ on A induced by d_n. Now if $(x_\gamma;\ \gamma\in\Gamma)$, $(y_\gamma;\ \gamma\in\Gamma)$ are Cauchy nets in A and the nets $(x_\gamma y_\gamma;\ \gamma\in\Gamma)$ converge to 0 then for $0<\varepsilon\in R$, and sufficiently large $\gamma\in\Gamma$ we have

$$d_n(x_\gamma y_\gamma,0),\ d_n(y_\gamma x_\gamma,0)<\varepsilon/2,$$

hence

$$n(x_\gamma y_\gamma) + n(y_\gamma x_\gamma) < \varepsilon ,$$

i.e.

$$d_n(x_\gamma y_\gamma) < \varepsilon .$$

It follows that $(A; \cdot, 0; \mathcal{U})$ is a strongly uniform BCK-algebra. □

1.2. In [1], Aló and Deeba showed that if $(A; \cdot, 0)$ is a BCK-algebra then $\mathcal{I}_A = \{\theta_I : I \in \mathrm{Id}(A)\}$ is a base for a uniformity on A. Actually $\mathcal{I}_A$ is a base for the discrete uniformity on A, since $\mathrm{id}_A = \theta_{\{0\}} \in \mathcal{I}_A$. However, nontrivial uniformities on A may be obtained in the same way if we replace $\mathcal{I}_A$ by any non-empty subset $\mathcal{B}$ of Con(A) which is closed under finite intersections. We prove a more general proposition. (First recall that a universal algebra (A; F) (of given type) is said to be subdirectly reducible if $\bigcap \mathrm{Con}_F(A) \setminus \{\mathrm{id}_A\} = \mathrm{id}_A$, where $\mathrm{Con}_F(A)$ is the complete lattice of all F-congruences on A, in which case (A; F) may be decomposed nontrivially as a subdirect product of algebras of the same type. It can be shown that a BCK-algebra $(A; \cdot, 0)$ is subdirectly reducible (as an algebra of type (2,0)) if and only if $\bigcap \mathrm{Id}(A) \setminus \{\{0\}\} = \{0\}$). □

1.3. Proposition. (i) Let (A; F) be an algebra of type t and let $\mathcal{B}$ be a subset of $\mathrm{Con}_F(A)$ which is closed under finite intersections. Then $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A and $(A; F; \mathcal{U})$ is a uniform algebra of type t. If $(A; \mathcal{U})$ is Hausdorff but $\mathcal{U}$ is not discrete then (A; F) is subdirectly reducible.

(ii) Let $(A; \cdot, 0)$ be a BCK-algebra and $\mathcal{B}$ a non-empty subset of $\mathcal{I}_A$ which is closed under finite intersections. Then $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A and $(A; \cdot, 0, \mathcal{U})$ is a strongly uniform BCK-algebra.

Proof. (i) Each element θ of $\mathcal{B}$, being an equivalence relation on A, satisfies $\mathrm{id}_A \subseteq \theta^{-1} = \theta = 2\theta$, where $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A. If $f \in F^+$ with $\mathrm{ar}(f) = m$, then

$$\bar{\theta} = \bigcap \left\{ \pi_i^{-1}(\theta) : i=1,\dots,m \right\}$$

is an element of the product uniformity on A^m, and since θ is a congruence relation, we have, for $x,y \in A^m$,

$$(x,y) \in \bar{\theta} \Longrightarrow (f(x),f(y)) \in \theta$$

whence f is uniformly continuous. If the uniformity on A is not discrete then $\bigcap \mathrm{Con}_F(A) \setminus \{id_A\} \subseteq \mathcal{B} = \bigcap \mathcal{U}$ and the last assertion follows.

(ii) It follows from (i) that $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A such that (A; ., 0, $\mathcal{U}$) is a uniform BCK-algebra. Now let $(x_\gamma; \gamma \in \Gamma)$ and $(y_\gamma; \gamma \in \Gamma)$ be Cauchy nets in A with $(x_\gamma y_\gamma; \gamma \in \Gamma)$ and $(y_\gamma x_\gamma; \gamma \in \Gamma)$ each converging to 0. Let $\theta_I \in \mathcal{B}$. For sufficiently large γ we have $(x_\gamma y_\gamma, 0)$, $(y_\gamma x_\gamma, 0) \in \theta_I$, whence $(x_\gamma, y_\gamma) \in \theta_I$, (since A/θ_I is a BCK-algebra). We deduce that (A; ., 0; $\mathcal{U}$) is a strongly uniform BCK-algebra. □

1.4. A complete Hausdorff uniform BCK-algebra (A; ., 0; $\mathcal{U}$) is strongly uniform. Indeed if $(x_\gamma; \gamma \in \Gamma)$, $(y_\gamma; \gamma \in \Gamma)$ are Cauchy nets in A, then they converge, say to x and y respectively. Since $\cdot$ is uniformly continuous, the nets $(x_\gamma y_\gamma; \gamma \in \Gamma)$, $(y_\gamma x_\gamma; \gamma \in \Gamma)$ converge to xy and yx respectively. If these latter nets also converge to 0, then since (A; $\mathcal{U}$) is Hausdorff, we have $xy = 0 = yx$, hence $x = y$. Therefore the net $((x_\gamma, y_\gamma); \gamma \in \Gamma)$ is eventually in every element of $\mathcal{U}$. □

Another class of strongly uniform BCK-algebras will be identified in Corollary 14. In section 15 we shall present examples which show that neither a complete uniform BCK-algebra nor a Hausdorff uniform BCK-algebra need be strongly uniform, and that a Hausdorff strongly uniform BCK-algebra need not be complete.

