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Elżbieta Łuczak-Kumorek

THE EXISTENCE THEOREM FOR NEUTRAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS

1. Introduction

The aim of this paper is to present the existence theorem for functional-differential inclusions of the form

$$\frac{d}{dt} D(t,x_t) \in F(t,x_t)$$

where F is a multivalued mapping having a Carathéodory selector and taking as its values nonempty closed compact but not necessarily convex or nonempty closed convex subsets of R^n and D is a single-valued mapping with values in R^n. We extend the results of J.K. Hale [4] on the functional-differential inclusions of neutral type.

2. Notations and definitions

Suppose $r \geq 0$ is a given real number, $R = (-\infty, +\infty)$, R^n is a n-dimensional linear vector space with a norm $|\cdot|$, $C([a,b], R^n)$ is the Banach space of continuous functions endowed with the topology of uniform convergence. If $[a,b] = [-r,0]$ we let $C_{or} = C([-r,0],R^n)$ and denote the norm of an element $\phi \in C_{or}$ by $\|\phi\| = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|$.

If $\sigma \in R$, $a \geq 0$ and $x \in C([\sigma - r, \sigma+a], R^n)$, then for any $t \in [\sigma, \sigma+a]$, we define $x_t \in C_{or}$ by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

By P(X), CCl(X) and Comp(X) we denote the spaces of all nonempty, nonempty closed convex and nonempty closed compact, respectively, subsets of metric spaces X.

Let $(T,\mathcal{F})$ be a measurable space, X be a metric space. A set-valued function $F: T \longrightarrow P(X)$ is said to be measurable (weakly measurable), if $\{t \in T: F(t) \cap U \neq \emptyset\} \in \mathcal{F}$ for every closed (open) set $U \subset X$ (see [3]).

If $F: Y \longrightarrow P(X)$, where Y is a topological space, then the assertion that F is measurable (weakly measurable) means that F is measurable (weakly measurable) when Y is assigned by σ-algebra $\beta(Y)$ of Borel subsets of Y.

If $F: T \times Y \longrightarrow P(X)$ then the various kinds of measurability of F are defined in terms of the product σ-algebra $\mathcal{F} \otimes \beta(Y)$ on $T \times Y$. (For T = R we will consider T together with the σ-algebra of all its Lebesgue measurable subsets).

A multivalued mapping $F: X \longrightarrow P(Y)$ is lower semicontinuous if the set $\{x \in X: F(x) \cap Z \neq \emptyset\}$ is open for every open subset Z of Y.

Let $CCl(R^n)$ be the family of all nonempty closed convex subsets of R^n, $\Omega \subseteq R \times C_{or}$ be an open set. Assume that $F: \Omega \longrightarrow CCl(R^n)$ satisfies the following conditions:

(i) $F: \Omega \longrightarrow CCl(R^n)$ is weakly measurable,

(ii) $F(t,\cdot)$ is lower semicontinuous for each fixed $t \in R$,

(iii) there is a Lebesgue integrable function $m: R \longrightarrow R^+$ such that $h(F(t,\phi), \{0\}) \leq m(t)$ for $(t,\phi) \in \Omega$, where h denotes the Hausdorff metric in $CCl(R^n)$.

Definition 1. Suppose Ω is an open set in $R \times C_{or}$, $D: \Omega \longrightarrow R^n$ is continuous, $\phi \longrightarrow D(t,\phi)$ has a continuous Fréchet derivative $D'_{\phi}(t,\phi)$ and

$$D'_{\phi}(t,\phi)\psi = \int_{-r}^{0} [d_{\theta}\eta(t,\phi,\theta)]\, \psi(\theta)$$

for $(t,\phi)\in\Omega$, $\psi\in C_{or}$, where $\eta(t,\phi,\theta)$ is an nxn matrix function with elements of bounded variation in $\theta\in[-r,0]$. For any $\beta\in[-r,0]$ we say that $D(t,\phi)$ is atomic at β on Ω, if

$$\eta(t,\phi,\beta^+) - \eta(t,\phi,\beta^-) = A(t,\phi,\beta),\ \det A(t,\phi,\beta) \neq 0$$

for some nxn matrix function $A(t,\phi,\beta)$ continuous in (t,ϕ) and there is a scalar function $\gamma(t,\phi,s,\beta)$ continuous for $(t,\phi)\in\Omega$, $s\geq 0$, $\gamma(t,\phi,0,\beta) = 0$ such that $\left|\int_{\beta-s}^{\beta+s} [d_\theta\eta(t,\phi,\theta)]\psi(\theta) - A(t,\phi,\beta)\psi(\beta)\right| \leq$ $\leq \gamma(t,\phi,s,\beta)\|\psi\|$ for $(t,\phi)\in\Omega$, $s\geq 0$, $\psi\in C_{or}$.

