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CONTROLLABILITY OF THE NEUTRAL FUNCTIONAL-DIFFERENTIAL INCLUSION WITH M-DISSIPATIVE RIGHT HAND SIDE

1. Notation and definitions

Let C_{or}, (resp. AC_{or}, L_{or}) denote the space of all continuous (resp. absolutely continuous, integrable) functions x acting from the interval $[-r,0]$ to $\mathbb{R}^n$ with the standard norm:

$$\|x\|_C = \sup_{-r\le t\le 0} |x(t)|$$

$$\text{(resp. } \|x\|_{AC} = x(-r) + \int_{-r}^{0} |\dot{x}(t)|\,dt$$

$$\text{and } \|x\|_L = \int_{-r}^{0} |x(t)|\,dt).$$

We introduce the following measurability condition (M):

Let D be a nonempty open subset of $\mathbb{R}\times C_{or}\times L_{or}$ and $F: D \longrightarrow \mathrm{Comp}(\mathbb{R}^n)$ be a set-valued function such that:

(M) For every interval $I\subset \mathbb{R}^1$ and sets $K\subset C_{or}$, $S\subset L_{or}$ with $I\times K\times S\subset D$ and for every $\varepsilon>0$ there exists a compact set $E\subset I$ with $\mu(I\setminus E)\le\varepsilon$ such that the restriction of F to $E\times K\times S$ is Borel measurable.

The space of all set-valued functions $F: D\longrightarrow \mathrm{Comp}(\mathbb{R}^n)$ satisfying condition (M) will be denoted by $\mathcal{M}(D, \mathrm{Comp}(\mathbb{R}^n))$.

For every x belonging to $C([\delta,\delta+\rho],R^n)$, $x(\delta)=0$, every z belonging to $L([\delta,\delta+\rho],R^n)$ and ϕ from AC_{or} we define $\phi \oplus x$ and $\dot{\phi} \boxplus z$ by setting:

$$(\phi \oplus x)(t) = \begin{cases} \phi(t-\delta) & \text{for } t\in[\delta-r,\delta] \\ \phi(0)+x(t) & \text{for } t\in[\delta,\delta+\rho], \end{cases}$$

$$(\dot{\phi} \boxplus z)(t) = \begin{cases} \dot{\phi}(t-\delta) & \text{for } t\in[\delta-r,\delta] \\ z(t) & \text{for } t\in[\delta,\delta+\rho]. \end{cases}$$

A pair (D,Ω) of nonempty sets $D\subset R\times C_{or}\times L_{or}$ and $\Omega\subset R\times AC_{or}$ will be called conformable if for every $(t,x),(t,x),(t,y)\in\Omega$ one has $(t,x,\dot{y})\in D$.

Given a conformable pair (D,Ω), a set $C\subset R\times C_{or}\times R\times C_{or}$ and a set-valued function $F:D\longrightarrow \mathrm{Comp}(R^n)$ we say that NFDI(D,F) is (Ω,C)-controllable if there exist numbers $\delta\in R$, $\rho>0$ and an absolutely continuous function $x:[\delta-r,\delta+\rho]\longrightarrow R^n$ such that

$$\begin{cases} \dot{x}(t)\in F(t,x_t,\dot{x}_t) & \text{for a.e. } t\in[\delta,\delta+\rho] \\ (t,x_t) & \text{for a.e. } t\in[\delta,\delta+\rho] \\ (\delta,x_\delta,\delta+\rho,x_{\delta+\rho})\in C & \end{cases}$$

where $x_t(s)=x(t+s)$ for $t\in[\delta,\delta+\rho]$, $s\in[-r,0]$.

We say that (D,Ω,C) has a nonempty intersection property if the following conditions are satisfied:

(i) (D,Ω) is a conformable pair,

(ii) $(\Omega\times\Omega)\cap C$ is nonempty,

(iii) there are $(\delta,\phi)\in\Omega$ and a number $\rho>0$ such that the set $K_\phi(\Omega,C)=\{x\in AC([\delta,\delta+\rho],R^n):x(\delta)=0,\ (t,(\phi\oplus x)_t)\in\Omega$ for $t\in[\delta,\delta+\rho]$ and $(\delta,x_\delta,\delta+\rho,x_{\delta+\rho})\in C\}$ is nonempty.