UA shall denote the category of all uniform algebras of a given type t, where for $\mathcal{A} = (A; F; \mathcal{U})$ and $\mathcal{B} = (B; F; \mathcal{V})$ in UA, a map $h: \mathcal{A} \to \mathcal{B}$ is a UA-morphism if and only if

(i) h: $(A; F) \longrightarrow (B; F)$ is a homomorphism of algebras, and

(ii) h: $(A; \mathcal{U}) \longrightarrow (B; \mathcal{V})$ is a uniformly continuous function.

We shall also consider the following full subcategories of UA:

HA, whose objects are the Hausdorff uniform algebras of type t;

CA, whose objects are the complete uniform algebras of type t;

CHA, whose objects are the complete Hausdorff uniform algebras of type t.

2. Lemma. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t. Then $\cap \mathcal{U}$ is a congruence relation on $(A; F)$. If $A^* = A/\cap \mathcal{U}$, and $\mathcal{U}^* = \{U^*: U \in \mathcal{U}\}$ where $U^* = \{(X,Y) \in A^* x A^*: (x,y) \in U$ for all $x \in X$, $y \in Y\}$ then the canonical surjection $\lambda = \lambda_{\mathcal{A}} : A \longrightarrow A^*$ is an HA-reflection of $\mathcal{A}$ onto $\mathcal{A}^* = (A^*; F; \mathcal{U}^*)$.

Proof. Let $\mathcal{D}$ be a symmetric base for $\mathcal{U}$. Then $\cap \mathcal{D} = \cap \mathcal{U}$. For each $D \in \mathcal{D}$, we have $id_A \subseteq D = D^{-1}$ so $\cap \mathcal{U}$ if reflexive and symmetric. Transitivity of $\cap \mathcal{U}$ follows from the fact that each $D \in \mathcal{D}$ contains 2E for some $E \in \mathcal{D}$, while the uniform continuity of each $f \in F$ ensures that $\cap \mathcal{U}$ is an F-congruence relation on A. There is no difficulty in checking that $\mathcal{A}^*$ is also a uniform algebra of type t and since $\cap \mathcal{U}^* = id_{A^*}$, we have $\mathcal{A}^* \in HA$. Clearly $\lambda \in Hom_{UA}(\mathcal{A}, \mathcal{A}^*)$.

If $\mathcal{B} = (B; F; \mathcal{V}) \in HA$ and $\phi \in Hom_{UA}(\mathcal{A}, \mathcal{B})$, define $\phi^*: A^* \longrightarrow B$ by $\phi^*(X) = \phi(x)$, where $x \in X \in A^*$. ϕ^* is properly defined since ϕ is uniformly continuous and $(B, \mathcal{V})$ is Hausdorff. Since ϕ is a UA-morphism, it follows that ϕ^* is an HA-morphism, unique such that $\phi^* \lambda = \phi$. Thus λ is an HA-reflection. □

Note that in case $\mathcal{A}$ is a pseudo-metric algebra, $\mathcal{A}^*$ is the metric identification of $\mathcal{A}$ and the above result specializes to [6, Lemma 3].

The following result is an algebraic variant of the "Metrization Lemma".

3. **Lemma.** Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t. Let $(D_n; n \in \omega)$ be a sequence of elements of $\mathcal{U}$ such that:

(1) $D_0 = A \times A$,

(2) $3D_{n+1} \subseteq D_n$ and

(3) $\bigcap \left\{ \pi_i^{-1}(D_{n+1}) : i=1,\ldots,m \right\} \subseteq f^{-1}(D_n)$

for all $n \in \omega$ and all $f \in F^+$ with $ar(f) = m$. Then there exists a pseudo-metric d on A such that

(i) $D_n \subseteq \left\{ (x,y) \in A \times A : d(x,y) < 2^{-n} \right\} \subseteq D_{n-1}$ for $0 < n \in \omega$

(ii) $(A; F; d)$ is a pseudo-metric algebra of type t.

Proof. The conditions of the Metrization Lemma [4, Lemma 6.12] hold so there exists a pseudo-metric d on A satisfying (i). Let $f \in F^+$ with $ar(f) = m$ and let $0 < \varepsilon \in R$ be given. Choose $n \in \omega$ such that $2^{-n} < \varepsilon$. Let $\delta = 2^{-n-2}$. For $x, y \in A^m$, we have

$$\begin{aligned} d(x,y) < \delta &\Longrightarrow d(x(i),y(i)) < \delta && (i=1,\ldots,m) \\ &\Longrightarrow (x(i),y(i)) \in D_{n+1} && (i=1,\ldots,m) \\ &\Longrightarrow (f(x),f(y)) \in D_n \\ &\Longrightarrow d(f(x),f(y)) < 2^{-n} < \varepsilon \end{aligned}$$

so f is uniformly continuous in d, and (ii) follows. □

If $(A, \mathcal{U})$ is a uniform space and Q is a family of pseudo-metrics on A, we define $V(d,\varepsilon) \subseteq A \times A$ by

$$V(d,\varepsilon) = \left\{ (x,y) \in A \times A : d(x,y) < \varepsilon \right\}$$

for $d \in Q$ and $0 < \varepsilon \in R$. The set

$$\left\{ V(d,\varepsilon) : d \in Q,\ 0 < \varepsilon \in R \right\}$$

is a subbase for a uniformity $\mathcal{U}(Q)$ on A. If $\mathcal{U} = \mathcal{U}(Q)$, we say that is generated by Q.

4. Corollary. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t and suppose F^+ is finite. Then $\mathcal{U}$ is generated by the family Q of all uniformly continuous pseudo-metrics d such that $f_{\mathcal{A}}: (A^{ar(f)},d) \to (A,d)$ is uniformly continuous for each $f \in F$.