Definition 2. Suppose $\Omega\subseteq R\times C_{or}$ is an open set, $D:\Omega\to R^n$ is a given continuous function atomic at zero. The relation

$$\frac{d}{dt}D(t,x_t)\in F(t,x_t) \tag{1}$$

is called the neutral functional-differential inclusion (NFDI).

Definition 3. For a given NFDI a function $x:[\sigma-r,\sigma+a)\to R^n$ is said to be a solution of (1) if there are $\sigma\in R$, $a>0$, such that $x\in C([\sigma-r,\sigma+a),R^n)$, $(t,x_t)\in\Omega$, $t\in[\sigma,\sigma+a)$, $t\to D(t,x_t)$ is continuously differentiable and satisfies (1) a.e. on $[\sigma,\sigma+a)$.

Definition 4. For a given $\sigma\in R$, $\phi\in C_{or}$ and $(t,\phi)\in\Omega$ we say that x is a solution of inclusion (1) with initial value ϕ at σ or simply a solution through (σ,ϕ), if there is an $a>0$ such that x is a solution of (1) on $[\sigma-r,\sigma+a)$ and $x_\sigma = \phi$.

Definition 5. Let T be measurable space and X be a metric space. A function $f: T\times T\to R^n$ is said to be a Carathéodory selector of a set-valued function $F: T\times X\to Cl(R^n)$, if f is a selector for F and f is such that $f(\cdot,x)$ is measurable for $x\in X$ and $f(t,\cdot)$ is continuous for $t\in T$.

Definition 6. A set-valued function $F:T\times X\to P(R^n)$, where T and X are as above, is called an M-mapping if every its lower semicontinuous restriction has a continuous selector.

3. Existence theorem

Theorem 1. If $\Omega \subseteq R \times C_{or}$ is an open set, (1) is NFDI and $F: \Omega \to CCl(R^n)$ satisfies conditions (i)-(iii) then for any $(\sigma, \phi) \in \Omega$ there exists a solution of (1) through (σ, ϕ).

Proof. A function x is a solution of (1) through (σ, ϕ) if there is an $a > 0$ such that $x \in C([\sigma - r, \sigma + a), R^n)$ and

$$(2) \qquad \begin{cases} \frac{d}{dt} D(t, x_t) \in F(t, x_t) & \text{for a.e. } t \in [\sigma, \sigma + a], \\ x_\sigma = \phi. \end{cases}$$

Since $F: \Omega \to CCl(R^n)$ satisfies the conditions (i)-(iii) then, in virtue of [2], there exists a mapping $f: \Omega \to R^n$ such that:

(a) $f(t, \cdot)$ is continuous for each fixed $t \in R$,

(b) f is measurable,

(c) $f(t, z) \in F(t, z)$ for $(t, z) \in \Omega$.

Clearly, f is a Carathéodory selection for F. Then x is a solution of (1) through (σ, ϕ), whenever there is an $a > 0$ such that x satisfies

$$(3) \qquad \begin{cases} D(t, x_t) = D(\sigma, \phi) + \int_\sigma^t f(\tau, x_\tau) d\tau & \text{for a.e. } t \in [\sigma, \sigma + a], \\ x_\sigma = \phi, \end{cases}$$

where f is a Carathéodory selection of a set-valued map F. Let $\hat{\phi}: [-r, \infty) \to R^n$ be defined by

$$\hat{\phi}(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, 0], \\ \phi(0) & \text{for } t \in [0, \infty), \end{cases}$$

and let $C^o([\sigma, \sigma + a], R^n)$ denote the Banach space of all continuous functions $z: [\sigma, \sigma + a] \to R^n$ such that $z(\sigma) = 0$. For every $z \in C^o([\sigma, \sigma + a], R^n)$ let

$$\hat{z}(t) = \begin{cases} z(t) & \text{for } t \in [\sigma, \sigma+a], \\ 0 & \text{for } t \in [\sigma-r, \sigma]. \end{cases}$$

We have, of course, $\hat{z} \in C([\sigma-r, \sigma+a], R^n)$ and $\hat{z}\big|_{[\sigma,\sigma+a]} = z$.

