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Andrzej Gębarowski

ON A CLASS OF HOMOGENEOUS CONFORMALLY RECURRENT MANIFOLDS

1. Introduction

Let (M,g) be a Riemannian manifold with a (possibly indefinite) metric g. A tensor field $T^{i_1\ldots i_p}{}_{j_1\ldots j_q}$ of type (p,q) on M will be called recurrent if

$$(1)\qquad T^{h_1\ldots h_p}{}_{t_1\ldots t_q} T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k} = T^{h_1\ldots h_p}{}_{t_1\ldots t_q,k} T^{i_1\ldots i_p}{}_{j_1\ldots j_q},$$

where the comma denotes covariant differentiation with respect to g. Relation (1) states that at any point $x \in M$ such that $T(x) \neq 0$ there exists a (unique) covariant vector $\varkappa$ (called the recurrence vector of T) which satisfies the condition

$$(2)\qquad T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k}(x) = \varkappa_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q}(x).$$

A Riemannian manifold (M,g) will be called recurrent (Ricci-recurrent) if its curvature tensor (Ricci tensor) is recurrent. Throughout this paper we assume that the Ricci tensor of a Ricci-recurrent manifold is not parallel.

According to Adati and Miyazawa [1], an n-dimensional ($n \geq 4$) Riemannian manifold (M,g) will be called conformally recurrent if its Weyl conformal curvature tensor

$$(3)\quad C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \\ + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{ik}g_{hj})$$

is recurrent.

If $C_{hijk,l} = 0$ everywhere on M and dim $M \geq 4$, then (M,g) is said to be conformally symmetric [2]. Such a manifold is called essentially conformally symmetric if it is neigher conformally flat ($C_{hijk} = 0$) nor locally symmetric ($R_{hijk,l} = 0$). Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as recurrent manifolds of dimension $n \geq 4$. The existence of essentially conformally recurrent manifolds, i.e. of conformally recurrent manifolds which lie beyond the two classes mentioned above, has been established in [8].

Let (M,g) be an n-dimensional Riemannian manifold whose metric g need not be definite. If $\bar{g}$ is another metric on M, and there exists a function p on M such that $\bar{g} = (\exp 2p)g$, then g and $\bar{g}$ are said to be conformally related or conformal to each other. A special class of conformally recurrent manifolds which is a natural extension of the class of essentially conformally symmetric ones have been defined and studied by W. Roter in his paper [11]. Namely, a conformally recurrent manifold (M,g) is called simple (s.c.r. in short) if its metric is locally conformal to a non-conformally flat conformally symmetric one, i.e. if for each point $x \in M$ there exists a neighbourhood V of x and a function p on V such that $\bar{g} = (\exp 2p)g$ is a non-conformally flat conformally symmetric metric. In [11] W. Roter gave a characterization of s.c.r. manifolds (Theorem 1) and proved

(Theorem 5) that every non-locally symmetric s.c.r. manifold is Ricci--recurrent or it admits a unique recurrent absolute exterior 2-form ω satisfying

$$C_{hijk} = e\omega_{hi}\omega_{jk} \tag{4}$$

with $|e| = 1$, rank $\omega = 2$ and $\omega_{ri}\omega^{r}_{j} = 0$.

Although the Ricci-recurrent ones do not exhaust the whole class of essentially s.c.r. manifolds (see [5]), they form a remarkable subclass.

We shall restrict our consideration to manifolds which are essentially s.c.r. and whose Weyl conformal curvature tensor is not of the form (4). In [11] W. Roter proved the existence and gave a complete local description (at generic points) of those manifolds. The present paper deals with a global classification problem for homogeneous s.c.r. manifolds of class C^∞ or analytic. First we consider homogeneous s.c.r. manifolds $\overline{M}^n$ (determined in Theorem 1), which are universal in the sense that the pseudo-Riemannian universal coverings of their homogeneous open submanifolds exhaust, up to isometry, all simply connected homogeneous s.c.r. manifolds of the considered type. Next using the same method as in [3] we obtain some information about the global structure of essentially s.c.r. manifolds. We prove there (Theorem 3) that such a manifold is always diffeomorphic to a product $\mathbf{R}^2 \times M$, M being flat and homogeneous.

Throughout this paper, by a manifold we shall mean a connected paracompact manifold either of class C^∞ or analytic. Concerning Riemannian manifolds, we shall often write M instead of (M,g).