Proof. If $d \in Q$ and $0 < \varepsilon \in R$ then $V(d,\varepsilon)$ contains an element of $\mathcal{U}$ by [4, Theorem 6.11, p.183]. On the other hand if $D \in \mathcal{U}$, define

$$D_0 = A \times A, \quad D_1 = D$$

and choose $E \in \mathcal{U}$ such that $3E \subseteq D$. Noting that for each $f \in F^+$ (with $ar(f) = m$), $f^{-1}(D)$ is a member of the product uniformity on A^m, we may choose $G_{f,1} \dots G_{f,m} \in \mathcal{U}$ such that

$$\cap \left\{\pi_i^{-1}(G_{f,i}) : i=1,\dots,m\right\} \subseteq f^{-1}(D).$$

Let $D_2 = E \cap (\cap \left\{G_{f,i}: i=1,\dots,ar(f);\ f \in F^+\right\})$. Then for each $f \in F^+$,

$$\cap \left\{\pi_i^{-1}(D_2): i=1,\dots,ar(f)\right\} \subseteq f^{-1}(D), \text{ and } 3D_2 \subseteq 3E \subseteq D.$$

Continuing in this fashion, we obtain a sequence $(D_n;\ n \in \omega)$ satisfying the conditions of Lemma 3. It follows from Lemma 3 that there exists $d \in Q$ such that $V(d, \frac{1}{4}) \subseteq D$, hence the result. □

If $\mathcal{A} = (A; F; \mathcal{U})$ is a uniform algebra of a given type and $(B; F)$ is a subalgebra of $(A; F)$, we call $\mathcal{B} = (B; F; \mathcal{U}_B)$ a uniform subalgebra od $\mathcal{A}$, where

$$\mathcal{U}_B = \left\{U \cap (B \times B): U \in \mathcal{U}\right\}.$$

5. Theorem. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t. Then $\mathcal{A}$ is isomorphic to a uniform subalgebra of a product of pseudo-metric algebras of type t. If $\mathcal{A}$ is Hausdorff and F^+ is finite then $\mathcal{A}$ is isomorphic to a uniform subalgebra of a product of metric algebras of type t.

Proof. Let Q be as in Corollary 4. For each $d \in Q$, set $\mathcal{A}_d = (A; F; d)$ and let $\lambda_d : \mathcal{A}_d \longrightarrow \mathcal{A}_d^*$ be the metric identification of $\mathcal{A}_d$. Let $\mathcal{A}' = \Pi(\mathcal{A}_d;\ d \in Q)$, $\mathcal{A}'' = \Pi(\mathcal{A}_d^*;\ d \in Q)$. Define $\phi : \mathcal{A} \longrightarrow \mathcal{A}'$ by

$$\pi_d \phi(x) = x \quad (x \in A,\ d \in Q).$$

ϕ is clearly a one-to-one F-homomorphism and since $\mathcal{U}$ contains each $V(d, \varepsilon)$, $\mathcal{U}$ is the smallest uniformity on A making each $\pi_d\, \phi$ (and hence ϕ itself) uniformly continuous. Now suppose $\mathcal{A}$ is Hausdorff and F^+ is finite. Then the map $\psi : \mathcal{A} \longrightarrow \mathcal{A}''$ defined by

$$\pi_d \psi(x) = \lambda_d(x) \quad (x \in A,\ d \in Q)$$

is one-to-one (since by Corollary 4, any two distinct points $x, y \in A$ must satisfy $d(x,y) \neq 0$ for some $d \in Q$), and an F-homomorphism. Since each λ_d is uniformly continuous, so is ψ. □

It is not clear whether the restriction on F^+ in the latter assertion of Theorem 5 may be dropped. We consider Theorem 5 to be of interest in its own right: in particular it implies that any uniform BCK-algebra is uniformly BCK-embeddable in a product of pseudo-metric BCK-algebras. As a case of special interest we may consider the strongly uniform BCK-algebras whose uniformities are induced by sets of ideals. Let A, $\mathcal{D}$, $\mathcal{U}$ and $\mathcal{A} = (A;\ .,\ 0;\ \mathcal{U})$ be as in Proposition 1.3(ii) and let

$$Q = \left\{ d_n : n \text{ is a pseudo-norm on A and } d_n : A \times A \longrightarrow R \text{ is uniformly continuous} \right\}.$$

We may show that $\mathcal{U}$ is generated by Q. If $d_n \in Q$ then

$$J = J_n = \left\{ x \in A : n(x) = 0 \right\}$$

is an ideal of A [6, Lemma 1] and it is easy to see that $\Theta_J \subseteq V(d_n, \varepsilon)$ for each real $\varepsilon > 0$. Conversely, given $\Theta_J \in \mathcal{D}$, define $n : A \longrightarrow R$ by

$$n(x) = \begin{cases} 0 & \text{if } x \in J \\ 1 & \text{otherwise}. \end{cases}$$

By [6, Lemma 1], n is a pseudo-norm on A and

$$\begin{aligned} d_n(x,y) < 1 &\Leftrightarrow n(xy) = n(yx) = 0 \\ &\Leftrightarrow xy,\ yx \in J \end{aligned}$$

so $V(d_n,1) = \theta_J$. Hence $d_n \in Q$ and $\mathcal{U}$ is generated by Q. It follows in the spirit of Theorem 5 that $\mathcal{A}$ is uniformly embeddable in $\Pi((A;\ .,\ 0;\ d_n)\ :\ d_n \in Q)$. Furthermore the metric identification $(A_n^*;\ .,\ 0,\ d_n^*)$ of a pseudo-normed BCK-algebra $(A;\ .,\ 0,\ d_n)$ is a normed BCK-algebra [6, Theorem 10(ii)] so if $(A;\ \mathcal{U})$ is Hausdorff (i.e. $\bigcap \mathcal{B} = id_A$), the method of Theorem 5 may be applied to prove that $\mathcal{A}$ is uniformly embeddable in $\Pi((A_n^*;\ .,\ 0;\ d_n^*)\ :\ d_n \in Q)$. In summary: a uniform (resp. Hausdorff uniform) BCK-algebra whose uniformity is induced by a set of ideals is isomorphic to a uniform subalgebra of a product of pseudo-normed (resp. normed) BCK-algebras.