Now, we can define for each fixed $\sigma \in R$ and $a > 0$ a mapping $\hat{\phi} \oplus \hat{z}$ by setting

$$(4) \qquad (\hat{\phi} \oplus \hat{z})(t) = \begin{cases} \hat{\phi}(t-\sigma) & \text{for } t \in [\sigma-r, \sigma], \\ \hat{\phi}(0) + \hat{z}(t) & \text{for } t \in [\sigma, \sigma+a]. \end{cases}$$

In what follows, we shall denote $\hat{\phi} \oplus \hat{z}$ by $\phi \oplus z$.

Let us observe that the functional-integral equation (3) is equivalent to the following one

$$(3') \qquad \begin{cases} D(t, (\phi \oplus z)_t) = D(\sigma, \phi) + \int_\sigma^t f(\tau, (\phi \oplus z)_\tau)\, d\tau \\ \qquad\qquad \text{for a.e. } t \in [\sigma, \sigma+a], \\ z_\sigma = 0. \end{cases}$$

In this way x is a solution of (1) through (σ, ϕ), if there is an $a > 0$ and $z \in C([\sigma, \sigma+a], R^n)$ such that z satisfies (3′) and $x = \phi \oplus z$.

Since $D(t, \phi)$ is continuously differentiable in ϕ, then

$$(5) \qquad D(t, (\phi \oplus z)_t) = D(t, (\phi \oplus 0)_t) +$$

$$+ D'_\phi(t, (\phi \oplus 0)_t)\left[(\phi \oplus z)_t - (\phi \oplus 0)_t\right] + g(t, (\phi \oplus 0)_t, \left[(\phi \oplus z)_t - (\phi \oplus 0)_t\right]),$$

where $g(t, \varphi, 0) = 0$, $|g(t, \varphi, \psi) - g(t, \varphi, \xi)| \leq \bar{\varepsilon}(t, \varphi, \delta)\|\psi - \xi\|$ for $(t, \varphi) \in \Omega$, $\|\psi\|, \|\xi\| \leq \delta$ and $\bar{\varepsilon}(t, \varphi, \delta)$ is continuous in t, φ, δ for $(t, \varphi) \in \Omega$, $\delta \geq 0$ and $\bar{\varepsilon}(t, \varphi, 0) = 0$. By (4), we have $(\phi \oplus z)_t - (\phi \oplus 0)_t = z_t$. Hence, by (5), it follows that

$$D'_{\phi}(t,(\phi \oplus 0)_t)z_t = D(t,(\phi \oplus z)_t) - D(t,(\phi \oplus 0)_t) - g(t,(\phi \oplus 0)_t, z_t).$$

Therefore, by (3'), x is a solution of (1) through (σ,ϕ), if $x(t) = (\phi \oplus z)(t)$ for $t \in [\sigma-r,\sigma+a]$ and z satisfies

$$(6)\quad \begin{cases} D'_{\phi}(t,(\phi \oplus 0)_t)z_t = D(\sigma,\phi) - D(t,(\phi \oplus 0)_t) - \\ \qquad - g(t,(\phi \oplus 0)_t, z_t) + \int_{\sigma}^{t} f(\tau,(\phi \oplus z)_\tau)d\tau \text{ for a.e. } t \in [\sigma,\sigma+a], \\ z_\sigma = 0. \end{cases}$$

Using the fact that D is atomic at 0 on Ω, we have (as long as $(t,(\phi \oplus 0)_t) \in \Omega$)

$$A(t,(\phi \oplus 0)_t,0)z_t(0) + \int_{-r}^{0-} [d_\theta \eta(t,(\phi \oplus 0)_t,\theta)]\, z_t(\theta) =$$

$$= D(\sigma,\phi) - D(t,(\phi \oplus 0)_t) - g(t,(\phi \oplus 0)_t,z_t) + \int_{\sigma}^{t} f(\tau,(\phi \oplus z)_\tau)\, d\tau .$$