2. The general form of universal models

In this section each Latin index runs over 1,2,...,n, and each Greek index - over 2,3,...,n-1. Given a pseudo-Riemannian manifold (M,g), by a *local isometry* of M we shall mean any isometry between open connected subsets of M. First we formulate certain important examples of simply connected, complete, essentially simple conformally recurrent manifolds whose Weyl conformal curvature tensor is not of the form (4) described as follows.

Theorem 1. (i) Let $\bar{M}$ denote the Euclidean n-space ($n \geqslant 4$) endowed with the metric $\bar{g}$ given by

$$\bar{g}_{ij}dx^i dx^j = \Phi(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2\,dx^1 dx^n,$$

where the function Φ is defined by

$$\Phi(x^1,\dots,x^n) = (A(x^1)k_{\lambda\mu} + B(x^1)c_{\lambda\mu})x^\lambda x^\mu,$$

A, B being a non-constant analytic functions on $\mathbb{R}$ and $[k_{\lambda\mu}]$, $[c_{\lambda\mu}]$ non-zero symmetric matrices such that $[k_{\lambda\mu}]$ is non-singular and $k^{\lambda\mu}c_{\lambda\mu} = 0$, rank $c_{\lambda\mu} > 1$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$. Then $\bar{M}$ is a simply connected, complete, analytic, n-dimensional simple conformally recurrent manifold whose Weyl conformal curvature tensor is not of the form (4).

(ii) In the Cartesian coordinates, any local isometry $f = (f^1,\dots,f^n)$ of $\bar{M}$ is of the form

$$(5)\quad \begin{cases} f^1(x^1,\dots,x^n) = \varepsilon x^1 + T \\ f^\lambda(x^1,\dots,x^n) = H^\lambda_\mu x^\mu + C^\lambda(x^1), \quad \lambda = 2,\dots,n-1 \\ f^n(x^1,\dots,x^n) = -\varepsilon k_{\lambda\mu}\dot{C}^\lambda(x^1)\left[H^\mu_\nu x^\nu + \frac{1}{2}C^\mu(x^1)\right] + \varepsilon x^n + z \end{cases}$$

where $[H^\lambda_\mu]$ is an $(n-2)\times(n-2)$ matrix and z, ε, T are real numbers satisfying the conditions

$$(6)\quad |\varepsilon| = 1, \quad A(t) = A(\varepsilon t + T),$$

$$(7)\quad \begin{cases} \text{a)} \quad k_{\nu\tau}H^\nu_\lambda H^\tau_\mu = k_{\lambda\mu} \\ \text{b)} \quad c_{\lambda\mu}B(t) = B(\varepsilon t + T)\, c_{\nu\tau}H^\nu_\lambda H^\tau_\mu \end{cases}$$

for any real t, and the functions C^λ, $\lambda = 2,3,\dots,n-1$, form a solution of the following system of ordinary differential equations

$$\ddot{E}^\lambda(t) = A(t)E^\lambda(t) + B(t)k^{\lambda\nu} c_{\nu\tau}E^\tau(t), \tag{8}$$

with $E^\lambda(t) = H^\lambda_\tau C^\tau(t)$.

(iii) Conversely, given z, ε, T and H^λ_μ, C^λ satisfying (6)-(8), formulae (5) define a global isometry of $\overline{M}$ onto itself.

Proof. (i) From Roter's consideration (cf. [11], proof of Th.7) it follows that $\overline{M}$ is a non-conformally symmetric s.c.r. Ricci-recurrent manifold whose Weyl conformal curvature tensor is not of the form (4). By an explicit computation we verify that the geodesic equations for $\overline{M}$ reduce to a system of linear differential equations, so that $\overline{M}$ is complete.

(ii) Let f be a local isometry of $(\overline{M},\overline{g})$. The differential dx^1 is the unique (up to a constant factor) parallel covariant vector field in $\overline{M}$ (cf. [9] p.54). Hence $f^*dx^1 = \varepsilon dx^1$, ε being a non-zero constant. Raising indices, we obtain $f_* \frac{\partial}{\partial x^n} = \varepsilon^{-1} \frac{\partial}{\partial x^n}$ i.e. $\partial_n f^n = \varepsilon^{-1}$ and $\partial_n f^i = 0$ for $i<n$. One can explicity compute that, the essential components of Ricci tensor R and ∇R are