At the same time the purely topological counterpart of Theorem 5 [4, Theorem 6.16, p.188] plays a key role in the construction of completions of uniform spaces. Indeed, if $(A;\ \mathcal{U})$ is a uniform space embedded in a product of (pseudo-) metric spaces $(A_i;d_i)$ then the closure of $(A;\ \mathcal{U})$ in a product of (pseudo-) metric completions of the $(A_i;d_i)$ is a uniform completion of $(A;\ \mathcal{U})$ [4, pp.196-197]. Naturally we would like to be able to embed Hausdorff uniform algebras $(A;\ F;\ \mathcal{U})$ into complete Hausdorff uniform algebras in a similar manner without imposing restrictions on the cardinality of F^+, but Theorem 5 will not lead to such a general result. Fortunately we may still obtain a general embedding theorem if we first consider a (purely topological) completion $(\bar{A};\ \bar{\mathcal{U}})$ of the underlying uniform space $(A;\ \mathcal{U})$ and extend each operation f to a uniformly continuous operation on $\bar{A}$.

Recall that the closure $\tilde{B}$ of a subset B of a uniform space $(A,\mathcal{U})$ consists of all points $x \in A$ to which some Cauchy net $(x_\gamma : \gamma \in \Gamma)$ in B converges [4, Theorem 6.21]. If $A = \tilde{B}$ then B is called a dense subset of $(A, \mathcal{U})$.

6. Lemma. Let $\mathcal{B} = (B;F;\mathcal{U}_B)$ be a uniform subalgebra $\mathcal{A} = (A;F;\mathcal{U})$. Let $f \in F$ with $ar(f) = m$ and suppose that for $i = 1,\dots,m$ there exists a (Cauchy) net $(x_\alpha; \alpha \in \Gamma_i)$ in B which converges to $x_i \in A$. Then there exists a (Cauchy) net $(z_\gamma; \gamma \in \Gamma)$ in B which converges to $f_{\mathcal{A}}(x_1,\dots,x_m)$.

Proof. Let $\Gamma = \Pi(\Gamma_i : i=1,\dots,m)$ and define, for $\gamma, \gamma' \in \Gamma$,

$$\gamma < \gamma' \text{ iff } \gamma(i) < \gamma'(i) \quad \text{for} \quad i=1,\dots,m.$$

Then Γ is a directed set. For $\gamma \in \Gamma$, define

$$x_\gamma = (x_{\gamma(1)},\dots,x_{\gamma(m)}); \quad z_\gamma = f_{\mathcal{B}}(x_\gamma); \quad x = (x_1,\dots,x_m).$$

Let $D \in \mathcal{U}$ and choose $E \in \mathcal{U}$ such that $\bigcap\left\{\pi_i^{-1}(E) : i=1,\dots,m\right\} \subseteq \subseteq f_{\mathcal{A}}^{-1}(D)$. For $i = 1,\dots,m$, choose $\alpha_i \in \Gamma_i$ such that $\alpha_i < x \in \Gamma_i \Rightarrow \Rightarrow (x_\alpha, x_i) \in E$. Let $\alpha = (\alpha_1,\dots,\alpha_m)$. Then

$$\alpha < \gamma \in \Gamma \Longrightarrow (x_\gamma, x) \in \bigcap\left\{\pi_i^{-1}(E); \; i=1,\dots,m\right\}$$
$$\Longrightarrow (z_\gamma, f_{\mathcal{A}}(x)) \in D,$$

and it follows that the net $(z_\gamma; \gamma \in \Gamma)$ converges to $f_{\mathcal{A}}(x)$. □

7. Corollary. Let $\mathcal{B} = (B;F;\mathcal{U}_B)$ be a uniform subalgebra of $\mathcal{A} = (A;F;\mathcal{U})$ and let $\mathcal{C} = (C;F;\mathcal{V})$ be a Hausdorff uniform algebra of the same type. Let $\phi : (\tilde{B};\mathcal{U}_{\tilde{B}}) \longrightarrow (C;\mathcal{V})$ be a uniformly continuous function.

(i) $\tilde{B} = (B;F;\mathcal{U}_{\tilde{B}})$ is a uniform subalgebra of $\mathcal{A}$.

(ii) If $\phi|_B$ is an F-homomorphism then so is ϕ.

(iii) If $\mathcal{C}$ is complete then every UA-morphism form $\mathcal{B}$ to $\mathcal{C}$ may be extended uniquely to a UA-morphism from $\tilde{\mathcal{B}}$ to $\mathcal{C}$.
($\tilde{\mathcal{B}}$ will be called the closure of $\mathcal{B}$ in $\mathcal{A}$).

Proof. (i) follows immediately from the previous lemma.

