DEMONSTRATIO MATHEMATICA

Vol. XXIII No 2 1990

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BV-SOLUTIONS OF NONLINEAR FUNCTIONAL EQUATIONS OF HIGHER ORDER

Introduction

In [1] we have considered BV-solutions of the nonlinear functional equations of first order, i.e. the case $m = 1$. In this paper we shall be concerned with BV-solutions of a nonlinear functional equations of the order $m \geq 2$, i.e.

$$\varphi(x) = h(x,\varphi[f_1(x)],\dots,\varphi[f_m(x)]), \tag{1}$$

$$\varphi[f^m(x)] = g(x,\varphi[f^{m-1}(x)],\dots,\varphi[f(x)],\varphi(x)), \tag{2}$$

where φ is an unknown function and f^i denotes i-th iterate of the function f.

Let (X,d) be a metric space. By $P(I)$ we denote the set of all the finite partitions p: $x_0 < x_1 < \dots < x_s$ of an interval $I \subset R$, where $x_i \in I$, $i = 0,1,\dots,s$. Similarly as in [1], for the function $\varphi: I \to X$ we define

$$\operatorname*{Var}_I \varphi := \sup_{P(I)} \sum_{i=1}^{s} d(\varphi(x_i), \varphi(x_{i-1})). \tag{3}$$

By $BV(I)$ we denote the set of all functions $\varphi: I \to X$ such that $\operatorname{Var} \varphi < \infty$.

1. In this section we assume that

(i) (X,d) is a complete metric space,

(ii) $h: I\times X^m \longrightarrow X$,

(iii) there are a function $H \in BV(I)$ and a positive constants l_i, $i = 1,2,\dots,m$, such that $0 < \sum_{i=1}^{m} l_i < 1$ and $d(h(x,y_1,\dots,y_m), h(\bar{x},\bar{y}_1,\dots,\bar{y}_m)) \leq \sum_{i=1}^{m} l_i d(y_i,\bar{y}_i) + d(H(x),H(\bar{x}))$, for $(x,y_1,\dots,y_m)$, $(\bar{x},\bar{y}_1,\dots,\bar{y}_m) \in I\times X^m$,

(iv) $f_i: I \longrightarrow I$, $i = 1,2,\dots,m$, are continuous and strictly increasing in I.

Theorem 1. If hypotheses (i)-(iv) are fulfilled then the equation (1) has a unique solution $\varphi \in BV(I)$ given by formula $\varphi = \lim_{n\to\infty} \varphi_n$ (uniformly in I), where

$$\varphi_{n+1}(x) = h(x,\varphi_n[f_1(x)],\dots,\varphi_n[f_m(x)]), \quad x \in I, \quad n = 0,1,\dots$$

and $\varphi_o \in BV(I)$ is arbitrarily choosen. Moreover

$$\operatorname*{Var}_I \varphi \leq \frac{\operatorname*{Var}_I H}{1 - \sum_{i=1}^{m} l_i}. \tag{4}$$

Proof. Let us take an arbitrary $M \geq \dfrac{\operatorname*{Var}_I H}{1 - \sum_{i=1}^{m} l_i}$ and define $A_M := \left\{\varphi \in BV(I) : \operatorname*{Var}_I \varphi \leq M\right\}$. The set A_M with the metric $\rho(\varphi_1,\varphi_2) := \sup_{x\in I} d(\varphi_1(x),\varphi_2(x))$, $\varphi_1,\varphi_2 \in A_M$ is a complete metric space. In the space A_M let us consider the transformation $\varphi \longrightarrow T[\varphi]$ defined by the formula

$$T[\varphi](x) = h(x,\varphi[f_1(x)],\dots,\varphi[f_m(x)]), \quad x \in I. \tag{5}$$

We shall prove that (5) maps A_M into itself. By (iii) and (iv), we have

$$\operatorname*{Var}_{I} T[\varphi] = \sup_{P(I)} \sum_{j=1}^{s} d(h(x_j,\varphi[f_1(x_j)],\ldots,\varphi[f_m(x_j)]),$$

$$,h(x_{j-1},\varphi[f_1(x_{j-1})],\ldots,\varphi[f_m(x_{j-1})]))\leqslant$$

$$\leqslant \sup_{P(I)} \sum_{j=1}^{s} \left\{ \sum_{i=1}^{m} l_i d(\varphi[f_i(x_j)],\varphi[f_i(x_{j-1})]) + d(H(x_j),\ H(x_{j-1})) \right\} \leqslant$$

$$\leqslant \sum_{i=1}^{m} l_i \sup_{P(I)} \sum_{j=1}^{s} d(\varphi[f_i(x_j)],\varphi[f_i(x_{j-1})]) +$$

$$+ \sup_{P(I)} \sum_{j=1}^{s} d(H(x_j),H(x_{j-1})) = \sum_{i=1}^{m} l_i \operatorname*{Var}_{f_i(I)} \varphi + \operatorname*{Var}_{I} H \leqslant$$

$$\leqslant \sum_{i=1}^{m} l_i \operatorname*{Var}_{I} \varphi + \operatorname*{Var}_{I} H \leqslant M \sum_{i=1}^{m} l_i + M - M \sum_{i=1}^{m} l_i = M.$$