For $F: D \longrightarrow \mathrm{Comp}(\mathbb{R}^n)$ we define set-valued functions G^F and $\mathcal{F}(G^F)$ by the formulas:

$$G^F(t,x) = F(t,(\phi \oplus x)_t,(\dot{\phi} \boxplus \dot{x})_t) \quad \text{for} \quad t \in [\delta,\delta+\varrho] \text{ and } x \in K_\phi(\Omega,C).$$

$$\mathcal{F}(G^F)(x) = \left\{ f \in L([\delta,\delta+\varrho],\mathbb{R}^n) : f(t) \in G^F(t,x) \text{ for a.e. } t \in [\delta,\delta+\varrho] \right\}$$

$$\mathrm{Dom}\, \mathcal{F}(G^F) = \left\{ x \in K_\phi(\Omega,C) : \mathcal{F}(G^F)(x) \text{ is nonempty} \right\}.$$

We will say that $\mathcal{F}(G^F)$ has strong-weak closed graph if for every sequence (x_n) in $\mathrm{Dom}\, \mathcal{F}(G^F)$ and every sequence (u_n) of $L([\delta,\delta+\varrho],\mathbb{R}^n)$ such that $u_n \in \mathcal{F}(G^F)(x_n)$ for $n = 1,2,\dots$, x_n tending to x strongly and u_n tending to u weakly in $L([\delta,\delta+\varrho],\mathbb{R}^n)$ we also have $u \in \mathcal{F}(G^F)(x)$.

Definition. Let $\Omega \subset \mathbb{R} \times AC_c$ and $C \subset \mathbb{R} \times C_{or} \times \mathbb{R} \times C_{or}$ be given. A set-valued function $F \in \mathcal{M}(D,\mathrm{Comp}(\mathbb{R}^n))$ is said to be m-dissipative with respect to (Ω,C) if (D,Ω,C) has a nonempty intersection property and the following hold:

(i) for every $x,y \in \mathrm{Dom}\, \mathcal{F}(G^F)$, every $u \in \mathcal{F}(G^F)(x)$, $v \in \mathcal{F}(G^F)(y)$ and $\lambda > 0$ we have $\|x-y\|_{L_\delta} \leq \|x-y-\lambda(u-v)\|_{L_\delta}$,

(ii) $\mathrm{Rang}(I-\mathcal{F}(G^F)) = L([\delta,\delta+\varrho],\mathbb{R}^n)$,

where I denotes the identity operator and $\|\ \|_{L_\delta}$ denotes the norm in the space $L([\delta,\delta+\varrho],\mathbb{R}^n)$.

2. Main result

Let us collect together some known consequences of m-dissipativity in the following Lemma 1 (see e.g. [1]).

Lemma 1. Let F be m-dissipative with respect to (Ω,C). For every $\lambda > 0$ let us define the following operators

$$J_\lambda(x) = (I - \lambda \mathcal{F}(G^F))^{-1}(x),$$

$$\mathcal{F}_\lambda(x) = \frac{1}{\lambda}(J_\lambda - I)(x).$$

Then:

(i) J_λ and $\mathcal{F}_\lambda$ are well defined single-valued operators acting from whole space $L([\delta,\delta+\rho],R^n)$ to itself.

(ii) $\mathcal{F}_\lambda(x) \in \mathcal{F}(G^F)(J_\lambda(x))$.

(iii) If $\mathrm{dist}(0,\mathcal{F}(G^F)(x)) = \inf\{\|y\|_{L_\delta} : y \in \mathcal{F}(G^F)(x)\}$ is bounded by K for each x then

$$\lim_{\lambda\to 0} \|J_\lambda(x) - x\|_{L_\delta} = 0 \quad \text{uniformly with respect to } x.$$

(iv) $\|\mathcal{F}_\lambda(x)\|_{L_\delta} \leq \mathrm{dist}(0, \mathcal{F}(G^F)(x))$ for $x \in \mathrm{Dom}\, \mathcal{F}(G^F)$.