(ii) Let $f \in F$ with $ar(f) = m$ and let $x = (x_1,\dots,x_m) \in \tilde{\mathcal{B}}^m$. For $i = 1,\dots,m$, choose a Cauchy net $(x_\alpha; \alpha \in \Gamma_i)$ in B converging to x_i. Since ϕ is uniformly continuous, the net $(\phi(x_\alpha); \alpha \in \Gamma_i)$ converges to $\phi(x_i)$ for each i. Let Γ be the directed set $\Pi\, \Gamma_i$. For $\gamma \in \Gamma$, define x_γ and z_γ as in Lemma 6, and also

$$y_\gamma = (\phi(x_{\gamma(1)}),\dots,\phi(x_{\gamma(m)})); \quad w_\gamma = f_{\mathcal{C}}(y_\gamma).$$

By the proof of Lemma 6, $(z_\gamma; \gamma \in \Gamma)$ converges to $f_{\mathcal{A}}(x)$ and $(w_\gamma; \gamma \in \Gamma)$ converges to $f_{\mathcal{C}}(\phi(x_1),\dots,\phi(x_m))$. Since ϕ is uniformly continuous, the net $(\phi(z_\gamma); \gamma \in \Gamma)$ must converge to $\phi(f_{\mathcal{A}}(x))$. Now since $\phi|_B$ is an F-homomorphism, we have

$$\phi(z_\gamma) = \phi(f_{\mathcal{B}}(x_\gamma)) = f_{\mathcal{C}}(y_\gamma) = w_\gamma$$

$(\gamma \in \Gamma)$. Since $\mathcal{C}$ is Hausdorff, it follows that

$$\phi(f_{\mathcal{A}}(x)) = f_{\mathcal{C}}(\phi(x_1),\dots,\phi(x_m))$$

and so ϕ is an F-homomorphism.

(iii) Let $\mathcal{C}$ be complete and let $\eta : \mathcal{B} \longrightarrow \mathcal{C}$ be a UA-morphism. By [7, Theorem 39.10], $\eta : (B; \mathcal{U}_B) \longrightarrow (C; \mathcal{T})$ may be extended uniquely to a uniformly continuous function $\tilde{\eta} : (\tilde{B}; \mathcal{U}_{\tilde{B}}) \longrightarrow (C; \mathcal{T})$ and $\tilde{\eta}$ is a UA-morphism by (ii). □

8. Theorem. Let $\mathcal{A}$ be a uniform algebra of type t. Then $\mathcal{A}$ is isomorphic to a dense subalgebra of a complete uniform algebra $\hat{\mathcal{A}}$ of type t.

Proof. By Theorem 5, $\mathcal{A}$ is isomorphic to a uniform subalgebra of a product of pseudo-metric algebras of type t, each of which is isomorphic to a uniform subalgebra of a complete pseudo-metric algebra of type t [6, Lemma 4]. The product $\mathcal{C}$ of these complete algebras is a complete uniform algebra of type t and contains the closure $\hat{\mathcal{A}}$ of the image of $\mathcal{A}$ as a complete uniform subalgebra (by Corollary 7(i)). □

9. Theorem. Let $\mathcal{A}$ be a Hausdorff uniform algebra of type t. Then $\mathcal{A}$ is isomorphic to a dense subalgebra of a complete Hausdorff uniform algebra $\bar{\mathcal{A}}$ which is unique in the sense that if $\mathcal{B}$ is any complete Hausdorff uniform algebra containing $\mathcal{A}$ as a dense uniform subalgebra then the identity map on $\mathcal{A}$ extends to a unique isomorphism from $\bar{\mathcal{A}}$ to $\mathcal{B}$. The embedding $\mathcal{A} \to \bar{\mathcal{A}}$ is a simple CHA-reflection of $\mathcal{A}$.

Proof. Let $\mathcal{A} = (A;F;\mathcal{U})$. By [4, Theorem 6.28], $(A;\mathcal{U})$ is uniformly isomorphic to a dense subspace of a complete Hausdorff uniform space $(\bar{A};\bar{\mathcal{U}})$. For each $f \in F$ (with $ar(f) = m$, say), $f_{\mathcal{A}}$ is a uniformly continuous function from A^m to A (with respect to $\mathcal{U}$). Since $(\bar{A}^m; \bar{\mathcal{U}})$ is a complete Hausdorff uniform space containing A^m as a dense subspace, it follows from [7, Theorem 39.10] that $f_{\mathcal{A}}$ may be extended uniquely to a uniformly continuous function $f_{\bar{\mathcal{A}}} : \bar{A}^m \to \bar{A}$, so we obtain a CHA-object $\bar{\mathcal{A}} = (\bar{A};F;\bar{\mathcal{U}})$ and an HA-embedding $\mathcal{A} \to \bar{\mathcal{A}}$. If $\mathcal{B} \in$ CHA and $\phi : \mathcal{A} \to \mathcal{B}$ is an HA-morphism then by Corollary 7, ϕ may be extended to a unique CHA-morphism $\psi : \bar{\mathcal{A}} \to \mathcal{B}$, so $\mathcal{A} \mapsto \bar{\mathcal{A}}$ is a simple CHA-reflection. In particular, if $\mathcal{A}$ is a dense subalgebra of $\mathcal{B}$ then id_A extends to CHA-morphism $\psi : \bar{\mathcal{A}} \to \mathcal{B}$ and $\eta : \mathcal{B} \to \bar{\mathcal{A}}$. Since $\eta\psi|_{\mathcal{A}} = id_{\mathcal{A}} = id_{\bar{\mathcal{A}}}|_{\mathcal{A}}$, it follows from Corollary 7 than $\eta\psi = id_{\bar{\mathcal{A}}}$ and similarly $\psi\eta = id_{\mathcal{B}}$. Thus ψ is an isomorphism. □

If $\mathcal{M}$ is a class of universal algebras of type t, then a uniform algebra $\mathcal{A} = (A;F;\mathcal{U})$ will be called a uniform $\mathcal{M}$-algebra if $(A;F) \in \mathcal{M}$. Bearing in mind the construction of the (Hausdorff) uniform completion of a (Hausdorff) uniform space described in Theorem 8 and the remarks following Theorem 5, the following theorem is an obvious consequence of Lemma 2 and Theorems 8 and 9. The symbols $\mathcal{A}^*$, $\hat{\mathcal{A}}$ and $\bar{\mathcal{A}}$ are used in the sense of the statements and/or proofs of these three results.