Further, we shall prove that (5) is a contraction map. Let us take arbitrary $\varphi_1,\varphi_2 \in A_M$. Taking into account (iii) and (iv), we have

$$\rho(T[\varphi_1],T[\varphi_2]) = \sup_{x\in I} d(h(x,\varphi_1[f_1(x)],\ldots,\varphi_1[f_m(x)]),$$

$$,h(x,\varphi_2[f_1(x)],\ldots,\varphi_2[f_m(x)]))\leqslant$$

$$\leqslant \sum_{i=1}^{m} l_i \sup_{x\in I} d(\varphi_1(x),\varphi_2(x)) = \rho(\varphi_1,\varphi_2) \sum_{i=1}^{m} l_i.$$

Thus, in virtue of the Banach principle there exists a unique fixed point of (5) in the space A_M. In other words there exists a unique solution $\varphi \in A_M$ of the equation (1). Because M is arbitrarily large, this is a unique BV(I) solution of equation (1).

Next, we shall prove that estimation (4) holds. Using successively hypotheses (iii) and (iv), we have

$$\operatorname*{Var}_I \varphi = \sup_{P(I)} \sum_{j=1}^{s} d(h(x_j,\varphi[f_1(x_j)],\dots,\varphi[f_m(x_j)]),$$

$$,h(x_{j-1},\varphi[f_1(x_{j-1})],\dots,\varphi[f_m(x_{j-1})]))\leq$$

$$\leq \sup_{P(I)} \sum_{j=1}^{s}\left\{\sum_{i=1}^{m} l_i\, d(\varphi[f_i(x_j)]\,,\,\varphi[f_i(x_{j-1})]) + d(H(x_j),H(x_{j-1}))\right\}\leq$$

$$\leq \sum_{i=1}^{m} l_i \operatorname*{Var}_{f_i(I)} \varphi + \operatorname*{Var}_I H \leq \operatorname*{Var}_I \varphi \sum_{i=1}^{m} l_i + \operatorname*{Var}_I H,$$

which proves inequality (4). This completes the proof of Theorem 1.

2. Now let us consider equation (2) and assume that

(i) (X,d) is a complete metric space,

(ii) $g: (a,b\rangle \times X^m \longrightarrow X$,

(iii) there are function $G \in BV(a,b\rangle$ and positive constants l_i $(i=1,2,\dots,m)$ such that $d(g(x,y_1,\dots,y_m),g(\bar{x},\bar{y}_1,\dots,\bar{y}_m))\leq$ $\leq\sum_{i=1}^{m} l_i\, d(y_i,\bar{y}_i) + d(G(x),G(\bar{x}))$ for $(x,y_1,\dots,y_m),(\bar{x},\bar{y}_1,\dots,\bar{y}_m)\in$ $\in (a,b\rangle \times X^m$,

(iv) $f: (a,b\rangle \longrightarrow (a,b\rangle$ is continuous and strictly increasing in $(a,b\rangle$, and $a<f(x)<x$ for $x \in (a,b\rangle$,

(v) all the roots of the polynomial $p(z) = z^m - l_1 z^{m-1} - \dots - l_m$ have absolute values less than 1.

T h e o r e m 2. By hypotheses (i)-(v), the equation (2) has a solution $\varphi \in BV(a,b\rangle$ depending on an arbitrary function. More precisely: for any system of functions $\{\varphi_i\}_{i=1,2,\dots,m}$ such that $\varphi_i \in BV(I_i)$, where $I_i = \langle f^i(b), f^{i-1}(b)\rangle$ and fulfilling the conditions $\varphi_{i+1}[f^i(b)] = \varphi_i[f^i(b)]$, $i = 1,2,\dots,m-1$, there exists the unique function $\varphi \in BV(a,b\rangle$ satisfying the equation (2) in $(a,b\rangle$ and such that $\varphi(x) = \varphi_k(x)$ for $x \in I_k$, $k = 1,2,\dots,m$.