Then, since $z_t(0) = z(t)$, we have

$$(7)\quad \begin{cases} z(t) = [A(t,(\phi \oplus 0)_t,0)]^{-1} \Big\{ -\int_{-r}^{0-} [d_\theta \eta(t,(\phi \oplus 0)_t,\theta)]\, z_t(\theta) + \\ + D(\sigma,\phi) - D(t,(\phi \oplus 0)_t) - g(t,(\phi \oplus 0)_t,z_t) + \int_{\sigma}^{t} f(\tau,(\phi \oplus z)_\tau)d\tau \Big\} \\ \qquad \text{for a.e. } t \in [\sigma,\sigma+a], \\ z_\sigma = 0. \end{cases}$$

If we let

$$(8)\quad (\dot{T}z)(t) = \begin{cases} 0 \quad \text{for} \quad t \in [\sigma-r,\sigma], \\ [A(t,(\phi \oplus 0)_t,0)]^{-1} \Big\{ -\int_{-r}^{0-} [d_\theta \eta(t,(\phi \oplus \theta)_t,\theta)]\, z_t(\theta) + \\ + D(\sigma,\phi) - D(t,(\phi \oplus 0)_t) - g(t,(\phi \oplus 0)_t,z_t) \Big\} \\ \text{for a.e. } t \in [\sigma,\sigma+a], \end{cases}$$

$$(9)\qquad (Sz)(t) = \begin{cases} 0 & \text{for} \quad t\in[\sigma-r,\sigma], \\ [A(t,(\phi\oplus 0)_t,0)]^{-1}\displaystyle\int_\sigma^t f(\tau,(\phi\oplus z)_\tau)\,d\tau & \\ \text{for a.e.} \quad t\in[\sigma,\sigma+a], & \end{cases}$$

then (7) is equivalent to $z(t) = (Tz)(t) + (Sz)(t)$ where $z\in C([\sigma-r,\sigma+a],R^n)$, $z_\sigma = 0$.

The proof of Theorem 1 will be complete if we show the existence of a fixed point of $T + S$ in $\mathcal{A}(a,b) = \{\xi\in C([\sigma-r,\sigma+a],R^n): \xi_\sigma = 0.\ \|\xi_t\|\le b$ for $t\in[\sigma,\sigma+a]\}$. For this purpose we will show the following lemma.

Lemma 1. There are positive real numbers a, b and $\bar a,\bar b$, $0<\bar a<a$, $0<\bar b<b$, such that $T:\mathcal{A}(\bar a,\bar b)\to C([\sigma-r,\sigma+a],R^n)$ and $S:\mathcal{A}(\bar a,\bar b)\to C([\sigma-r,\sigma+a],R^n)$, have the properties

1° T is a contraction,

2° S is completely continuous,

3° $T + S:\mathcal{A}(\bar a,\bar b)\to\mathcal{A}(\bar a,\bar b)$,

where T and S are mappings defined by (8) and (9), respectively.

Proof. For any ν, $0<\nu<\frac{1}{4}$, there are $a>0$, $b>0$, such that $(t,(\phi\oplus 0)_t)\in\Omega$ and

$$\left|[A(t,(\phi\oplus 0)_t,0)]^{-1}\right|\,\varepsilon(t,(\phi\oplus 0)_t,b)<\nu,$$

$$\left|[A(t,(\phi\oplus 0)_t,0)]^{-1}\right|\,\gamma(t,(\phi\oplus 0)_t,a,0)<\nu$$

for $t\in[\sigma,\sigma+a]$ where $\gamma(t,\phi,s,\beta)$ is the function from Definition 1 and $\varepsilon(t,\varphi,\delta)$ is the function in the proof of Theorem 1. For any non-negative real $\bar a$ and $\bar b$ let

$$\mathcal{A}(\bar a,\bar b) = \left\{\xi\in C([\sigma-r,\sigma+\bar a],R^n) : \xi_\sigma = 0,\ \|\xi_t\|\le\bar b \text{ for } t\in[\sigma,\sigma+a]\right\}.$$