Lemma 2. Let $F \in \mathcal{M}(D,\mathrm{Comp}(R^n))$ be bounded and such that $\mathcal{F}(G^F)$ has a strong-weak closed graph. Then for every $\varrho>0$,

$$\mathcal{F}(G^F_\varrho):\mathrm{Dom}\, \mathcal{F}(G^F) \longrightarrow 2^{L([\delta,\delta+\rho],R^n)} \quad \text{with} \quad G^F_\varrho(t,x) = G^F(t,x) + \varrho B$$

(where B is a closed unit ball) has also a strong-weak closed graph.

Proof. Suppose (x_n) and (v_n) are arbitrary sequences in $\mathrm{Dom}\, \mathcal{F}(G^F)$ and $L([\delta,\delta+\rho],R^n)$ respectively and such that $v_n \in \mathcal{F}(G^F_\varrho)(x_n)$ for $n = 1,2,\ldots$, $\|x_n - x\|_{L_\delta} \longrightarrow 0$ and v_n tends weakly to v in $L([\delta,\delta+\rho],R^n)$.

Since $v_n \in \mathcal{F}(G^F_\varrho(x_n))$ then $\mathrm{dist}(v_n(t),G^F(t,x_n)) \leq \varrho$ for a.e. $t \in [\delta,\delta+\rho]$ and $n = 1,2,\ldots$. It is known ([2]) that for every $n = 1,2,\ldots$ there exists $u_n \in \mathcal{F}(G^F)(x_n)$ such that

$$|v_n(t) - u_n(t)| = \mathrm{dist}(v_n(t), G^F(t,x_n)) \leq \varrho \quad \text{for a.e. } t \in [\delta,\delta+\rho].$$

By the boundedness of F the set $\bigcup_{n=1}^{\infty} \mathcal{F}(G^F)(x_n)$ is uniformly integrable on $[\delta,\delta+\rho]$.

Therefore, by Dunford Theorem the set $\{u_n\}_{n=1}^{\infty}$ is relatively weakly compact in $L((\delta,\delta+\rho],R^n)$. Thus there exists u and a subsequence, say (u_k) of (u_n) such that u_k tends weakly to u as k tends to ∞.

We have of course $|v_k(t) - u_k(t)| \leq \eta$ for $k = 1,2,\ldots$ and a.e. $t \in [\delta,\delta+\varrho]$.

By Banach-Mazur Theorem there exists a set of real numbers $C_{n_k} \geq 0$, $k=1,2,\ldots,N$, $N=1,2,\ldots$ with $\sum_{k=1}^{N} C_{n_k} = 1$ and such that $\sum_{k=1}^{N} C_{n_k}\left[v_{n_k}(t) - u_{n_k}(t)\right]$ tends to $v(t) - u(t)$ for a.e. $t \in [\delta,\delta+\varrho]$ as $N \to \infty$. But for every $N = 1,2,\ldots$ and a.e. $t \in [\delta,\delta+\varrho]$ we have

$$\left|\sum_{k=1}^{N} C_{n_k}\left[v_{n_k}(t) - u_{n_k}(t)\right]\right| \leq \sum_{k=1}^{N} C_{n_k}\eta = \eta .$$

Then we obtain $|v(t) - u(t)| \leq \eta$ for a.e. $t \in [\delta,\delta+\varrho]$.

Now by the assumption on the graph of $\mathcal{F}(G^F)$ we have $u \in \mathcal{F}(G^F)(x)$, i.e. $u(t) \in G^F(t,x)$ for a.e. t. Therefore for a.e. $t \in [\delta,\delta+\varrho]$ we have $v(t) \in G^F(t,x) + \eta B$ i.e. $v \in \mathcal{F}(G^F_\eta)(x)$.