P r o o f . Let $I_i = \langle f^i(b), f^{i-1}(b)\rangle$, $i = 1,2,\dots$ By (iv), we have $\bigcup_{i=1}^{\infty} I_i = (a,b\rangle$. Let us take arbitrary functions $\varphi_i : I_i \to R$, $i = 1,2,\dots,m$, fulfilling the conditions of the theorem. Notice that if $x \in I_{1+m}$, then $f^{-i}(x) \in I_{1+m-i}$, $i = 1,2,\dots,m$. In particular $f^{-m}(x) \in I_1$. So let us define the function $\varphi_{1+m} : I_{1+m} \to R$ for $x \in I_{1+m}$ as follows

$$\varphi_{1+m}(x) = g(f^{-m}(x), \varphi_{1+m-1}[f^{-1}(x)], \dots, \varphi_1[f^{-m}(x)]). \tag{6}$$

We shall prove that $\varphi_{1+m} \in BV(I_{1+m})$. Using successively (6), (iii) and (iv), we obtain

$$\operatorname*{Var}_{I_{1+m}} \varphi_{1+m} = \sup_{P(I_{1+m})} \sum_{j=1}^{s} d(g(f^{-m}(x_j), \varphi_{1+m-1}[f^{-1}(x_j)], \dots, \varphi_1[f^{-m}(x_j)]),$$

$$, g(f^{-m}(x_{j-1}), \varphi_{1+m-1}[f^{-1}(x_{j-1})], \dots, \varphi_1[f^{-m}(x_{j-1})])) \leq$$

$$\leq \sup_{P(I_{1+m})} \sum_{i=1}^{m} \sum_{j=1}^{s} l_i \, d(\varphi_{1+m-i}[f^{-i}(x_j)], \varphi_{1+m-i}[f^{-i}(x_{j-1})]) +$$

$$+ \sup_{P(I_{1+m})} \sum_{j=1}^{s} d(G[f^{-m}(x_j)], G[f^{-m}(x_{j-1})]) \leq \sum_{i=1}^{m} l_i \operatorname*{Var}_{I_{1+m-i}} \varphi_{1+m-i} + \operatorname*{Var}_{I_1} G.$$

Hence $\varphi_{1+m} \in BV(I_{1+m})$. More generally, let us take a function $\varphi_{k+m-1}: I_{k+m-1} \to R$ and let $\varphi_{k+m-1} \in BV(I_{k+m-1})$, $k = 2,3,\dots$. Evidently, if $x \in I_{k+m}$ then $f^{-m}(x) \in I_k$. Therefore the function $\varphi_{k+m}: I_{k+m} \to R$ given by the formula

$$(7) \quad \varphi_{k+m}(x) = g(f^{-m}(x), \varphi_{k+m-1}[f^{-1}(x)], \dots, \varphi_k[f^{-m}(x)]), \quad k=2,3,\dots$$

is well defined. Moreover, by a similar calculation as above, one can prove that $\varphi_{k+m} \in BV(I_{k+m})$ and that

$$(8) \quad \operatorname*{Var}_{I_{k+m}} \varphi_{k+m} \leq \sum_{i=1}^{m} l_i \operatorname*{Var}_{I_{k+m-i}} \varphi_{k+m-i} + \operatorname*{Var}_{I_k} G, \qquad k = 2,3,\dots$$

Now we introduce the following notations

$$a_k := \operatorname*{Var}_{I_k} \varphi_k, \quad b_k := \operatorname*{Var}_{I_k} G, \qquad k = 1,2,\dots$$

Using these notations we can write (8) as follows

$$(9) \qquad a_{k+m} \leq \sum_{i=1}^{m} l_i a_{k+m-i} + b_k, \qquad k = 1,2,\dots$$

We define the function $\varphi: (a,b\rangle \to R$ in the following way

$$(10) \qquad \varphi(x) = \begin{cases} \varphi_i(x) & \text{for} \quad x \in I_i, \quad i=1,2,\dots,m, \\ \varphi_i(x) & \text{for} \quad x \in I_i, \quad i=m+1,m+2,\dots \end{cases}$$

From (6), (7) and (10) it follows that φ satisfies equation (2) in $(a,b\rangle$. The uniqueness of the solution is obvious. To prove that (10) belongs to the class $BV(a,b\rangle$, we write $\operatorname*{Var}_{(a,b\rangle} \varphi = \sum_{n=1}^{\infty} a_n$. Since the sequence a_n, $n \in N$, satisfies inequality (9) it follows from Lemma 5.1 in [2] that the series $\sum_{n=1}^{\infty} a_n$ converges. This completes the proof.

REFERENCES

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Received November 7, 1988.