$$R_{11} = (n-2)A, \qquad \nabla_j R_{11} = (n-2)\nabla_j A$$

(see [11], formulae (36)) so that the relation $f^*\nabla R = \nabla R$ yields $f^1 = x^1 \circ f = \varphi(x^1)$. Moreover, f leaves invariant the orthogonal complement D of $\frac{\partial}{\partial x^n}$, which is an integrable codimension one distribution on $\overline{M}$. This parallel (n-1)-plane field determines a foliation whose leaves are totally geodesic submanifolds of $\overline{M}$. Any leaf $\mathcal{L}$ of D is given by $x^1 = \text{const}$ and inherits from $\overline{M}$ a symmetric connection (as a totally geodesic submanifold), which is flat since $dx^2,\dots,dx^n$ are parallel along $\mathcal{L}$ (see [7], pp.56-59) so that $x^2,\dots,x^n$ are affine coordinates for α. Our local isometry f, whenever defined, sends (local) leaves of D affinely into leaves. Thus, f^λ and f^n are affine functions of $x^2,\dots,x^n$, and f is of the form

$$(9)\qquad \begin{cases} f^1(x^1,\dots,x^n) = \varphi(x^1), \\ f^\lambda(x^1,\dots,x^n) = f^\lambda_\mu(x^1)x^\mu + C^\lambda(x^1), \\ f^n(x^1,\dots,x^n) = f^n_\mu(x^1)x^\mu + \varepsilon^{-1}x^n + C^n(x^1), \end{cases}$$

with

$$(10)\qquad \det\left[f^\lambda_\mu(x^1)\right] \neq 0.$$

Comparing now the components of $\bar{g}$ with those transformed by f, we obtain

$$1 = \bar{g}_{1n} = (f^*\bar{g})_{1n} = \varepsilon^{-1}\dot{\varphi}(x^1),$$

which implies $f^1(x^1,\dots,x^n) = \varepsilon x^1 + T$ for some real T.

Next we have

$$0 = \bar{g}_{1\lambda} = (f^*\bar{g})_{1\lambda} = k_{\nu\mu}\dot{f}^\nu_\varrho(x^1)x^\varrho f^\mu_\lambda(x^1) + k_{\nu\mu}\dot{C}^\nu(x^1)f^\mu_\lambda(x^1) + \varepsilon f^n_\lambda(x^1).$$

The right-hand side of this equality is a polynomial in variables $x^2,\dots,x^{n-1}$, so that $k_{\lambda\mu}\dot{f}^\lambda_\varrho(x^1)f^\mu_\nu(x^1) = 0$, hence $\dot{f}^\lambda_\mu = 0$ by (10) i.e. f^λ_μ is constant, say $f^\lambda_\mu(x^1) = H^\lambda_\mu$, and

$$(11)\qquad f^n_\lambda(x^1) = -\varepsilon^{-1}k_{\nu\mu}\dot{C}^\nu(x^1)H^\mu_\lambda.$$

It is also easy to see that $k_{\lambda\mu} = (f^*\bar{g})_{\lambda\mu} = k_{\nu\varrho}H^\nu_\lambda H^\varrho_\mu$, which implies (7)(a). Evaluating the equality $f^*\bar{g} = \bar{g}$ for the component $\bar{g}_{11}$ we obtain certain equality between polynomials in variables x^λ, $\lambda = 2,\dots,n-1$, namely

$$\left[A(x^1)k_{\lambda\mu} + B(x^1)\,c_{\lambda\mu}\right]x^\lambda x^\mu = \varepsilon^2\left[A(\varepsilon x^1 + T)k_{\nu\tau} + B(\varepsilon x^1 + T)\,c_{\nu\tau}\right]\cdot$$
$$\cdot\left[H^\nu_\lambda H^\tau_\mu x^\lambda x^\mu + 2H^\nu_\lambda C^\tau(x^1)x^\lambda + C^\nu(x^1)C^\tau(x^1)\right] + 2\varepsilon\dot{f}^n_\mu(x^1)x^\mu +$$
$$+ 2\varepsilon\dot{C}^n(x^1) + k_{\lambda\mu}\dot{C}^\lambda(x^1)\dot{C}^\mu(x^1),$$

and by comparing their coefficients and using (7) a) we get

(12) $A(x^1)k_{\lambda\mu} + B(x^1)\, c_{\lambda\mu} = \varepsilon^2\left[A(\varepsilon x^1+T)k_{\lambda\mu} + B(\varepsilon x^1+T)\, c_{\nu\tau}H^\nu_\lambda H^\tau_\mu\right]$,

(13) $\dot{f}^n_\mu(x^1) + \varepsilon\left[A(\varepsilon x^1+T)k_{\nu\tau} + B(\varepsilon x^1+T)\, c_{\nu\tau}\right]H^\nu_\mu C^\tau(x^1) = 0,$