Before formulating our main result we introduce the following notation

$$\Lambda_\alpha = \left\{ u \in L([\delta,\delta+\varrho],R^n) : |u(t)| < \alpha \quad \text{almost everywhere}\right\},$$

$K_\alpha = \mathcal{T}(\Lambda_\alpha)$ where $\mathcal{T}$ is an operator defined by the formula

$$(\mathcal{T}u)(t) = \int_\delta^t u(t)dt;$$

$$T_H(u) = \bigcap_{\varepsilon>0}\bigcap_{\alpha>0}\ \bigcup_{0<h<\alpha}\left[\frac{1}{h}(H-u) + \varepsilon B\right]$$

is Bouligand's contingent cone to H at $u \in H$.

The following viability theorem given in [3] will be used in the proof of our main result:

Viability Theorem ([3], Th.4): Let $\phi \in AC_{or}$ and a compact set $H \subset R^n$ be such that $\mathcal{T}(u)(t) \in H - \phi(0)$ for every $t \in [\delta,\delta+\varrho]$ and $u \in \Lambda_\alpha$ with some constant $\alpha = 1+M$.

Let $F : [\delta,\delta+\rho]\times C_{or}\times L_{or} \to \mathrm{Comp}\ R^n$ be such that:

a) there exist a constant $\eta>0$ and a continuous function $q : K_\alpha \to L([\delta,\delta+\rho],R^n)$ such that $q(x)(t)\in G_\eta^F(t,x)$ for a.e. $t\in[\delta,\delta+\rho]$, $x\in K_\alpha$ and such that a function $g : [\delta,\delta+\rho]\times K_\alpha \to R^n$ defined by $g(t,x) = q(x)(t)$ has Volterra's property with respect to $x\in K_\alpha$,

b) $G^F([\delta,\delta+\rho]\times K_\Phi(\Omega,C))\subset MB$,

c) for every $x\in K_\alpha$ and a.e. $t\in[\delta,\delta+\rho]$ we have

$$G^F(t,x)\subset T_H[(\Phi\oplus x)_t(0)] + \eta B.$$

Then for a given $\eta>0$ there exists $x^\eta\in K_\alpha$ such that $x^\eta(t)\in H - \Phi(0)$ and $\dot{x}^\eta(t)\in G^F(t,x^\eta)$ for a.e. $t\in[\delta,\delta+\rho]$.

Theorem. Let $\Omega\subset R\times AC_{or}$, $C\subset R\times C_{or}\times R\times C_{or}$ and $F\in\mu(D,\mathrm{Comp}(R^n))$ be such that the following conditions (i)-(iv) hold:

(i) F is bounded and m-dissipative with respect to (Ω,C),

(ii) $\mathcal{F}(G^F)$ has a strong-weak closed graph,

(iii) there exist numbers $l>0$, $M>0$ and a compact set $H\subset R^n$ such that

(a) $\mathcal{T}(u)(t)\in H - \Phi(0)$ for every $t\in[\delta,\delta+\rho]$ and $u\in\Lambda_\alpha$ with $\alpha = l + M$,

(b) $G^F([\delta,\delta+\rho]\times K_\Phi(\Omega,C))\subset MB$,

(c) $K_\alpha\subset K_\Phi(\Omega,C)$,

(d) for every $\eta>0$ there exists $\delta_\eta>0$ such that for every $x\in K_\alpha$ and $y\in K_\Phi(\Omega,C)$ satisfying $\|x-y\|_L\le\delta_\eta$ we have

$$F(t,(\Phi\oplus y)_t,(\dot{\Phi}\boxplus\dot{y})_t)\subset T_H[(\Phi\oplus x)_t(0)] + \eta B \text{ for a.e. } t\in[\delta,\delta+\rho],$$

(iv) for every $x,y\in K_\Phi(\Omega,C)$, every $\lambda>0$ and $u\in\mathcal{F}(G^F)(x)$, $v\in\mathcal{F}(G^F)(y)$ we have for all $t\in[\delta,\delta+\rho]$

$$\sup_{\delta\le\tau\le t}|x(\tau)-y(\tau)| \le \sup_{\delta\le\tau\le t}|x(\tau)-y(\tau)-\lambda(u(\tau)-v(\tau))|.$$

Then $NFDI(D,F)$ is (Ω,C)-controllable.