<u>10.</u> Theorem. Let $\mathcal{U}$ be a class of universal algebras of type t and let $\mathcal{A}$ be a uniform $\mathcal{U}$-algebra.

(i) If $\mathcal{U}$ is closed under the formation of homomorphic images then $\mathcal{A}^*$ is a Hausdorff uniform $\mathcal{U}$-algebra.

(ii) If $\mathcal{U}$ is closed under the formation of direct products, isomorphic images and subalgebras then $\hat{\mathcal{A}}$ is a complete uniform $\mathcal{U}$-algebra.

(iii) If $\mathcal{U}$ is closed under the formation of direct products, homomorphic images and subalgebras and $\mathcal{A}$ is Hausdorff then $\bar{\mathcal{A}}$ is a complete Hausdorff uniform $\mathcal{U}$-algebra. □

We shall now consider classes of BCK-algebras and obtain a BCK-analogue of Theorem 10. Having already noted that the $\{.,0\}$-quasi-variety of all BCK-algebras is not a variety, we should not expect the Hausdorff completion of every Hausdorff uniform BCK-algebra to be a BCK-algebra. The situation is better in the case of strongly uniform Hausdorff BCK-algebras.

<u>11</u>. Lemma. Let $\mathcal{B} = (B;.,0;\mathcal{U}_B)$ be a strongly uniform BCK-algebra and a uniform subalgebra of the Hausdorff uniform algebra $\mathcal{A} = (A;.,0;\mathcal{U})$ of type (2,0). Then the closure $\tilde{\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{A}$ satisfies the quasi-identity

$$xy = 0 = yx \Longrightarrow x = y.$$

Proof. Let $x,y \in \tilde{B}$ with $xy = 0 = yx$. Choose nets $(x_\alpha;\ \alpha \in \Gamma_1)$ and $(y_\beta;\ \beta \in \Gamma_2)$ in B, converging to x and y respectively. Let

$\Gamma = \Gamma_1 \times \Gamma_2$ and define $(\alpha,\beta) < (\alpha',\beta')$ iff $\alpha < \alpha'$ and $\beta < \beta'$. Define $x_{\alpha,\beta} = x_\alpha$, $y_{\alpha,\beta} = y_\beta$ $(\alpha \in \Gamma_1, \beta \in \Gamma_2)$. Then the nets $(x_\gamma;\ \gamma \in \Gamma)$ and $(y_\gamma;\ \gamma \in \Gamma)$ converge to x and y respectively, and by the proof of Lemma 6, the nets $(x_\gamma y_\gamma;\ \gamma \in \Gamma)$ and $(y_\gamma x_\gamma;\ \gamma \in \Gamma)$ each converge to 0. Since $\mathcal{B}$ is strongly uniform, the net $((x_\gamma, y_\gamma);\ \gamma \in \Gamma)$ is eventually in every element of $\mathcal{U}_B$ (hence in every element of $\mathcal{U}_{\tilde{B}}$). For $D \in \mathcal{U}_{\tilde{B}}$ choose $E \in \mathcal{U}_{\tilde{B}}$ such that $3E \subseteq D$, and $\gamma' \in \Gamma$ such that for $\gamma' < \gamma \in \Gamma$,

$$(x, x_\gamma),\ (x_\gamma, y_\gamma),\ (y_\gamma, y) \in E.$$

Then $(x,y) \in 3E \subseteq D$, and so $(x,y) \in \bigcap \mathcal{U}_{\tilde{B}} = id_{\tilde{B}}$, i.e., $x = y$. □

We omit a routine proof of the following lemma.

12. Lemma. If UA is the category of all uniform algebras of type (2,0) then the class of all strongly uniform BCK-algebras is closed under the formation of UA-direct products, UA-subalgebras and UA-isomorphic images. □

13. Theorem. Let $\mathcal{A} = (A;\cdot,0;\mathcal{U})$ be a uniform BCK-algebra.

(i) $\hat{\mathcal{A}}$ is a complete uniform BCK-algebra. $\hat{\mathcal{A}}$ is strongly uniform if and only if $\mathcal{A}$ is.

(ii) If $\mathcal{A}$ is a strongly uniform then $\mathcal{A}^*$ is a Hausdorff strongly uniform BCK-algebra.

(iii) If $\mathcal{A}$ is Hausdorff then the following conditions are equivalent:

(1) $\mathcal{A}$ is strongly uniform;

(2) $\bar{\mathcal{A}}$ is a uniform BCK-algebra;

(3) $\bar{\mathcal{A}}$ is a complete Hausdorff strongly uniform BCK-algebra.

Proof. (i) The class of all BCK-algebras is closed under direct products, subalgebras and isomorphic images, so the first assertion follows from Theorem 10(ii). The second assertion follows from Lemma 12.