For any $0<\bar{b}<b$, there is an $\bar{a}$, $0<\bar{a}<a$, so that $\|(\phi\oplus 0)_t-\phi\|<b-\bar{b}$. Further, we restrict $\bar{a}$ so that for $t\in[\sigma,\sigma+\bar{a}]$ we have

$$|[A(t,(\phi\oplus 0)_t,0)]^{-1}|\ |D(\sigma,\phi)-D(t,(\phi\oplus 0)_t)|\leqslant \nu\bar{b},$$

$$|[A(t,(\phi\oplus 0)_t,0)]^{-1}|\ |\int_{\sigma}^{t} m(\tau)d\tau|\leqslant \nu\bar{b}.$$

We now show that T and S satisfy the conditions 1° - 3°. In fact, by (8) and the above restrictions on $\bar{a}$ and $\bar{b}$, for any $z,y\in\mathcal{A}(\bar{a},\bar{b})$ we have

$$|(Tz)(t)-(Ty)(t)| =$$

$$= \Big|[A(t,(\phi\oplus 0)_t,0)]^{-1}\Big\{-\int_{-r}^{0-}[d_\theta\eta(t,(\phi\oplus 0)_t,\theta)]z_t(\theta)-D(t,(\phi\oplus 0)_t)+$$

$$+D(\sigma,\phi)-g(t,(\phi\oplus 0)_t,z_t)+\int_{-r}^{0-}[d_\theta\eta(t,(\phi\oplus 0)_t,\theta)]y_t(\theta)+$$

$$+D(t,(\phi\oplus 0)_t)-D(\sigma,\phi)+g(t,(\phi\oplus 0)_t,y_t)\Big\}\Big|\leqslant$$

$$\leqslant\Big|[A(t,(\phi\oplus 0)_t,0)]^{-1}\Big|(\ell(t,(\phi\oplus 0)_t,\bar{b})\|z_t-y_t\|+$$

$$+\gamma(t,(\phi\oplus 0)_t,\bar{a},0)\cdot\|z_t-y_t\|)\leqslant\nu\|z_t-y_t\|+\nu\|z_t-y_t\|<\frac{1}{2}\|z_t-y_t\|.$$

Hence, T is a contraction.

Let $z,y\in\mathcal{A}(\bar{a},\bar{b})$ and let $\mu=Tz+Sy$. Since, for $t\in[\sigma-r,\sigma]$, by (8), (9), $(Tz)(t)+(Sy)(t)=0$ and for $t\in[\sigma,\sigma+\bar{a}]$ we have

$$|\mu(t)| = \Big|[A(t,(\phi\oplus 0)_t,0]^{-1}\Big\{-\int_{-r}^{0-}[d_\theta\eta(t,(\phi\oplus 0)_t,\theta)]\ z_t(\theta)+$$

$$+D(\sigma,\phi)-D(t,(\phi\oplus 0)_t)-g(t,(\phi\oplus 0)_t,z_t)+\int_{\sigma}^{t}f(\tau,(\phi\oplus y)_\tau)d\tau\Big\}\Big|\leqslant$$

$$\leqslant\nu\bar{b}+\nu\bar{b}+\nu\bar{b}+\nu\bar{b}<\bar{b},$$

therefore $T + S : \mathcal{A}(\bar{a},\bar{b}) \to \mathcal{A}(\bar{a},\bar{b})$. It is not difficult to show that S is continuous. Moreover, S is compact, since for every $t_1, t_2 \in$ $\in [\sigma - r, \sigma + \bar{a}]$

$$|(Sz)(t_1) - (Sz)(t_2)| \leq \frac{1}{2}\bar{b}.$$

This completes the proof of Lemma 1.

Lemma 1 yields the existence of a fixed point of T + S in $\mathcal{A}(\bar{a},\bar{b})$ (see [5], Lemma 2.1) and thus a solution of (1) through (σ,ϕ). Then the proof of Theorem 1 is completed.

Remark. Theorem 1 will be true in the case of the multivalued mapping $F : \Omega \to \mathrm{Comp}(R^n)$ if F is an M-mapping such that F is measurable on Ω, $F(t,\cdot)$ is lower semicontinuous for each fixed t and $pr_R(\Omega)$ has a finite measure, where $pr_R(\Omega)$ denotes the projection of on the real line. Then, in virtue of [1]ΩF has a Carathéodory selector f and the proof of Theorem 1 will be analogous.

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INSTITUTE OF MATHEMATICS AND PHYSICS, HIGHER COLLEGE OF ENGINEERING, 65-246 ZIELONA GÓRA, POLAND

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