(14) $$2s\dot{C}^n(x^1) + k_{\lambda\mu}\dot{C}^\lambda(x^1)\dot{C}^\mu(x^1) + \varepsilon^2\left[A(\varepsilon x^1+T)k_{\nu\tau} + \right.$$
$$\left. + B(\varepsilon x^1+T)\, c_{\nu\tau}\right]C^\nu(x^1)C^\tau(x^1) = 0.$$

Transvecting (12) with $k^{\lambda\mu}$ and taking account of $k^{\lambda\mu}c_{\lambda\mu} = 0$ and $k^{\lambda\mu}H^\nu_\lambda H^\tau_\mu = k^{\nu\tau}$, we have

(15) $$A(t) = \varepsilon^2 A(\varepsilon t + T)$$

for any real t. Using the very same argument as in [4] (proof of Theorem 2) we obtain $|\varepsilon| = 1$ and (15) implies (6). Obvious consequence of (12) is formula (7) b).

Combining (11) with (13) and taking into account (6)and 7 a) b) we obtain

$$H^\lambda_\tau\ddot{C}^\tau(t) = A(t)H^\lambda_\tau C^\tau(t) + B(t)k^{\lambda\nu}\, c_{\nu\tau}H^\tau_\sigma C^\sigma(t),$$

which by setting $E^\lambda(t) = H^\lambda_\tau C^\lambda(t)$ yields (8).

From (14) and (8) it follows immediately that

$$d(C^n(t) + \tfrac{1}{2}\varepsilon k_{\lambda\mu}\dot{C}^\lambda(t)C^\mu(t))/dt = d(C^n(t) + \tfrac{1}{2}\varepsilon k_{\lambda\mu}\dot{E}^\lambda(t)E^\mu(t))/dt = 0$$

hence

$$C^n(t) = -\frac{1}{2}\varepsilon k_{\lambda\mu}\dot{C}^\lambda(t)C^\mu(t) + z$$

for some real z. In view of (11), this completes the proof of assertion (ii).

(iii) As for the inverse one, we can immediately verify that formulae (5) together with (6)-(8) define global isometry of $\overline{M}$ onto itself. This completes the proof.

In the sequel we shall need the following lemma:

Lemma 1 (see [3], Lemma 1). Let (M,g) and $(\bar{M},\bar{g})$ be two homogeneous pseudo-Riemannian manifolds, locally isometric to each other. If M is simply connected and $\bar{M}$ has the property

(16) any local isometry of $\bar{M}$ can be extended to a global isometry of $\bar{M}$ onto itself,

then there exists an isometric immersion $f: M \to \bar{M}$ such that

(i) the image $f(M)$ is homogeneous (as an open submanifold of $\bar{M}$),

(ii) $f: M \to f(M)$ is a covering.

As mentioned in the introduction the name "universal model" for $\bar{M}^n$ can now be justified as follows.

Theorem 2. Any simply connected homogeneous s.c.r. manifold whose Weyl conformal curvature tensor is not of the form (4) is isometric to the pseudo-Riemannian universal covering of an open homogeneous submanifold of universal model.

Proof. Our argument is a replica of the proof of Theorem 2 in [3]. To prove the statement, observe that any homogeneous simply connected s.c.r. manifold whose Weyl conformal curvature tensor is not of the form (4) is locally isometric to a universal model $\bar{M}$ (cf. [11] Theorem 7). Theorem 1 implies clearly that $\bar{M}$ enjoys the property (16). By Lemma 1, there exists an isometric immersion $f: M \to f(M) \subset \bar{M}$ which is nothing but the universal covering projection. This completes the proof.

3. Some global properties

We are now going to derive some consequences of the above results. It will be convenient to adopt the following notations and conventions. Each element h of the isometry group $I(\bar{M})$ determined in Theorem 1 will be identified with quintuples $h = (\varepsilon, T, H, C(t), z)$ where (as we know) ε belongs to the multiplicative group $Z_2 = \{-1,1\}$, T satisfies (6) and so its range is a discrete subset of $\mathbb{R}$, $H = [H^\lambda_\mu] \in G$ the group of all $(n-2)\times(n-2)$ matrices satisfying (7), the curve $t \to C(t) = [C^2(t),\dots,C^{n-1}(t)]$ in $\mathbb{R}^{n-2}$ is an element of the vector space V of all solutions of (8) and z is an arbitrary real number. On the space V we define the exterior 2-form ω by

$$\omega(C_1,C_2) = \frac{1}{2} k_{\lambda\mu}[\dot{C}^\lambda_1(t)C^\mu_2(t) - C^\lambda_1(t)\dot{C}^\mu_2(t)].$$