(ii) Let $(X_\gamma;\ \gamma \in \Gamma)$ and $(Y_\gamma;\ \gamma \in \Gamma)$ be Cauchy nets in $\mathcal{A}^*$ such that $(X_\gamma Y_\gamma;\ \gamma \in \Gamma)$ and $(Y_\gamma X_\gamma;\ \gamma \in \Gamma)$ each converge to $0_{\mathcal{A}^*}$ in $\mathcal{U}^*$. Choose $x_\gamma \in X_\gamma$ and $y_\gamma \in Y_\gamma$ for each $\gamma \in \Gamma$. Then $(x_\gamma y_\gamma;\ \gamma \in \Gamma)$ and $(y_\gamma x_\gamma;\ \gamma \in \Gamma)$ each converges to 0 in $\mathcal{U}$. Since $\mathcal{A}$ is strongly uniform, the net $((x_\gamma,\ y_\gamma);\ \gamma \in \Gamma)$ is eventually in every element of $\mathcal{U}$, whence the net $((X_\gamma,\ Y_\gamma);\ \gamma \in \Gamma)$ is eventually in every element of $\mathcal{U}^*$. In particular, for $X, Y \in \mathcal{A}^*$, we have

$$XY = 0_{\mathcal{A}^*} = YX \Longrightarrow (X,Y) \in \bigcap \mathcal{U}^* \Longrightarrow X = Y.$$

Hence $\mathcal{A}^*$ is a strongly uniform BCK-algebra. $\mathcal{A}^*$ is Hausdorff by Lemma 2.

(iii) (1) $\Longrightarrow$ (2). A product of homomorphic images of BCK-algebras is a universal algebra of type (2,0) satisfying the four defining $\{.,0\}$-identities of a BCK-algebra. $\bar{\mathcal{A}}$ is the closure in such an algebra of an isomorphic image of $\mathcal{A}$ and so, by Lemma 11, satisfies the quasi-identity

$$xy = 0 = yx \Longrightarrow x = y$$

also. It follows that $\bar{\mathcal{A}}$ is a uniform BCK-algebra.
(2) $\Longrightarrow$ (3). Let $\bar{\mathcal{A}}$ be a uniform BCK-algebra. $\bar{\mathcal{A}}$ is also complete and Hausdorff by Theorem 9. Therefore $\bar{\mathcal{A}}$ is strongly uniform by 1.4.
(3) $\Longrightarrow$ (1) follows from Lemma 12. □

<u>14</u>. Corollary. Let $\mathcal{A} = (A;.,0;\mathcal{U})$ be a Hausdorff uniform BCK-algebra and suppose $(A;.,0)$ is a member of some $\{.,0\}$-variety of BCK-algebras. Then $\mathcal{A}$ is a strongly uniform BCK-algebra.

Proof. By Theorem 10(iii) $\bar{\mathcal{A}}$ is a (complete Hausdorff) uniform BCK-algebra. Now by Theorem 13(iii), $\mathcal{A}$ is strongly uniform. □

In the next section we show, inter alia, that we may not drop from the above corollary the condition that $(A;.,0)$ is a member of a BCK-variety.

15. Examples

15.1. We exhibit a complete uniform BCK-algebra which is not strongly uniform. (Hence, in 1.4, the condition that $(A;.,0;\mathcal{U})$ be Hausdorff may not be dropped). We exploit Wroński's example from [8]. Let B and C be countably infinite sets such that the sets B, C and ω are mutually disjoint. Order B and C into one-to-one sequences $(b_i;\ i\in\omega)$ and $(c_i;\ i\in\omega)$ respectively. For $i,j\in\omega$, define

$$ij = i \dot{-} j = \max\{i \dot{-} j, 0\}$$

$$ib_j = i_{cj} = 0$$

$$b_i j = b_{i+j},\ c_i j = c_{i+j}$$

$$b_i b_j = c_i c_j = j \dot{-} i$$

$$b_i c_j = c_i b_j = (j+1) \dot{-} i.$$

Let $A = B\cup C\cup\omega$. By [8], $(A;\ .,0)$ is a BCK-algebra and $\Theta = B^2\cup C^2\cup\omega^2\in \mathrm{Con}(A)$. By Proposition 1.3, $\{\Theta\}$ is a base for a uniformity $\mathcal{U}$ on A and $\mathcal{A} = (A;.,0.,\mathcal{U})$ us a uniform BCK-algebra. ($\mathcal{A}$ is not Hausdorff since $\cap\,\mathcal{U} = \Theta$). Now if $(x_\gamma;\ \gamma\in\Gamma)$ is a Cauchy net in $(A;\mathcal{U})$ then there exists $\gamma\in\Gamma$ such that

$$\gamma<\alpha,\beta\in\Gamma \Longrightarrow (x_\alpha,x_\beta)\in\Theta\ .$$

Hence there exists $X\in\{B,C,\omega\}$ such that

$$\gamma<\alpha\in\Gamma\ x_\alpha\in X.$$

This implies that the net $(x_\gamma;\ \gamma\in\Gamma)$ converges to any given element of X. We conclude that $\mathcal{A}$ is complete. However, for any $i,j\in\omega$, we have

(1) $(b_i,b_j),\ (c_i,c_j)\in\Theta$,

(2) $(b_ic_i,0) = (c_ib_i,0) = (1,0)\in\Theta$,

(3) $(b_i,c_i)\notin\Theta$.

Now (1) implies that $(b_i\ ;\ i\in\omega)$ and $(c_i;\ i\in\omega)$ are Cauchy sequences in $(A;\mathcal{U})$ while (2) implies that the sequences $(b_ic_i;\ i\in\omega)$ and $(c_ib_i;\ i\in\omega)$ each converge to 0. However, by (3), the sequence $((b_i,c_i);\ i\in\omega)$ is never in Θ, so $\mathcal{A}$ is not a strongly uniform BCK-algebra. □

15.2. We show that a Hausdorff uniform BCK-algebra need not be strongly uniform. Let B, C, A, and Θ be exactly as in 15.1. Let $(E,\cdot,0_E)$ be the direct product BCK-algebra $\Pi((A;\cdot,0):\ i\in\omega)$. For each $n\in\omega$, define

$$\Theta_n = \Big\{(x,y)\in E\times E\ :\ x(i) = y(i) \text{ for } 0\leq i\leq n \text{ and } (x(i),y(i))\in\Theta \text{ for all but finitely many } i\in\omega\Big\}.$$