Proof. Let J_λ and $\mathcal{F}_\lambda$ be operators defined in Lemma 1. For every fixed $\lambda>0$ we define $f_\lambda:[\delta,\delta+\rho]\times K_\phi(\Omega,C) \to R^n$ by setting $f_\lambda(t,x) = \mathcal{F}_\lambda(x)(t)$. We have $f_\lambda(\cdot,x) = \mathcal{F}_\lambda(x) \in L([\delta,\delta+\rho],R^n)$ for every $x \in K_\phi(\Omega,C)$.

Now let $z_1 = J_\lambda(x)$ and $z_2 = J_\lambda(y)$ for fixed $x,y \in K_\phi(\Omega,C)$. By the definition of J_λ we obtain $z_1,z_2 \in K_\phi(\Omega,C)$ and $x \in z_1-\lambda\mathcal{F}(G^F)(z_1)$, $y \in z_2-\lambda\mathcal{F}(G^F)(z_2)$. Select now $u_1 \in \mathcal{F}(G^F)(z_1)$ and $u_2 \in \mathcal{F}(G^F)(z_2)$ such that $x = z_1 - \lambda u_1$, $y = z_2 - \lambda u_2$.

By (iv) for fixed $t \in [\delta,\delta+\rho]$ we obtain

$$\sup_{\delta\le\tau\le t} |z_1(\tau)-z_2(\tau)| \le \sup_{\delta\le\tau\le t} |z_1(\tau)-z_2(\tau)-\lambda(u_1(\tau)-u_2(\tau))|$$

i.e. $\sup\limits_{\delta\le\tau\le t} |J_\lambda(x)(\tau) - J_\lambda(y)(\tau)| \le \sup\limits_{\delta\le\tau\le t} |x(\tau)-y(\tau)|$.

Hence, by the definition of $\mathcal{F}_\lambda$ if follows that

$$\sup_{\delta\le\tau\le t} |\mathcal{F}_\lambda(x)(\tau) - \mathcal{F}_\lambda(y)(\tau)| = \frac{1}{\lambda}|[J_\lambda(x)(\tau)-J_\lambda(y)(\tau)]-[x(\tau)-y(\tau)]| \le$$

$$\le \frac{2}{\lambda} \sup_{\delta\le\tau\le t} |x(\tau)-y(\tau)| \quad \text{for } t \in [\delta,\delta+\rho].$$

Therefore

$$|f_\lambda(t,x)-f_\lambda(t,y)| \le \frac{2}{\lambda}|x-y|_t \quad \text{where} \quad |x-y|_t = \sup_{\delta\le\tau\le t}|x(\tau)-y(\tau)|.$$

Hence, in particular it follows that f_λ is for every fixed $\lambda>0$ continuous and has Volterra's property with respect to its second variable (it means that for every $u,v \in K_\phi(\Omega,C)$ and $t \in [\delta,\delta+\rho]$ such that $u(\tau) = v(\tau)$ for $\tau \in [\delta,t]$ we have $f_\lambda(t,u) = f_\lambda(t,v)$).

By virtue of Lemma 1 (ii) and (iv) we have $f_\lambda(\cdot,x) \in \mathcal{F}(G^F)(J_\lambda(x))$ and $|f_\lambda(t,x)| \le K$ for $x \in K_\phi(\Omega,C)$ and a.e. $t \in [\delta,\delta+\rho]$. By assumption (iii)(b) we can use Lemma 1 (iii) to obtain that for every $\delta_o>0$ there is $\lambda_o>0$ such that $\|J_\lambda(x)-x\|_{L_\delta} \le \delta_o$ for $x \in K_\phi(\Omega,C)$ and $\lambda \in (0,\lambda_o)$.