Differentiating $\omega(E_1,E_2)$ and taking into account (8) we get

$$d\omega(E_1,E_2)/dt = \frac{1}{2} k_{\lambda\mu}[\ddot{E}^\lambda_1(t)E^\mu_2(t) - E^\lambda_1(t)\ddot{E}^\mu_2(t)] = 0,$$

which implies that ω is a constant independent of t. The 2-form ω is the group operation of $I(\bar{M})$, namely

$$(\varepsilon_1,T_1,H_1,C_1(t),z_1)(\varepsilon_2,T_2,H_2,C_2(t),z_2) =$$

$$= [\varepsilon_1\varepsilon_2,\ \varepsilon_1T_2 + T_1,\ H_1H_2,\ H_1C_2(t) + C_1(\varepsilon_2t + T_2),$$

$$\varepsilon_1\varepsilon_2\omega(H_1C_2(t),\ C_1(\varepsilon_2t + T_2)) + \varepsilon_1z_2 + z_1],$$

where the courves $t \to H_1C_2(t)$ and $t \to C_1(\varepsilon_2t + T_2)$ are easily seen to lie in space V again.

Points of our manifold $\bar{M}$, whose underlying set is just $\mathbb{R}^n$, will be described as triples (x,w,u), $x,u \in \mathbb{R}$, $w \in \mathbb{R}^{n-2}$, so that for an isometry $h = (\varepsilon,T,H,C(t),z) \in I(\bar{M})$ we have

(17) $$h(x,w,u) = (\varepsilon x + T,\ Hw + C(x),\ -\varepsilon\langle\dot{C}(x),\ Hw + \frac{1}{2}C(x)\rangle + \varepsilon u + z),$$

$\langle\cdot,\cdot\rangle$ being the (possibly indefinite) inner product in $\mathbb{R}^{n-2}$ determined by $k_{\lambda\mu}$.

Going on to a further study of the homogeneous s.c.r. manifolds we start with the following auxiliary fact (see [3], Lemma 4):

Lemma 2. Given real vector spaces V and W with a not necessarily definite inner product in W, let G be a Lie group transformations of $V \times W \times V$, each of which is of the form

$$(18) \quad (v_1, w, v_2) \longrightarrow (\varepsilon v_1 + T,\ Aw + C(v_1),\ P(v_1)w + \varepsilon v_2 + S(v_1))$$

for some linear isometry A of W, $|\varepsilon| = 1$, $T \in V$ and C^∞-mappings $C: V \longrightarrow W$, $P: V \longrightarrow L(W,V)$ and $S: V \longrightarrow V$. Then

(i) any open orbit U of G is diffeomorphic to the product $V \times U_o \times V$, U_o being an open subset of W on which a certain group G_o of affine isometries acts transitively;

(ii) any open orbit of G coincides with $V \times W \times V$ whenever the inner product in W is definite.

Using Lemma 2, we now proceed to prove

Theorem 3. Let (M,g) be an n-dimensional simply connected homogeneous s.c.r. manifold whose Weyl conformal curvature tensor is not of the form (4). Then M is diffeomorphic to $R^2 \times M_1$, where M_1 is a simply connected homogeneous flat pseudo-Riemannian manifold with a metric of index k, k = index g - 1.

Proof. By Theorem 2, M is the universal covering of an open homogeneous submanifold U of a universal model $\bar{M}$. Let G be a group of isometries acting on U transitively. G may be assumed to be connected (if necessary we take its connected component of the identity). We are now in the conditions of Lemma 2. In fact, we have the natural decomposition $\bar{M} = R \times R^{n-2} \times R$ with the inner product $\sum_{\lambda,\mu} k_{\lambda\mu} dx^\lambda dx^\mu$ in R^{n-2}, and, by (17) the transformations of G are of form (18). Therefore, by Lemma 2, U is diffeomorphic to $R^2 \times U_o$, U_o being a flat homogeneous manifold with a metric of index equal to k, k = index g-1. Hence M is diffeomorphic to $R^2 \times M_1$, M_1 being the universal covering of U_o. This completes the proof.

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MATHEMATICAL INSTITUTE, COLLEGE OF PEDAGOGICS,
35-950 RZESZÓW, POLAND

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