It is easily checked that $\Theta_n\in \mathrm{Con}(E)$ for all $n\in\omega$, that $\mathcal{B} = \{\Theta_n:\ n\in\omega\}$ is closed under finite intersections and that $\bigcap\mathcal{B} = \mathrm{id}_E$. It follows from Proposition 1.3(i) that $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on E and that $\mathcal{E} = (E;\cdot,0;\mathcal{U})$ is a Hausdorff uniform BCK-algebra. Now consider the sequences $(x_i;\ i\in\omega)$ and $(y_i;\ i\in\omega)$ defined by

$$x_i(k) = b_0\ ,$$

$$y_i(k) = \begin{cases} b_0 & \text{for } 0\leq k\leq i \\ c_0 & \text{for } k>i \end{cases}$$

$(i,k\in\omega)$. The constant sequence $(x_i;\ i\in\omega)$ is obviously Cauchy, while for any $n,i,j\in\omega$ with $n\leq i<j$, we have

$$y_i(k) \neq y_j(k) \Longleftrightarrow i\leq k\leq j,$$

and hence $(y_i,y_j)\in\Theta_n$. It follows that $(y_i;\ i\in\omega)$ is also Cauchy. Now if $n,\ i\in\omega$ with $\leq i$, we have

$$(x_iy_i)(k) = (y_ix_i)(k) = \begin{cases} 0 & \text{for } 0\leq k\leq i \\ 1 & \text{for } k>i \end{cases}$$

$(k \in \omega)$ and since $(0,1) \in \Theta$, it follows that $(x_i y_i, 0)$, $(y_i x_i, 0) \in \Theta_n$. Thus the sequences $(x_i y_i;\ i \in \omega)$ and $(y_i x_i;\ i \in \omega)$ each converge to 0_E. However $(b_0, c_0) \notin \Theta$ and therefore there are no n, $i \in \omega$ such that $(x_i, y_i) \in \Theta_n$. This implies that $\mathcal{E}$ is not a strongly uniform BCK-algebra. (Note that by 1.4, $(E, \mathcal{U})$ is not a complete space; more explicitly, $(y_i;\ i \in \omega)$ does not converge). □

15.3. We conclude with an example of Hausdorff strongly uniform BCK-algebra which is not complete. Let $\rho(\omega)$ denote the set of all subsets of ω. Then $(\rho(\omega), \setminus, \emptyset)$ is a BCK-algebra (and, in fact, a member of every variety of BCK-algebras). For each $n \in \omega$, define

$$\begin{aligned} I(n) &= \{z \in \rho(\omega) : z \text{ is finite and } 0,1,\dots,n \notin z\}, \\ \Theta_n &= \Theta_{I(n)} = \{(x,y) \in \rho(\omega) \times \rho(\omega) : x \setminus y,\ y \setminus x, \in I(n)\}, \\ \mathcal{B} &= \{\Theta_n : n \in \omega\}. \end{aligned}$$

It is easily checked that $I(n)$ is an ideal of $\rho(\omega)$ for each $n \in \omega$, that $\{I(n) : n \in \omega\}$ (and hence $\mathcal{B}$) is closed under finite intersections and that $\bigcap \{I(n) : n \in \omega\} = \{\emptyset\}$ (whence $\bigcap \mathcal{B} = \mathrm{id}_{\rho(\omega)}$). It follows from Proposition 1.3(ii) that $\mathcal{B}$ is a base for a (Hausdorff) uniformity $\mathcal{U}$ on $\rho(\omega)$ and that $\mathcal{A} = (\rho(\omega), \setminus, \emptyset, \mathcal{U})$ is a Hausdorff strongly uniform BCK-algebra. Consider the sequence $(x_n;\ n \in \omega)$ defined by

$$x_n = \{0,1,\dots,n\} \quad (n \in \omega).$$

Given $n,m,k \in \omega$ with $n \leq m < k$, we have

$$\begin{aligned} x_m \setminus x_k &= \emptyset \in I(n); \\ x_k \setminus x_m &= \{m+1,\ m+2,\dots,k\} \in I(n), \end{aligned}$$

and it follows that $(x_n;\ n \in \omega)$ is Cauchy. Suppose $(x_n;\ n \in \omega)$ converges to some $y \in \rho(\omega)$. For any $m \in \omega$ and for a sufficiently large integer $k > m$, we must have $x_k \setminus y \in I(m)$, whence $m \in y$. Thus y must be ω.

But for each $m \in \omega$, the set $\omega \setminus x_m$, being infinite, cannot be an element of any $I(n)$. Thus $(x_n;\ n \in \omega)$ does not converge to ω, and hence does not converge. □

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REFERENCES

[1] R.A. Aló, E.Y. Deeba: A note on uniformities of a BCK--algebra, Math. Japon., 30 (1985), 237-240.

[2] W.H. Cornish: On Iséki's BCK-algebras, Lecture Notes in Pure and Appl. Math. Vol. 74, 101-122, Marcel Dekker, New York, 1982.

[3] K. Iséki, S. Tanaka: An introduction to the theory of BCK-algebras, Math. Japon., 23 (1978), 1-26.

[4] J.L. Kelly: General Topology. Van Nostrand, Princeton, 1955.

[5] J.G. Raftery, T. Sturm: On ideal and congruence lattices of BCK-semilattices, Math. Japon., 32 (1987), 465-474.

[6] J.G. Raftery, T. Sturm: On completions of pseudo-normed BCK-algebras and pseudo-metric universal algebras, to appear in Mathematica Japonica, 33 (1988).

[7] S. Willard: General Topology. Addison Wesley, 1968.

[8] A. Wroński: BCK-algebras do not form a variety, Math. Japon., 28 (1983), 211-213.

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