Now let $\varepsilon > 0$ be given and suppose $\delta_o > 0$ is such taken that

$$F(t,(\phi \oplus y)_t,(\dot{\phi} \boxplus \dot{y})_t) \subset T_H[(\phi \oplus x)_t(0)] + \varepsilon B$$

for a.e. $t \in [\delta,\delta+\varrho]$, every $x \in K_\alpha$ and $y \in K_\phi(\Omega,C)$ satisfying $\|x-y\|_{L_\delta} \leq \delta_o$. Then we have $G^F(t,J_\lambda(x)) \subset T_H[(\phi \oplus x)_t(0)] + \varepsilon B$ and $f_\lambda(t,x) \in T_H[(\phi \oplus x)_t(0)] + \varepsilon B$ for every $x \in K_\alpha$, $\lambda \in (0,\lambda_o)$ and a.e. $t \in [\delta,\delta+\varrho]$.

Now by Viability Theorem 4 given in [3] we obtain that for every $\varepsilon > 0$ and every $\lambda \in (0,\lambda_o)$ there is $x_\lambda \in K$ such that

$$\dot{x}_\lambda(t) \in G^F(t,x_\lambda) + \varepsilon B \quad \text{for a.e.} \quad t \in [\delta,\delta+\varrho].$$

Let $\lambda = \frac{1}{k}$ and select N such that $\frac{1}{k} < \lambda_o$ for $k \geq N$. We put $x^k = x_{1/k}$ for $k \geq N$. Since K_α is bounded, closed and uniformly absolutely continuous in AC then it is compact in C and therefore in L^1 too. Since $|x(t)| < 1+M$ for almost all t and $k \geq N$ then the set $\{\dot{x}^k\}$ is uniformly integrable so by Dunford's theorem L^1-relative sequential weakly compact. Therefore there exists $x_\varepsilon \in K_\alpha$ and a subsequence, say again (x^k) of (x^k) such that $\|x^k - x_\varepsilon\|_{L_\delta} \longrightarrow 0$ and $\dot{x}^k \longrightarrow \dot{x}_\varepsilon$ weakly in $L([\delta,\delta+\varrho],R^n)$.

We have $\dot{x}^k \in \mathcal{F}(G_\varepsilon^F)(x^k)$ for $k = 1,2,\ldots$ where $G_\varepsilon^F(t,x) = G^F(t,x) + \varepsilon B$ for $(t,x) \in I \times K$.

By virtue of Lemma 2 it follows that $\dot{x}_\varepsilon \in \mathcal{F}(G_\varepsilon^F)(x_\varepsilon)$ for every $\varepsilon > 0$.

Taking ε tending monotonically to zero we can again find $x \in K_\alpha$ and a subsequence of (x_ε) say (x_{ε_n}) such that $\|x_{\varepsilon_n} - x\|_{L_\delta} \longrightarrow 0$ and $\dot{x}_{\varepsilon_n} \longrightarrow \dot{x}$ weakly in $L([\delta,\delta+\varrho],R^n)$. Since $\dot{x}_{\varepsilon_n} \in \mathcal{F}(G^F_{\varepsilon_n})(x_{\varepsilon_n})$ for every $n = 1,2,\ldots$ then $\dot{x} \in \mathcal{F}(G^F_{\varepsilon_n})(x)$ for every ε_n. Therefore

$\dot{x} \in \mathcal{F}(G^F)(x)$. Put now $y = \Phi \oplus x$. We have $\dot{y}(t) \in F(t, y_t, \dot{y}_t)$ for $t \in [\delta, \delta+\varrho]$, $(t, y_t) \in \Omega$ and $(\delta, y_\delta, \delta+\varrho, y_{\delta+\varrho}) \in C$ because $x \in K_\alpha \subset K_\Phi(\Omega, C)$. Then NFDI(D,F) is (Ω, C)-controllable.

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